

Homogeneous polynomially parameter-dependent state feedback controllers for finite time stabilization of linear time-varying systems

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Abstract—This paper investigates the problem of parameter-dependent state feedback control of continuous-time systems in the context of finite time stability. The controller is designed in order to guarantee that the closed-loop system is finite time stable. The system is considered time varying with the parameters modeled within a unit simplex. The design conditions obtained by means of Lyapunov functions are expressed as linear matrix inequalities. The finite time stability is assessed by using homogeneous polynomially parameter-dependent state feedback gains with arbitrary degree g . LMI relaxations are proposed based on Pólya's theorem. A controller is obtained by the solution of a factibility problem and the effect of the relaxation procedures analyzed by an optimization problem. Numerical examples are provided.

I. INTRODUCTION

Finite time stability and stabilization of time-varying dynamic systems have attracted considerable attention of specialists in control theory. The main idea is to provide tools for analysis and synthesis of systems that can assure an upper bound to the state space trajectories over a finite time interval given pre-specified constraints on the initial conditions. As presented in [1], a time-varying linear system

$$\dot{x}(t) = A(t)x(t), \quad t \in [0, T]$$

is said to be finite time stable with respect to (c_1, c_2, T, R) , with $c_2 > c_1$ and $R > 0$, if

$$x(0)'Rx(0) \leq c_1 \Rightarrow x(t)'Rx(t) \leq c_2, \quad \forall t \in [0, T].$$

This condition does not consider any other requirement with respect to the system asymptotic stability [2]. It is noteworthy to mention that an asymptotically stable nonlinear system that can reach an equilibrium point within a finite time interval is also denominated as finite time stable in the literature. This case is not considered in this paper, and the interested readers are encouraged to see [3] and references therein.

By considering the former definition given above, the problem of designing a controller that assures finite time stabilization reduces to the problem of finding a control law that bounds the magnitude of the closed-loop system states over a finite time interval. Note that it is possible to constraint the system behavior within the transient time, even if it is not asymptotically stable, what may be used, as for instance to set control strategies to avoid the excitation of nonlinear dynamics [4].

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In this context, it is worth mentioning the works [5] concerned with finite-time stabilization of precisely known systems affected by norm-bounded exogenous disturbances, [6] that provides necessary and sufficient conditions for finite time stabilization of linear time-varying systems and [7] that deals with the problem of finite time stability and stabilization of nonlinear quadratic in the state systems. In [8], the author provides sufficient conditions to design gain scheduling state feedback controllers assuring finite time stability of the closed-loop system. In [4] and [9], necessary and sufficient conditions for finite time stability and robust finite time stabilization are presented in terms of differential linear matrix inequalities.

Although necessary and sufficient conditions are presented in some works, as in [4, 9, 10], they may not be useful from a practical point of view, or even computationally prohibitive. To overcome this, conservatism is introduced in the conditions to turn them numerically tractable, for instance, in [10], these can be relaxed by matrix sum of squares.

Thus, the trade-off between conservatism and numeric efficiency encourages continued research. In this sense, the present work proposes an extension of the method appeared in [8] to deal with finite time stabilization of linear parameter varying systems (LPV) by using homogeneous polynomially parameter-dependent state feedback controllers of arbitrary degree. LPV models belong to a class of linear systems whose dynamics are described as a function of time-varying parameters, and it is useful, as for instance, to describe nonlinear systems in different operation points [11, 12]. The finite time stability condition proposed in [1] is used to obtain the main results.

Different from previous results seen in the literature, LMI relaxations based on Pólya's theorem are also considered. The idea is to increase the polynomial degree, or the relaxation index, providing less conservative design conditions [13]. Moreover, the time-varying parameters are described using a polytope, and it is assumed to be available online. The resolution of the sensors is considered high enough so that the measurement errors can be ignored. It is also assumed that the parameters may vary arbitrarily, or their variation rates are unknown. Numerical examples illustrate the main results.

