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Stability of continuous-time consensus algorithms for switching networks with bidirectional interaction.

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***Abstract*— We consider the problem of reaching consensus in a network of the first-order continuous-time agents with switching topology. The coupling gains are neither piecewise-continuous, nor separated from zero, nor symmetric, though the network topology is assumed to be undirected. The main result of the paper provides conditions on the network topology that are necessary and sufficient for reaching consensus.**

I. INTRODUCTION

The phenomenon of *synchronization* or *consensus* in multi-agent networks caused by local interactions between the agents has recently attracted enormous attention in various research communities. This interest is motivated by numerous natural phenomena and engineering designs based on synchronism between the heterogeneous components (*agents*) of complex systems. Examples include, but are not limited to, flocking, swarming, and other forms of coordinated behaviors of complex biological and technical systems [13], [17] and synchronization in networks of coupled dynamical systems [22] and oscillator ensembles. We refer the reader to [10], [13], [17] for excellent surveys of recent research on distributed consensus protocols.

Networks of first-order agents have earned a reputation of relatively simple but instructive models for basic study of consensus dynamics. In particular, many high-order consensus algorithms are direct extensions of first-order prototypes [10], [17] or are even formally based on them [19]. The effect of the time-variant topology and couplings on the overall dynamics of the network is one of the major concerns in the recent consensus literature. The major research techniques include the use of relevant algebraic results [10], [13], [18] on convergence of infinite matrix products and Lyapunov-like methods [1], [3], [9], [11], [12], [20] based on shrinking properties of the convex hull of the agents states.

Despite essential progress, some issues in this area remains open. This partly pertains to a gap between necessary and sufficient conditions for consensus in the part concerned with time-varying interaction topology. A popular sufficient condition, often referred to as the network uniform connectivity [9], [19], requires that the union of the interaction graphs over any period of a fixed length is connected (whereas the graph alterations are separated by a fixed dwell time). Though this condition is sometimes characterized as "the weakest possible" [20], it is in fact far from being necessary unless consensus is understood in a strong "uniform" sense [9]. On the other hand, most of known necessary conditions are far from being sufficient in general, though some of them may acquire sufficiency in some special cases (e.g., for constant network topology and time-invariant linear couplings).

For discrete-time networks, the above gap was recently filled in [3], [12] in the case of *bidirectional interactions*, which means that the graph is undirected but the couplings may be non-symmetric. The concerned criterion relies on connectivity of the graph whose arcs connect agents that interact infinitely many times. However this result cannot be applied even to sampled-data continuous-time systems since even exact discretization may destroy key companion requirements, e.g., strict convex hull shrinking [12] or strict positivity of coupling gains [3].

In this paper, we address requirements to the time-varying interaction graph that are necessary and sufficient for consensus in general bidirectional networks of first-order agents. Unlike the recent paper [4], we discard the symmetry condition. This required to develop a new technique, partly since that from [4] is essentially based on the use of a quadratic Lyapunov function, specific for the case of fully symmetric networks. The requirement of symmetry is not only non-robust against small perturbations of coupling gains but also essentially restricts the potential for extension of the result on nonlinear systems. To reduce them to the linear case, coupling gains are commonly introduced as the coefficients of linear decomposition of the generalized velocity vector with respect to the relative positions of the neighbors. Symmetry of the resultant coefficients is an open and doubtful issue. Getting rid of the symmetry requirement also opens the door for study of second-order agents [15]. We also discard some technical restrictions typical for the entire previous research in the area, such as piecewise-continuity and non-chattering (elapsing no less than a fixed dwell time between switchings) of interaction topology and couplings.

After this paper had been accepted, the authors became aware that the consensus criterion for linear systems (Theorem 1) appeared (in somewhat extended form) in the very recent publication [6]. In this paper, we not only offer an alternative proof of this particular result, but also provide a detailed discussion of its applications to nonlinear networks and networks of double integrators.

II. PRELIMINARIES AND NOTATIONS.

Throughout the paper, $m:n := \{m, m+1, \dots, n\}$ (where m, n are naturals, $m < n$) and "a.a." means "almost all" (all except the elements of a set of zero Lebesgue measure). For any set $S \subset \mathbb{R}^M$, the symbol $co\, S$ denotes the convex hull

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of S, and $Cone\,S$ is the smallest convex cone containing S. If S is convex, $\mathrm{ri}\,S$ symbolizes the relative interior of S.

A *(directed) graph* is a pair $G = (V, E)$ of finite sets V and $E \subset V \times V$, called the sets of *nodes* and *arcs*, respectively. The graph is *undirected* if $(v, w) \in E \Leftrightarrow (w, v) \in E$. A sequence of nodes $v_1, \dots, v_k$ with $(v_i, v_{i+1}) \in E\ \forall i$ is called the *path* between v_1 and v_k. A *root* is a node connected to any other one with a path, a graph is *rooted* if at least one root exists, and is (strongly) *connected* if all its nodes are roots. The symbol $\mathbb{G}_N$ stands for the class of all graphs $G = (1 : N, E)$ (possibly disconnected) that contain no self-loops: $(v, v) \notin E\,\forall v \in 1 : N$. For $G \in \mathbb{G}_N$ and $j \in 1 : N$ let $G_j = \{k : (k, j) \in E\}$ be the set of nodes adjacent to j.