II. PROBLEM STATEMENT AND PRELIMINARY RESULTS

Consider a linear time-varying system with $t \in [0, T]$

$$\dot{x}(t) = A(\alpha(t))x(t) + B_1(\alpha(t))u(t) + B_2(\alpha(t))w(t) \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state space vector, $u(t) \in \mathbb{R}^p$ is the control input, and $w(t) \in \mathbb{R}^r$ is the noise input with bounded L_2 norm. All matrices are real, with appropriate dimensions, belonging to the polytope¹

$$\mathcal{P} \triangleq \left\{ \left[\begin{array}{c|c} A(\alpha) & B_1(\alpha) \\ \hline B_2(\alpha) & - \end{array} \right] = \sum_{i=1}^N \alpha_i \left[\begin{array}{c|c} A_i & B_{1i} \\ \hline B_{2i} & - \end{array} \right] \right\} \quad (2)$$

The vector of time-varying parameters $\alpha \in \mathbb{R}^N$ belongs to the unit simplex

$$\mathcal{U} = \left\{ \gamma \in \mathbb{R}^N : \sum_{i=1}^N \gamma_i = 1, \gamma_i \geq 0, i = 1, \dots, N \right\}$$

and the system matrices are given, for any time $t \geq 0$, by the convex combination of the known vertices of the polytope \mathcal{P} . It is also considered that no assumption is made on the parameter rate of variation, and that they are measured online.

The control law investigated here is given by

$$u(t) = K_g(\alpha)x(t) \quad (3)$$

where $K_g(\alpha)$ is an homogeneous polynomially parameter-dependent gain of arbitrary degree g generally written as

$$K_g(\alpha) = \sum_{j=1}^{J(g)} \alpha_1^{k_1} \alpha_2^{k_2} \dots \alpha_N^{k_N} K_{\mathcal{K}_j(g)}, \quad k_1 k_2 \dots k_N = \mathcal{K}_j(g) \quad (4)$$

where $\alpha_1^{k_1} \alpha_2^{k_2} \dots \alpha_N^{k_N}$, $\alpha \in \mathcal{U}$, $k_i \in \mathbb{Z}_+$, $i = 1, \dots, N$ are the monomials and $K_{\mathcal{K}_j(g)} \in \mathbb{R}^{p \times n}$, $j = 1, \dots, J(g)$ are matrix valued coefficients to be determined. $\mathcal{K}(g)$ is the set of N -tuples obtained as all possible combinations of $k_1 k_2 \dots k_N$, $k_i \in \mathbb{N}$, $i = 1, \dots, N$, such that $k_1 + k_2 + \dots + k_N = g$. $\mathcal{K}_j(g)$ is the j th N -tuples of $\mathcal{K}(g)$, which is lexically ordered, $j = 1, \dots, J(g)$. As an example, fixing $N = 3$ and $g = 2$ results in

$$K_3(\alpha) = \alpha_3^2 K_{002} + \alpha_2 \alpha_3 K_{011} + \alpha_2^2 K_{020} \\ + \alpha_1 \alpha_3 K_{101} + \alpha_1 \alpha_2 K_{110} + \alpha_1^2 K_{200}.$$

The closed-loop system is given by

$$\dot{x}(t) = \bar{A}(\alpha)x(t) + \bar{B}(\alpha)w(t) \quad (5)$$

where $\bar{A}(\alpha) = A(\alpha) + B_1(\alpha)K_g(\alpha)$ and $\bar{B}(\alpha) = B_2(\alpha)$

The problem to be solved is to find sufficient design conditions of state-feedback controller as in (3) with gain given by (4) such that the closed-loop system (5) is bounded over a finite time interval.

In the presence of external disturbances, the concept of finite time stability is extended to finite time boundedness as presented below.

Definition 1: [14] Given three positive scalars c_1, c_2 and T , with $c_2 > c_1$, a positive definite matrix R and a class of signals \mathcal{W} , the time-varying linear system

$$\dot{\zeta}(t) = \bar{A}(t)\zeta(t) + \bar{B}(t)w(t) \quad (6)$$

¹The time dependence of $\alpha(t)$ will be omitted to lighten the notation.

is finite time bounded with respect to $(c_1, c_2, \mathcal{W}, T, R)$, if

$$\zeta(0)'R\zeta(0) \leq c_1 \Rightarrow \zeta(t)'R\zeta(t) \leq c_2, \\ \forall w(t) \in \mathcal{W}, \forall t \in [0, T].$$

It is important to mention that variants of the class of signals \mathcal{W} give rise to different types of finite time boundedness (FTB) problem formulations. This fact can be explored in accordance with the knowledge of the system designer on the type of disturbances affecting the system. Some cases can be seen in [1, 8, 10, 15, 16].

The class \mathcal{W} to be considered in this work consist of all signals $w(t)$ satisfying the following constraint

$$w(t)'w(t) \leq d, \quad \forall t \in [0, T], \quad d \geq 0. \quad (7)$$

It is easy to note that for $w(t) = 0$, FTB implies finite time stability.

The control problem to be dealt with can be stated as follows.

Problem 1: Assume that α and $x(t)$ are available online for feedback through the control law (3), find an homogeneous polynomially parameter-dependent gain $K_g(\alpha) \in \mathbb{R}^{p \times n}$ in the form (4) such that the closed-loop system (5) is FTB with respect to $(c_1, c_2, \mathcal{W}, T, R)$ considering the class of signals (7).

The following lemma presents a sufficient condition for the analysis of FTB of a linear time-varying system, and is used in the solution of Problem 1.

Lemma 1: [1] System (6) is FTB with respect to $(c_1, c_2, \mathcal{W}, T, R)$, if there exist positive definite symmetric matrices $Q_1 \in \mathbb{R}^{n \times n}$, $Q_2 \in \mathbb{R}^{r \times r}$, and a positive scalar β such that

$$\begin{bmatrix} \bar{A}(t)\bar{Q}_1 + \bar{Q}_1\bar{A}'(t) - \beta\bar{Q}_1 & \bar{B}(t)Q_2 \\ Q_2\bar{B}'(t) & -\beta Q_2 \end{bmatrix} < 0 \quad (8)$$

$$\frac{c_1}{\lambda_{\min}(Q_1)} + \frac{d}{\lambda_{\min}(Q_2)} < \frac{c_2 e^{-\beta T}}{\lambda_{\max}(Q_1)} \quad (9)$$

in which $\bar{Q}_1 = R^{-1/2}Q_1R^{-1/2}$, and $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ indicate, respectively, the maximum and the minimum eigenvalue of the argument.

The proof will be presented for completeness since it differs from the one in [1] by showing that this lemma is also valid in the case where $w(t)$ is any signal, constrained only in its norm as specified by the class \mathcal{W} . As shown in [1], the signal $w(t)$ should be constant for all $t \in [0, T]$ or, if time varying, must be differentiable within the finite time interval.

Proof: Consider a Lyapunov function $V(\zeta) = \zeta'\bar{Q}_1^{-1}\zeta$, and assume that the inequality

$$\dot{V}(\zeta) < \beta V(\zeta) + \beta w'Q_2^{-1}w$$

is valid for all $t \in [0, T]$. Multiplying both sides by $e^{-\beta t}$ one has

$$e^{-\beta t}\dot{V}(\zeta) < \beta e^{-\beta t}V(\zeta) + \beta e^{-\beta t}w'Q_2^{-1}w \\ e^{-\beta t}\dot{V}(\zeta) - \beta e^{-\beta t}V(\zeta) < \beta e^{-\beta t}w'Q_2^{-1}w \\ \frac{d(e^{-\beta t}V(\zeta))}{dt} < \beta e^{-\beta t}w'Q_2^{-1}w.$$

Integrating the last inequality it follows that

$$e^{-\beta t}V(\zeta) - V(\zeta_0) < \beta \int_0^t e^{-\beta\tau} w' Q_2^{-1} w d\tau$$

$$V(\zeta) < e^{\beta t} V(\zeta_0) + \beta e^{\beta t} \int_0^t e^{-\beta\tau} w' Q_2^{-1} w d\tau.$$