III. Problem Setup and Assumptions.

We consider a team of N agents with states $x_j(t) \in \mathbb{R}^n$ coupled by means of the following distributed protocol

$$\dot{x}_j(t) = \sum_{k=1}^{N} a_{jk}(t)[x_k(t) - x_j(t)], \quad j \in 1 : N. \quad (1)$$

Here $t \geq t_0$, and $a_{jk}(t) \geq 0$ are measurable locally integrable functions. They are called *coupling gains* and determine the "strengths" of interactions between the coupled agents. If $a_{jk}(t) = 0$, the k-th agent has no influence on the j-th one at time t; we put $a_{jj} \equiv 0$ for convenience. Let $X(t) := col[x_1(t), \dots, x_N(t)]$ denote the joint state vector.

The following claim is well known (see e.g. [1], [9], [20] and Remark 2 in Section VI).

Proposition 1: The convex hull of the agents states $V(t) := co\{x_1(t), \dots, x_N(t)\}$ shrinks as time progresses: $V(t'') \subseteq V(t')$ whenever $t'' > t'$. In particular, if $x_j(t_*) \in D\forall j$, where D is a convex set, then $x_j(t) \in D\forall t \geq t_*, j$.

Our objective is to disclose conditions under which $V(t)$ collapses into a singleton, i.e. the system comes to *consensus*.

Definition 1: The networked system (1) *reaches the consensus* if for any initial data $X(t_0)$, a vector $c \in \mathbb{R}^n$ exists such that $x_j(t) \to c$ as $t \to +\infty$ for any $j \in 1 : N$.

A popular assumption under which consensus has been established up to now is the so-called *uniform connectivity* [19]: there exist $\varepsilon > 0$ and $T > 0$ such that the graph $(1 : N, E_t) \in \mathbb{G}_N$ with the set of arcs $E = \{(j, k) : \int_t^{t+T} a_{jk}(s)ds > \varepsilon\}$ is rooted. Though this property is sometimes characterized as "the weakest assumption on the graph connectivity such that consensus is guaranteed" [20], it is not necessary for consensus in the sense of Definition 1. Indeed, for two $N = 2$ scalar agents obeying (1), $\delta(t) := x_1(t) - x_2(t) = \exp\left(-\int_{t_0}^{t} \beta(s)ds\right)\delta(0)$, $\beta(s) := a_{12}(s) + a_{21}(s)$. This team reaches consensus if and only if $\delta(t) \to 0$ as $t \to \infty$ for any $x_1(t_0), x_2(t_0)$, which in turn is trivially equivalent to $\int_{t_0}^{+\infty} \beta(s)ds = \infty$. This is clearly less restrictive than the uniform connectivity.

The ideas underlying this example can be applied to general networks and give rise to the following.

Proposition 2: Whenever the system (1) reaches consensus, the following graph is rooted

$$\Gamma_\infty := [1 : N, E_\infty],\ E_\infty := \left\{(j, k) : \int_0^\infty a_{jk}(t)dt = \infty\right\}. \quad (2)$$

This proposition may be proved following lines given in [3], [12] for discrete-time systems; see [6].

For constant gains $a_{jk}(t) \equiv a_{jk}$, the graph Γ_∞ is rooted if and only if uniform connectivity holds since $a_{jk} > 0 \Leftrightarrow (j, k) \in E_\infty$. So existence of a root in Γ_∞ is not only necessary but also sufficient for consensus [18]. At the same time, this claim fails to be true for time-variant couplings if $N > 2$ (which can be shown by a slight extension of the counterexample from [12, p.174]). This a displays a gap between necessary and sufficient conditions.

To address conditions that necessary and sufficient for consensus, we focus on networks (1) with *bidirectional interaction*, which means that $a_{jk}(t) > 0 \Longleftrightarrow a_{kj}(t) > 0$ (whereas the symmetry $a_{jk}(t) \equiv a_{kj}(t)$ is not required). Moreover, we assume that this property does not degrade over time by imposing the following.

Assumption 1: A constant $K > 0$ exists such that

$$a_{jk}(t) \leq K a_{kj}(t) \quad \forall t, j, k \quad (3)$$

In fact, the constant may depend on the indices $K = K_{j,k}$. In this case, the assumption holds with $K := \max_{jk} K_{jk}$.

IV. Consensus in general networks

Theorem 1: Suppose that Assumption 1 holds. Then the following claims are valid:

i) Any solution of (1) has a finite limit: $c_j := \lim_{t\to\infty} x_j(t)$;
ii) Agents from a common connected component $C(\Gamma_\infty)$ of the graph Γ_∞ have a common limit;
iii) The network (1) reaches the consensus if and only if the graph Γ_∞ is connected.

The proofs of the theorems stated in this section, are given in Section VI. For discrete-time systems, a similar in spirit result was established in [3], [12]. For continuous-time systems, iii) was established in [4] under the additional requirement that the gains are symmetric $a_{jk}(t) \equiv a_{kj}(t)$ and piece-wise continuous. Dropping the symmetry assumption permits us to advance to nonlinear consensus protocols.

Specifically, we still consider a team of N agents with states $x_j(t) \in \mathbb{R}^n$ united in the joint state vector $X(t) = col[x_1(t), \dots, x_N(t)]$, interaction between the agents is determined by an *undirected* graph $\Gamma(t) := [1 : N, E(t)] \in \mathbb{G}_N$: agents j and k interact if and only if $(j, k) \in E(t)$.

Let the system evolution obey the equation of the form:

$$\dot{x}_j(t) = f_j[\Gamma(t), X(t), t], \quad j \in 1 : N,\ t \geq 0. \quad (4)$$

Here $f_i(\cdot) : \mathbb{G}_N \times D^N \times [0; +\infty) \to \mathbb{R}^n$ and $\Gamma(\cdot) : [0, +\infty) \to \mathbb{G}_N$ are given mappings, $D \subset \mathbb{R}^n$ is an open set, $\Gamma(\cdot)$ is measurable (i.e. $\Gamma^{-1}(G)$ is measurable for any $G \in \mathbb{G}_N$), and $\Gamma(t)$ is undirected for almost all t.

Assumption 2: For any $j \in 1:N$ and graph $\Gamma \in \mathbb{G}_N$, the function $f_i(\Gamma, \cdot) : D^N \times [0, +\infty) \to \mathbb{R}^n$ is continuous and

$$f_j(\Gamma, X, t) = \sum_{k \in \Gamma_j} \alpha_{jk}(t, X)(x_k - x_j), \tag{5}$$

where $\alpha_{jk}(t, X) > 0$ and moreover, for any compact subset $\mathfrak{K} \subset D$, there exist constants $\psi(\mathfrak{K})$ and $\Psi(\mathfrak{K})$ such that

$$0 < \psi(\mathfrak{K}) \le \alpha_{jk}(t, X) \le \Psi(\mathfrak{K}) < \infty \,\forall t \ge 0 \,\forall X \in \mathfrak{K}^N. \tag{6}$$

The condition (5) implies the inclusion

$$f_j(\Gamma, X, t) \in \text{ri } Cone \left\{x_k - x_j\right\}_{k \in \Gamma_j} \tag{7}$$

and essentially comes to it since (7) entails (5) with $\alpha_{jk}(t, X) > 0$. The difference between Assumption 2 and (7) basically comes to imposing the conditions (6), which prevent the decomposition coefficients from degradation. It is instructive to note here that (7) does not mean any symmetry.

We say that agents $j \ne k$ *essentially interact* if their cumulative time of interaction is infinite: $\text{mes}\left\{t \ge 0 : (j, k) \in E(t)\right\} = \infty$, where mes stands for the Lebesgue measure. The graph $\Gamma_{\text{ess}} \in \mathbb{G}_N$ with the set $1 : N$ of nodes whose arcs connect only essentially interacting agents is called the *graph of essential interactions*.

Nonlinear consensus is addressed in the following.

Theorem 2: Let Assumption 2 be true and the convex hull of the initial positions of the agents lie in the domain D: $co\{x_1(0), \ldots, x_N(0)\} \subset D$. Then the following claims hold:

1) Any solution $X(\cdot)$ of the system (4) can be extended on $t \to \infty$ and remains in D: $x_k(t) \in D$ for any $t \ge 0$;
2) Any solution $X(\cdot)$ has a limit: $\exists c_i = \lim_{t \to \infty} x_i(t) \,\forall i$;
3) Agents from a common connected component $C(\Gamma_{\text{ess}})$ of the graph Γ_{ess} have a common limit;
4) If the graph Γ_{ess} of essential interactions is connected, the network (4) reaches consensus.

Theorem 2 is close in spirit to the consensus criterion from [9]. However [9] assumes more restrictive uniform connectivity of the interaction graph, along with fixed dwell time separating graph switchings and time-invariant functions f_j, unlike this paper. At the same time, [9] deals with directed networks (4) and drops (6) by replacing Assumption 2 with the weaker assumption (7).

V. Applications to specific networks

In the present section we illustrate the potential of Theorem 2 by applying it to several classical models.