Knowing that $w(t)$ belongs to the class \mathscr{W} defined in (7), then

$$w' Q_2^{-1} w \leq \lambda_{\max}(Q_2^{-1}) w' w \leq d \lambda_{\max}(Q_2^{-1})$$

consequently

$$V(\zeta) < e^{\beta t} V(\zeta_0) + \beta e^{\beta t} d \lambda_{\max}(Q_2^{-1}) \int_0^t e^{-\beta\tau} d\tau$$

$$< e^{\beta t} [V(\zeta_0) + d \lambda_{\max}(Q_2^{-1}) (1 - e^{-\beta t})].$$

With the choice of the Lyapunov function, one has

$$\lambda_{\min}(Q_1^{-1}) \zeta' R \zeta \leq \zeta' R^{1/2} Q_1^{-1} R^{1/2} \zeta \leq \lambda_{\max}(Q_1^{-1}) \zeta' R \zeta$$

then

$$\lambda_{\min}(Q_1^{-1}) \zeta' R \zeta < e^{\beta t} [\zeta_0' R \zeta_0 \lambda_{\max}(Q_1^{-1})$$

$$+ d \lambda_{\max}(Q_2^{-1}) (1 - e^{-\beta t})].$$

Knowing that $(1 - e^{-\beta t}) < 1$, and $\zeta_0' R \zeta_0 < c_1$, it follows

$$\zeta' R \zeta < \frac{e^{\beta t}}{\lambda_{\min}(Q_1^{-1})} [c_1 \lambda_{\max}(Q_1^{-1}) + d \lambda_{\max}(Q_2^{-1})].$$

Consequently, the inequality

$$\frac{e^{\beta t}}{\lambda_{\min}(Q_1^{-1})} [c_1 \lambda_{\max}(Q_1^{-1}) + d \lambda_{\max}(Q_2^{-1})] < c_2$$

assures that $\zeta' R \zeta < c_2$. In particular, for $t = T$ one has

$$c_1 \lambda_{\max}(Q_1^{-1}) + d \lambda_{\max}(Q_2^{-1}) < c_2 e^{-\beta T} \lambda_{\min}(Q_1^{-1})$$

that is exactly the second inequality of Lemma 1. Moreover, evaluating the inequality

$$\dot{V}(\zeta) < \beta V(\zeta) + \beta w' Q_2^{-1} w$$

on the system trajectories, one obtains the first inequality of Lemma 1. ■

It is worth mentioning that the class of signals \mathscr{W} considered in [8], that is

$$\mathscr{W}_2 = \left\{ w : \int_0^T w'(t) w(t) dt \leq d, \quad d \geq 0 \right\} \quad (10)$$

can be handled by a modified version of Lemma 1 in which the element (2,2) of LMI (8) becomes $-Q_2$ as shown bellow.

Lemma 2: [8] System (6) is FTB with respect to $(c_1, c_2, \mathscr{W}_2, T, R)$, if there exist positive definite symmetric matrices $Q_1 \in \mathbb{R}^{n \times n}$, $Q_2 \in \mathbb{R}^{r \times r}$, and a positive scalar β such that

$$\begin{bmatrix} \bar{A}(t) \bar{Q}_1 + \bar{Q}_1 \bar{A}'(t) - \beta \bar{Q}_1 & \bar{B}(t) Q_2 \\ Q_2 \bar{B}'(t) & -Q_2 \end{bmatrix} < 0 \quad (11)$$

and LMI (9) are feasible.

Proof: The proof follows similar steps of Lemma 1. ■

III. MAIN RESULTS

The following theorem presents sufficient conditions for the synthesis of FTB parameter-dependent state feedback controllers.

Theorem 1: (HOMOGENEOUS PARAMETER-DEPENDENT STATE FEEDBACK CONTROLLER) Given a linear parameter varying continuous-time system (1), a degree g , and parameters $(c_1, c_2, d, T, R, \beta)$, if there exist symmetric positive definite matrices $Q_1 \in \mathbb{R}^{n \times n}$, $Q_2 \in \mathbb{R}^{r \times r}$, an homogeneous parameter-dependent matrix $L_g(\alpha) \in \mathbb{R}^{p \times n}$, and positive real scalars λ_1 , and λ_2 , such that²

$$\begin{bmatrix} \mathcal{M}_{11}(\alpha) + B_1(\alpha) L_g(\alpha) + L_g'(\alpha) B_1(\alpha)' & B_2(\alpha) Q_2 \\ (*) & -\beta Q_2 \end{bmatrix} < 0$$

$$\mathcal{M}_{11} = A(\alpha) \bar{Q}_1 + \bar{Q}_1 A(\alpha)' - \beta \bar{Q}_1 \quad (12)$$