A. Agreement protocols with static couplings.

One of the simplest consensus protocols, proposed firstly in [14], has the form

$$\dot{x}_j(t) = \sum_{j \in \Gamma_i(t)} \gamma_{jk}\left[x_k(t) - x_j(t)\right] \qquad j \in 1:N. \tag{8}$$

Here $x_j(t) \in \mathbb{R}$ is the value of some quantity generated by agent j at time t, the consensus is treated as asymptotical agreement between the agents on this quantity of interest. We suppose that $\gamma_{jk}(\cdot)$ are smooth, $x\gamma_{jk}(x) > 0$ for $x \ne 0$ and $\gamma'_{jk}(0) > 0$. A time-varying undirected graph describes the instantaneous information flow: j and k can communicate at time t only if j and k are connected. Examples of networks (8) include, just to mention a few, ensembles of identical Kuramoto oscillators with initial phases from $(-\pi/2; \pi/2)$ [5] and continuous-time analogue of the Krause opinion dynamics model [8].

Proposition 3: For the system (8), Assumption 2 and the conclusions 1-4 of Theorem 2 are true. In particular, if the graph Γ_{ess} of essential interactions is connected, the consensus (agreement) is established.

Proof: Taking $\alpha_{jk}(X) := \gamma_{jk}(x_k - x_j)/(x_k - x_j)$ if $x_k \ne x_j$ and $\alpha_{jk}(X) := \gamma'_{jk}(0)$ otherwise, one easily shows that (5) is valid. To prove (12), it suffices to note that $\alpha_{jk}(X) > 0$ and continuous. ■

It should be remarked that unlike known results on stability of the protocol (8), the foregoing Proposition 3 does not assume uniform connectivity of the graph [9], [20], however it deals with undirected graphs only. It also does not rely on the oddity of couplings imposed in [14].

B. Continuous-time Vicsek model

In [21] Vicsek et al. proposed a discrete-time planar model of self-organization in swarms of self-propelled units that move with a common speed but different headings. At any time every agent directs its heading along the vector of ”average” velocity (where averaging is over the ”nearest neighbors” of the agent that lie within the given distance from it). The simplified linearized version of this model assumes that averaging is performed directly on the headings. Convergence properties of the discrete-time Vicsek model are well investigated in discrete-time, see e.g. [7], [12]. Now we consider a continuous-time version of the Vicsek model.

Let the agents move on the plane ($x_j \in \mathbb{R}^2$) with a common speed $v > 0$, adjusting their speed vectors $v_j = v(\cos\theta_j, \sin\theta_j)$ via rotating towards the average velocity of the neighbors. The angular speed of rotation is proportional to the difference between the desired and current headings:

$$\theta'_i(t) = a_i \left[\arctan \frac{\sum_{j:|x_j(t)-x_i(t)|<R} \sin\theta_j(t)}{\sum_{j:|x_j(t)-x_i(t)|<R} \cos\theta_j(t)} - \theta_i(t)\right], \tag{9}$$

where $a_i > 0$ and $R > 0$ are given constants. To ensure that the denominator does not vanish, we assume that $|\theta_i| < \pi/2$.

The system (9) has the form (4), where $j \leftrightarrow k$ is the edge of $\Gamma(t)$ if and only if $|x_j(t) - x_k(t)| < R$.

Proposition 4: The system (9) satisfies Assumption 2. If the initial headings belong to $(-\pi/2; \pi/2)$, then $\theta_j(t)$ remain there afterwards and the conclusions 1-4 of Theorem 2 are true. In particular, if the graph Γ_{ess} of essential interactions is connected, the agents asymptotically move in a common direction: $\theta_j(t) \to \overline{\theta}$ as $t \to \infty$ for all j.

Proof: Let $D := \{\theta : |\theta| < \pi/2\}$. Given a compact subset $\mathfrak{K} \subset D$ and $X = col(\theta_1, \ldots, \theta_N) \in \mathfrak{K}^N$, we have

$$\cos\theta_i \ge \nu = \nu(\mathfrak{K}) > 0,\ \nu[\theta_j - \theta_i] \le \sin[\theta_j - \theta_i] \le \theta_j - \theta_i,$$

for ν sufficiently small. It is easy to show that

$$\frac{\sum\limits_{j\in\Gamma_i(t)}\sin\theta_j(t)}{\sum\limits_{j\in\Gamma_i(t)}\cos\theta_j(t)}-\tan\theta_i(t)=\frac{\sum\limits_{j\in\Gamma_i(t)}\sin(\theta_j(t)-\theta_i(t))}{\cos\theta_i(t)\sum\limits_{j\in\Gamma_i(t)}\cos\theta_j(t)}=$$
$$=\sum_{j\in\Gamma_i(t)}(\theta_j(t)-\theta_i(t))\beta_{ij}(t),\quad \beta_{ij}(t)\in[\nu;(\nu N)^{-1}],$$

which implies (5) and (6). Theorem 2 completes the proof.

C. Second-order consensus with velocity measurements.

Consensus problems for networks of double integrators have recently attracted considerable interest because of various applications to multi-vehicle formation control; see [17] and references therein. Now we apply Theorem 1 to a protocol proposed in [16]. It ensures rendezvous of second-order agents that have access to their absolute velocities and also measure their positions relative to the neighbors.

Consider a team of N agents with coordinates q_j and velocities p_j that obey the equations

$$\dot q_j(t)=p_j(t)\in\mathbb{R}^n, \dot p_j(t)=u_j(t),\quad j\in 1:N, \tag{10}$$

and are controlled in accordance with the protocol

$$u_j(t)=-\varrho p_j(t)+\sum_{k=1}^N a_{jk}(t)\big[(q_k(t)-q_j(t))+\gamma(p_k(t)-p_j(t))\big].$$

Here $a_{jk}(t)\geq 0$ are time-varying couplings gains and $\varrho,\gamma>0$. We recall that Γ_∞ is defined by (2).

Proposition 5: Let Assumption 1 hold and $\varrho\gamma>1$. Then the following claims are true:

i) The trajectories of the agents converge to equilibria: $q_j(t)\to c_j$ and $p_j(t)\to 0$ as $t\to+\infty$;
ii) The members j,k of any common connected component $C(\Gamma_\infty)$ reach a common point: $c_j=c_k$;
iii) The agents rendezvous for any initial states if and only if the graph Γ_∞ is connected.

Proof: The proof is based on the following elegant trick borrowed from [15]. Let us consider $\bar N:=2N$ "virtual agents" with the respective "states" $\bar x_j(t)$, where $\bar x_j(t):=q_j(t)+\gamma p_j(t)$ and $\bar x_{N+j}(t):=q_j(t)$ for $j\in 1:N$. For $j,k\in 1:N$, we put $\bar a_{jk}:=a_{jk}$, $a_{j,N+j}:=\gamma^{-1}$, $a_{N+j,j}:=\gamma^{-1}(\gamma\varrho-1)$ and $a_{j,N+k}=a_{N+j,k}=0$ for $j\neq k$ and $a_{N+j,N+k}:=0$. It is easy to see that the "virtual agents" obey equation (1) with N, x_j, a_{jk} replaced with $\bar N$, $\bar x_j$, $\bar a_{jk}$. The corresponding graph $\bar\Gamma_\infty$ has the set of arcs

$$\bar E_\infty := E_\infty\cup\{(j,N+j),(N+j,j): j\in 1:N\}$$

and is connected if and only if Γ_∞ is connected; in general, the connected components of these graphs are in a one-to-one correspondence: $C(\Gamma_\infty)\leftrightarrow C(\bar\Gamma_\infty):=C(\Gamma_\infty)\cup[C(\Gamma_\infty)+N]$, where the sum is in the Minkowski sense. Thanks to Theorem 1, every $\bar x_j(t)$ has a finite limit c_j, and $c_j=c_{N+j}$, which implies that $q_j(t)\to c_j$ and $p_j(t)\to 0$ as $t\to\infty$ for any $j\in 1:N$. Thus i) does hold. By the same theorem, $c_j=c_k$ for all $j,k\in C(\Gamma_\infty)\subset C(\bar\Gamma_\infty)$, i.e., ii) is true. Claim iii) is valid since rendezvous of the original agents means the consensus between the "virtual agents", which is achieved if and only if Γ_∞ is connected by Theorem 1. ■

VI. Proofs of Theorems 1,2

To start with, we introduce useful constructions that will play a central role in the proofs of these theorems.

A. Technical preliminaries.

We consider a collection $y_1(\cdot),\dots,y_N(\cdot)\in \mathrm{Lip}(\Delta)$, where $\Delta:=[t_0;+\infty)$ and $\mathrm{Lip}(\Delta)$ is the set of all locally Lipschitz functions $y(\cdot):\Delta\mapsto\mathbb{R}$ defined on Δ. The following fact is immediate from the Danskin Theorem [2].

Lemma 1: Let $y_-(t):=\min_{i\in[1:N]}y_i(t), y_+(t):=\max_{i\in[1:N]}y_i(t)$ and $j^\pm(t)$ be indices such that $y_\pm(t)=y_{j^\pm(t)}(t)$ for a.a. t. Then $y_\pm(\cdot)\in\mathrm{Lip}(\Delta)$ and $y_\pm'(t)=y_{j^\pm(t)}'(t)$ for a.a. t.

For any t, we introduce the *ordering permutation* $\mathcal{K}_N(t)=[k^1(t),\dots,k^N(t)]$ that rearranges the set $S_N(t):=\{y_1(t),\dots,y_N(t)\}$ in the ascending order: $z_1(t):=y_{k^1(t)}(t)\leq\dots\leq z_N(t):=y_{k^N(t)}(t)$. The resultant ensemble of functions $z_1(\cdot),\dots,z_N(\cdot)$ is called the *ordering* of $y_1(\cdot),\dots,y_N(\cdot)$.

Lemma 2: Firstly, $z_i(\cdot)\in\mathrm{Lip}(\Delta)$ and secondly, $z_i'(t)=y_{k^i(t)}'(t)$ for a.a. t.