$$\lambda_1 I < Q_1 < I, \quad \lambda_2 I < Q_2 \quad (13)$$

$$\begin{bmatrix} c_2 e^{-\beta T} & \sqrt{c_1} & \sqrt{d} \\ (*) & \lambda_1 & 0 \\ (*) & (*) & \lambda_2 \end{bmatrix} > 0 \quad (14)$$

with

$$L_g(\alpha) = \sum_{j=1}^{J(g)} \alpha_1^{k_1} \alpha_2^{k_2} \dots \alpha_N^{k_N} L_{\mathcal{L}_j(g)}, \quad k_1 k_2 \dots k_N = \mathcal{L}_j(g)$$

$$\bar{Q}_1 = R^{-\frac{1}{2}} Q_1 R^{-\frac{1}{2}}$$

then there exists a state-feedback control law in the form (3), such that the closed-loop system (5) is finite time bounded with respect to $(c_1, c_2, \mathscr{W}, T, R)$. The homogeneous polynomially parameter-dependent gain is given by

$$K_g(\alpha) = L_g(\alpha) \bar{Q}_1^{-1} = \sum_{j=1}^{J(g)} \alpha_1^{k_1} \alpha_2^{k_2} \dots \alpha_N^{k_N} L_{\mathcal{L}_j(g)} \bar{Q}_1^{-1} \quad (15)$$

Proof: Consider the change of variables $L_g(\alpha) = K_g(\alpha) \bar{Q}_1$ and the closed-loop system matrices, LMI (12) becomes

$$\begin{bmatrix} \bar{A}(\alpha) \bar{Q}_1 + \bar{Q}_1 \bar{A}(\alpha)' - \beta \bar{Q}_1 & B_2(\alpha) Q_2 \\ (*) & -\beta Q_2 \end{bmatrix} < 0$$

which guarantees the LMI (8). Moreover, it is easy to see that LMI (9) is satisfied if the conditions

$$\lambda_1 \mathbf{I} < Q_1 < \mathbf{I} \quad (16)$$

$$\lambda_2 \mathbf{I} < Q_2 \quad (17)$$

$$c_1 / \lambda_1 + d / \lambda_2 < c_2 e^{-\beta T} \quad (18)$$

are guaranteed. The LMIs in (13) are exactly the LMIs (16) and (17). Finally, applying Schur complement in the LMI (14) inequality (18) is obtained. The gain matrices are obtained from the change of variables done before. ■

As a first remark, it is important to mention that the conditions of Theorem 1 are described in terms of the parameter α and consequently have to be tested at all points of the unit simplex \mathscr{U} . Although this means infinite-dimensional design conditions, systematic procedures to construct finite dimensional LMI described in terms of the vertices of the

²The term (*) indicates symmetric blocks in the matrix inequality.

polytope (2) can be found in many works in the literature, as for instance [13]. However, the algebraic manipulation of polynomial parameter dependent matrices of arbitrary degree becomes as involved as the degree of the polynomials increases. In order to avoid these manipulations, the specialized parser ROLMIP³ [17] has been used to construct the LMIs.

In order to check the effect of the degree g in the robust FTB filter design conditions, the same optimization problem presented in [8] is considered here, that is, the minimization of c_2 under the conditions of Theorem 1. As stated in the next corollary, as g increases improvements in the value of c_2 may be obtained.

Corollary 1: Consider $c_2(g)$ as the solution of

$$\begin{aligned} \min \quad & c_2 \\ \text{such that} \quad & (12),(13),(14) \text{ hold} \end{aligned} \quad (19)$$

for a given g , then $c_2(g+1) \leq c_2(g)$.

Proof: The proof follows from the fact that if there exist a feasible set such that the conditions of Theorem 1 hold, yielding a guaranteed upper bound to $c_2(g)$, then it is immediate to see that the same set of variable with degree $g+1$ are a particular solution to the LMIs of Theorem 1, [13]. As a consequence, the minimization of c_2 under (12), (13), (14) for $g+1$ produces at least the same value obtained with g , implying that $c_2(g+1) \leq c_2(g)$. ■

As shown in Corollary 1, a sequence of less conservative LMI relaxations may be obtained in the conditions of Theorem 1 by increasing the degree g . As a consequence, the number of decision variables is also increased. By using an extension of Pólya's theorem, [13, 18], and based on the fact that the time-varying parameters α belong to the unit simplex, the conditions of Theorem 1 may also be improved using a sufficiently large positive integer f with no increase in the number of variables for a given degree g . This fact is summarized in the next corollary.

Corollary 2: Given a system (1), a degree g , a positive integer f , and parameters $(c_1, c_2, d, T, R, \beta)$, if there exist symmetric positive definite matrices $Q_1 \in \mathbb{R}^{n \times n}$, $Q_2 \in \mathbb{R}^{r \times r}$, an homogeneous parameter-dependent matrix $L_g(\alpha) \in \mathbb{R}^{p \times n}$, and positive real scalars λ_1 , and λ_2 , such that (13), (14) and

$$\left(\sum_{i=1}^N \alpha_i \right)^f \begin{bmatrix} \mathcal{M}_{11}(\alpha) & B_2(\alpha)Q_2 \\ (\star) & -\beta Q_2 \end{bmatrix} < 0 \quad (20)$$

$$\begin{aligned} \mathcal{M}_{11}(\alpha) = & A(\alpha)\tilde{Q}_1 + \tilde{Q}_1A(\alpha)' - \beta\tilde{Q}_1 \\ & + B_1(\alpha)L_g(\alpha) + L_g'(\alpha)B_1(\alpha)' \end{aligned}$$

are feasible then there exists a state-feedback control law in the form (3), such that the closed-loop system (5) is finite time bounded with respect to $(c_1, c_2, \mathcal{W}, T, R)$. The homogeneous polynomially parameter-dependent gain is given as in (15).

Proof: The proof follows straightly from the fact that $\alpha_1 + \dots + \alpha_N = 1$. ■

³Available for download at <http://www.dt.fee.unicamp.br/~agulhari/rolmip/rolmip.htm>

Considering the class of disturbance \mathcal{W}_2 , Lemma 2 can be applied in order to obtain the following theorem.

Theorem 2: Given a linear parameter varying continuous-time system (1), a degree g , and parameters $(c_1, c_2, d, T, R, \beta)$, if there exist symmetric positive definite matrices $Q_1 \in \mathbb{R}^{n \times n}$, $Q_2 \in \mathbb{R}^{r \times r}$, an homogeneous parameter-dependent matrix $L_g(\alpha) \in \mathbb{R}^{p \times n}$, and positive real scalars λ_1 , and λ_2 , such that (13), (14) and

$$\begin{aligned} \begin{bmatrix} \mathcal{M}_{11}(\alpha) + B_1(\alpha)L_g(\alpha) + L_g'(\alpha)B_1(\alpha)' & B_2(\alpha)Q_2 \\ (\star) & -Q_2 \end{bmatrix} < 0 \\ \mathcal{M}_{11} = A(\alpha)\tilde{Q}_1 + \tilde{Q}_1A(\alpha)' - \beta\tilde{Q}_1 \end{aligned} \quad (21)$$

are feasible, then there exists a state-feedback control law in the form (3), such that the closed-loop system (5) is finite time bounded with respect to $(c_1, c_2, \mathcal{W}_2, T, R)$. The state feedback gains are given as in (15).

Proof: The proof follows similar steps of Theorem 1. ■

Finally, it should be mentioned that the conditions of theorems 1 and 2 presented in [8] can be obtained as a particular case of Theorem 2 for $g=0$ and $g=1$, respectively. Furthermore, extensions of Corollaries 1 and 2 can be considered for Theorem 2. The next section provides some illustrative examples.

IV. NUMERICAL EXPERIMENTS

All the experiments have been performed in a PC equipped with: Linux Ubuntu 10.10, using the SDP solver SeDuMi [19] interfaced by the parser ROLMIP 2.0 [17], MATLAB 7.12.0.