Proof: The proof is via induction on N. For $N=1$, the claim is evident. Let it be true for some N, and let $y_1(\cdot),\dots,y_{N+1}(\cdot)\in\mathrm{Lip}(\Delta)$. The ordering of the first N functions $z_1(\cdot),\dots,z_N(\cdot)\in\mathrm{Lip}(\Delta)$ by the induction hypothesis. The recursion $y_{N+1}^0(t):=y_{N+1}(t)$,

$$y_{N+1}^\nu(t):=\max\{z_\nu(t);y_{N+1}^{\nu-1}(t)\},$$
$$\widehat z_\nu(t):=\min\{z_\nu(t);y_{N+1}^{\nu-1}(t)\},\ \nu\in 1:N$$

results in the ordering $\widehat z_1(t)\leq\dots\leq\widehat z_N(t)\leq\widehat z_{N+1}(t):=y_{N+1,N}(t)$ of the entire set $y_1(\cdot),\dots,y_{N+1}(\cdot)$. By applying Lemma 1 at every recursion step, we see that $\widehat z_\nu(\cdot)\in\mathrm{Lip}(\Delta)\ \forall\nu$. For any $\nu\in 1:N$, the sequence

$$\mathcal{Z}_\nu(t):=[\widehat z_1(t),\dots,\widehat z_\nu(t),z_{\nu+1}(t),\dots,z_N(t),y_{N+1}^\nu(t)] \tag{11}$$

is obtained from $\mathcal{Z}_{\nu-1}$ via a permutation $J_\nu(t)$ of indices, which either is the identity one or exchanges the places of the νth and $(N+1)$th entries. By Lemma 1, the sequence $\mathcal{Z}_\nu'(t)$ that results from replacement of any function in (11) by its derivative is related to $\mathcal{Z}_{\nu-1}'(t)$ by the same permutation for a.a. t. The sequence $\mathcal{Z}_0(t)$ is obtained from $\mathcal{Y}(t):=[y_1(t),\dots,y_{N+1}(t)]$ via a permutation of indices $\mathcal{K}(t)=[k^1(t),\dots,k^N(t),N+1]$. By the induction hypothesis, $\mathcal{K}(t)$ also relates $\mathcal{Z}_0'(t)$ and $\mathcal{Y}'(t):=[y_1'(t),\dots,y_{N+1}'(t)]$ for a.a. t. Then $\mathcal{K}_{N+1}(t)=J_N\circ\dots\circ J_1\circ\mathcal{K}(t)$ transforms $\mathcal{Y}'(t)$ into $\mathcal{Z}_N'(t)$ for a.a. t. ■

Definition 2: The ensemble of scalar functions $y_1(\cdot),\dots,y_N(\cdot)\in\mathrm{Lip}(\Delta)$ is said to be *weakly shrinking* if a constant $q>0$ exists such that for all $j\in 1:N$ and almost all $t\in\Delta$ the following statements hold:

$$\begin{aligned} y_j'(t)<0\Rightarrow\exists k: y_k(t)<y_j(t)\ \&\ y_k'(t)\geq -qy_i'(t)\\ y_j'(t)>0\Rightarrow\exists k: y_k(t)>y_j(t)\ \&\ y_k'(t)\leq -qy_i'(t)\end{aligned} \tag{12}$$

The following lemma establishes a key property of weakly shrinking ensembles.

Lemma 3: Each element y_j of a weakly shrinking ensemble has a summable derivative: $\int_\Delta |\dot{y}_j(t)|dt < \infty$. In particular, a finite limit $\lim_{t\to+\infty} y_j(t)$ exists. A minimal and maximal values $y_\pm(t)$ are respectively non-decreasing and non-increasing functions.

Proof: We start with a special case where the functions y_j keep the order as time progresses: $y_1(t) \leq y_2(t) \leq \ldots \leq y_N(t)$ for a.a. $t \in \Delta$. Due to (12), $\dot{y}_1(t) \geq 0, \dot{y}_N(t) \leq 0$ for a.a. t, which proves the last claim of the lemma and implies that $y_j(t) \in [y_1(0); y_N(0)] \,\forall t$. Now we are going to prove that $\int_\Delta |\dot{y}_r(t)|dt < \infty$ via induction on r. For $r = 1$, the claim is evident since $\dot{y}_1 \geq 0$. Let it be true for $i \in 1 : r-1$ and put $Q := \{t : y_r'(t) < 0\}$. By (12), $Q = Q_1 \cup \ldots \cup Q_{r-1}$, where $Q_i := \{t \in Q : y_i'(t) \geq -q y_r'(t)\}$. Hence

$$\int_Q |y_r'(t)|\, dt \leq \sum_{i=1}^{r-1} \int_{Q_i} |y_r'(t)|\, dt = -\sum_{i=1}^{r-1} \int_{Q_i} y_r'(t)\, dt \leq \leq q^{-1} \sum_{i=1}^{r-1} \int_{Q_i} y_i'(t)\, dt < \infty,$$

where the last inequality holds by the induction hypothesis. So were the integral $\int_{t:y_r'(t)\geq 0} y_r'(t)\, dt$ infinite, we would have $y_r(t) \to +\infty$ as $t \to \infty$, in violation of the inequalities $y_r(t) \leq y_N(0)$ for a.a. t. Hence $y_r'(\cdot)$ is summable.