Example 1

This first example, from [1], consists of an affine time-varying system that can be rewritten using a polytopic structure (2) with vertices as given bellow

$$\begin{aligned} A_1 = \begin{bmatrix} 0 & 1 \\ -12 & -11 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 8 & 9 \end{bmatrix}, \\ B_{11} = \begin{bmatrix} -10 \\ 1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0.25 \\ -1 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.25 \\ 1 \end{bmatrix}. \end{aligned}$$

As shown in [1], this system is FTB for $c_1 = 1$, $c_2 = 10$, $d = 0.5$, $T = 0.42s$, $\beta = 1.1$ and $R = \mathbf{I}$ by using a robust state-feedback controller. This is equivalent to apply Theorem 1 with $g=0$. It is important to stress that this system is not asymptotically stable as can be easily checked at the A_2 vertex. Considering a different scenario, where the time-varying parameters are available online, the method proposed in [1] can also be used. Assuming $c_2 = 9.8$ (slightly small than the previous case), the conditions in [1] are not able to provide a state feedback controller that guarantees FTB to the closed-loop system (5). On the other hand, Theorem 1 was applied with $g=1$ being able to provide a linear parameter varying gain in the form

$$K(\alpha) = \alpha_1 K_{10} + \alpha_2 K_{01}$$

with matrices

$$K_{10} = [-0.84124 \quad -3.1545], \quad K_{01} = [-0.70253 \quad -2.7773].$$

Considering now $T = 1$, $\beta = 0.6$ and the class of norm-bounded disturbances \mathcal{W}_2 given in (10), Corollary 1 is applied in order to investigate the effect of increasing degree g in the search of minimum upper bounds of c_2 attained by the conditions of Theorem 2. Note that by using the proposed approach with degree $g = 4$ it was possible to obtain an upper bound to c_2 approximately 49.43% smaller than the one provided by [8, Theorem 2]. The results are summarized in Table I. The number of scalar variables V , the number of LMI rows L and the time t to solve the feasibility problem are provided by the parser.

TABLE I

MINIMUM UPPER BOUNDS OF c_2 FOR DIFFERENT VALUES OF DEGREE g OBTAINED USING THEOREM 2 AND THE APPROACH OF [8].

Method	g	f	c_2	V	L	t
[8, Theorem 1]	0	0	112.49	9	14	0.22s
[8, Theorem 2]	1	0	61.60	11	17	0.19s
Theorem 2	1	1	44.66	11	28	0.26s
Theorem 2	1	2	44.66	11	39	0.34s
Theorem 2	2	0	44.66	13	20	0.21s
Theorem 2	2	1	36.21	13	31	0.31s
Theorem 2	2	2	33.46	13	42	0.33s
Theorem 2	3	0	36.21	15	23	0.24s
Theorem 2	3	1	31.15	15	34	0.32s
Theorem 2	3	2	26.81	15	45	0.40s
Theorem 2	4	0	31.15	17	26	0.26s
Theorem 2	4	1	27.79	17	37	0.35s

Example 2

The second example is adapted from [4] considering a time-varying $\theta(t)$ in matrix A

$$A(t) = \begin{bmatrix} \theta(t) & -0.4026 & -0.5796 & -0.5733 \\ 0.8984 & 0.0276 & -0.3622 & 0.9771 \\ -1.0667 & 2.8289 & -2.4719 & 1.0680 \\ -1.1599 & 0.1808 & -0.0779 & -1.2306 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.01 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where $\theta(t) = 0.0884|\sin(t)| + 1.0884(1 - |\sin(t)|)$.

This system can be described by a polytope as in (2) with $N = 2$. The vertices are obtained considering the minimum and maximum values of $\theta(t)$, 0.0884 and 1.0884 respectively, and the time-varying parameters modeled by the vector $\alpha(t) \in \mathcal{U}$ with

$$\alpha_1(t) = |\sin(t)|,$$

$$\alpha_2(t) = 1 - |\sin(t)|.$$

For zero initial conditions, a time simulation was performed in the interval $t \in [0, 11]$, considering that the system is affected by a unit step at time $t = 1$ s. The finite time

stability of this system is investigated for $T = 11$, $\beta = 0.5$, $c_1 = 0$, $c_2 = 50$, $d = 1$, and

$$R = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

The behavior of the state space variables of the open-loop system can be seen in Figure 1. From the curve shown, it is evident that the system is not finite time stable since the condition $x(t)'Rx(t) < c_2 = 50$ is violated for $t > 9.6$ s.