Let $z_1(\cdot), \ldots, z_N(\cdot)$ stand for the ordering of $y_1(\cdot), \ldots, y_N(\cdot)$, $\mathcal{K}(t) = [k^1(t), \ldots, k^N(t)]$ be the ordering permutation, and $\mathcal{M}(t) = [m^1(t), \ldots, m^N(t)]$ be its inverse. By Lemma 2 and Definition 2, the ensemble $z_1(\cdot), \ldots, z_N(\cdot)$ is weakly shrinking. So the claims of Lemma 3 hold with $y_i(\cdot) := z_i(\cdot)$ and thus $\int |z_i'(t)|\, dt < \infty \;\forall i$. By invoking Lemma 2 once more, we get

$$\int |y_j'(t)|\, dt = \int |z_{m^j(t)}'(t)|\, dt \leq \sum_{m=1}^{N} \int |z_m'(t)|\, dt < \infty,$$

and therefore $\exists \lim_{t\to+\infty} y_j(t)$, which completes the proof. ■

Remark 1: To prove the last claim of Lemma 3, we did not use (12) in full but employed only a simpler fact: $\dot{y}_j(t) \geq 0$ if $y_j(t) = y_-(t) = \min_k y_k(t)$ for a.a. t, and $\dot{y}_j(t) \leq 0$ if $y_j(t) = y_+(t) = \max_k y_k(t)$.

B. *Multi-agent system* (1) *and weakly shrinking ensembles.*

In this subsection, we suppose Assumption 1 to hold.

Let $X(\cdot) = \big[x_1(\cdot), \ldots, x_N(\cdot)\big]$ be a solution of (1) and $\xi \in \mathbb{R}^n$. We introduce the functions $y_j(t) := \xi^* x_j(t)$ and $\eta_{jk}(t) := a_{jk}(t)(y_k(t) - y_j(t))$, and observe that for a.a. t,

$$y_j'(t) = \sum_{k=1}^{N} \eta_{jk}(t), \tag{13}$$

Remark 2: Proposition 1 follows from (13) and Remark 1. Indeed, let us represent the convex polytope $V(t') = co\{x_1(t'), \ldots, x_N(t')\}$ as an intersection of hyper-planes: $V(t') = \{x \in \mathbb{R}^n : \xi_i^* x \leq c_i, i \in 1 : m\}$. By Remark 1 and (13) (applied to $\xi := \xi_i$), one has

$$\xi_i^* x_j(t) \leq \max_k \xi_i^* x_k(t') \leq c_i \Rightarrow x_j(t) \in V(t') \;\forall j \forall t \geq t'.$$

Since $V(t')$ is a convex set, $V(t) \subseteq V(t')$ whenever $t \geq t'$.

The following key lemma establishes a fundamental property of networks with bidirectional interaction and explains our interest to weakly shrinking ensembles.

Lemma 4: The ensemble of scalar-valued functions $y_1(\cdot), \ldots, y_N(\cdot)$ is weakly shrinking.

Proof: We prove the first implication in (12); the second one is established likewise. Note that $\dot{y}_j(t) < 0 \overset{(13)}{\Longrightarrow} \exists j_1 = j_1(t,j) : \eta_{jj_1}(t) \leq N^{-1}\dot{y}_j(t) < 0 \Longrightarrow y_{j_1}(t) < y_j(t)$. By Assumption 1, $\eta_{j_1 j}(t) \geq -K^{-1}\eta_{j_1 j}(t) \geq -\chi \dot{y}_j(t)$, where $\chi_1 := (KN)^{-1}$. Applying (13) to $j := j_1$ yields that

$$y_{j_1}'(t) = \sum_{k\in 1:N} \eta_{j_1 k}(t) = \eta_{j_1 j}(t) + \underbrace{\sum_{k\in 1:N, k\neq j_1} \eta_{j_1 k}(t)}_{s_1}. \tag{14}$$

If $s_1 \geq \chi/2 y_i'(t)$, the foregoing implies that

$$y_{j_1(t,j)}'(t) \geq -\frac{\chi}{2} y_j'(t). \tag{15}$$

Otherwise

$$0 > \frac{\chi}{2} y_j'(t) \geq \sum_{k\neq j_1} \eta_{j_1 k}(t) \Rightarrow \exists j_2 : \eta_{j_1 j_2}(t) \leq \frac{\chi}{2N} y_j'(t). \tag{16}$$

We note that $\eta_{j_2 j_1}(t) \geq -K^{-1}\eta_{j_1 j_2}(t) \geq -\chi^2/2\dot{y}_j(t)$. By retracing (14) with j_2 put in place of j_1 and using that $\eta_{j_2 j}(t) \geq 0 \Leftarrow y_{j_2}(t) < y_{j_1}(t) < y_j(t)$, we get

$$y_{j_2}'(t) \geq -\frac{\chi^2}{2} y_j'(t) + \underbrace{\sum_{k\neq j_1, j_2} \eta_{j_2 k}(t)}_{s_2}.$$