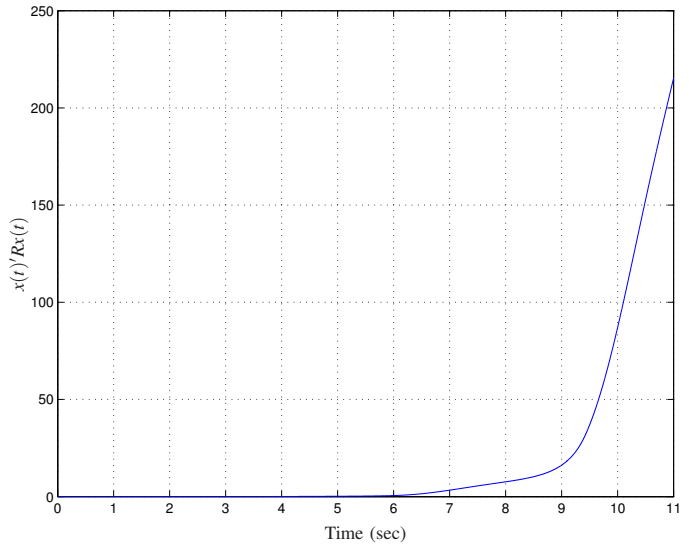


Fig. 1. Squared weighted norm of the open-loop system states.

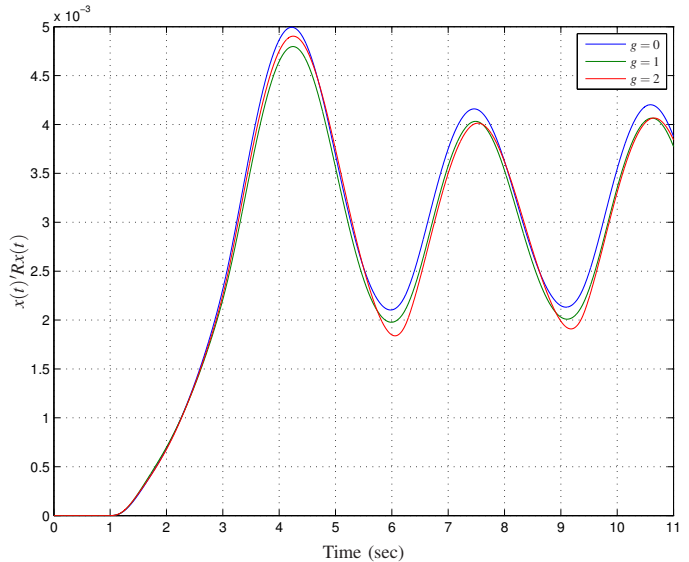


Fig. 2. Squared weighted norm of the closed-loop system state.

Theorem 1 was then applied with $g = 0, 1$ and 2, yielding

the following controllers of type (3)

$$K_0 = [11.4169 \quad -4.9106 \quad -1.4545 \quad -4.9372],$$

$$K_1(\alpha) = \alpha_1 K_{10} + \alpha_2 K_{01},$$

$$K_2(\alpha) = \alpha_1^2 K_{20} + \alpha_1 \alpha_2 K_{11} + \alpha_2^2 K_{02}$$

with matrices

$$K_{10} = [12.6621 \quad -5.1395 \quad -1.6034 \quad -5.2659],$$

$$K_{01} = [11.0537 \quad -4.7956 \quad -1.3772 \quad -4.8096],$$

$$K_{20} = [15.8084 \quad -6.4101 \quad -1.9485 \quad -7.9729],$$

$$K_{11} = [11.9927 \quad -6.0836 \quad -1.2193 \quad -0.4973],$$

$$K_{02} = [14.1894 \quad -6.0509 \quad -1.7219 \quad -7.5007].$$

As shown in Figure 2, the three designed controllers guarantee the finite time stability of the closed-loop system.

V. CONCLUSION

In this paper, LMI conditions for the design of homogeneous polynomially parameter-dependent state feedback controllers for time-varying systems were proposed. The time-varying parameters are modeled using a polytope and is considered to be measured online. The controller is obtained in such a way that the closed-loop system is guaranteed to be stable in finite time under the presence of bounded disturbance. As the degree g of the parameter-dependent gain matrices, or the relaxation parameter f introduced based on the Pólya's theorem, increases less conservative results can be obtained. The controller is obtained from the solution of a feasibility problem with LMIs constraints, providing a new methodology for the design of parameter-dependent FTB state feedback controllers. The numerical experiments performed illustrate the discussions presented throughout the text.

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