If $s_2 \geq \chi^2/4 y_j'(t)$, the foregoing implies that

$$y_{j_2(t,j)}'(t) \geq -\left(\frac{\chi}{2}\right)^2 y_j'(t). \tag{17}$$

Otherwise

$$0 > \left(\frac{\chi}{2}\right)^2 y_j'(t) \geq \sum_{k\neq j_1, j_2} \eta_{j_2 k}(t)$$

and therefore there exists $j_3 : \eta_{j_2 j_3}(t) \leq \frac{\chi^2}{4N} y_j'(t)$. Up to the apparent index and multiplier substitutions, this inequality is similar to (16). This permits us to iterate the foregoing. Since the chain $j, j_1, j_2, j_3, \ldots$ is finite, these iterations are necessarily terminated at some step ν. Termination holds at the situation similar to (15) and (17):

$$y_{j_\nu(t,j)}'(t) \geq -\left(\frac{\chi}{2}\right)^\nu y_j'(t).$$

By putting $q := \left(\frac{\chi}{2}\right)^N$ and observing that $\nu \leq N$ and $y_{j_q}(t) < \ldots < y_{j_2}(t) < y_{j_1}(t) < y_j(t)$, we arrive at the first inequality from (12). ■

Lemmas 4 and 3 give rise to the following.

Corollary 1: The functions $y_j'(\cdot)$ are summable: $\int_{t_0}^{\infty} |y_j'(t)|dt < \infty$. The same is true for $\eta_{jk}(\cdot)$.

Proof: The first claim directly follows from Lemma 4 by virtue of Lemma 3. To prove the second one, we introduce the ordering $z_1(\cdot), \ldots, z_N(\cdot)$ of the family $y_1(\cdot), \ldots, y_N(\cdot)$

and the ordering permutation $\mathcal{K}(t) = [k^1(t), \ldots, k^N(t)]$. Let $S_m(t) := \sum_{j=1}^N |\eta_{k_m(t)j}(t)|$. It suffices to show that each $S_m(\cdot)$ is summable since by the definition of $\mathcal{K}(t)$, we have

$$\sum_{m=1}^N S_m(t) = \sum_{j=1}^N \sum_{m=1}^N |\eta_{k_m(t)j}(t)| = \sum_{j=1}^N \sum_{k=1}^N |\eta_{kj}(t)|.$$

Note that $S_1(\cdot)$ is summable since $\eta_{k_1(t)j}(t) \geq 0 \, \forall j$ and

$$S_1(t) = \sum_{j=1}^N \eta_{k_1(t)j}(t) \overset{(13)}{=\!=} y'_{k_1(t)}(t) = z'_1(t).$$

Since $\eta_{k_2(t)j}(t) \geq 0 \, \forall j \neq k_1(t)$ and $|\eta_{k_2(t)k_1(t)}| \overset{(3)}{\leq} K|\eta_{k_1(t)k_2(t)}| \leq KS_1(t)$, we have

$$S_2(t) = -\eta_{k_2(t)k_1(t)} + \sum_{j \neq k_1(t)} \eta_{k_2(t)j}(t) =$$
$$= -2\eta_{k_2(t)k_1(t)}(t) + \sum_{j=1}^N \eta_{k_2(t)j}(t) \overset{(13),(3)}{\leq} 2KS_1(t) + z'_2(t)$$

By retracing these arguments, we see that

$$S_3(t) = -\eta_{k_3(t)k_1(t)} - \eta_{k_3(t)k_1(t)} + \sum_{j \neq k_1(t), k_2(t)} \eta_{k_3(t)j}(t) =$$
$$= -2(\eta_{k_3(t)k_1(t)}(t) + \eta_{k_3(t)k_2(t)}(t)) + \sum_{j=1}^N \eta_{k_2(t)j}(t) \overset{(13),(3)}{\leq}$$
$$\leq 2K(S_1(t) + S_2(t)) + z'_3(t),$$

and finally $S_N(t) \leq 2K(S_1(t) + \ldots + S_{N-1}(t)) + z'_N(t)$. ■

C. Proof of Theorem 1 and Proposition 2

Without any loss of generality, we may assume that $n = 1$, otherwise we may proceed componentwise. By Lemma 4 applied to $y_j := x_j$ ($\xi = 1$), the ensemble $x_1(\cdot), \ldots, x_N(\cdot)$ is weakly shrinking and thus finite limits $c_j := \lim_{t\to\infty} x_j(t)$ exist by virtue of Lemma 3. This proves i) in Theorem 1. To prove ii), we invoke Corollary 1, which states that $\int_{t_0}^\infty a_{jk}(t)|x_k(t) - x_j(t)|dt < \infty$ and so $c_j \neq c_k$ implies that $\int_{t_0}^\infty a_{jk}(t) < \infty$, i.e., $(j,k) \notin E_\infty$. Since any two nodes j and k from a common connected component $C(\Gamma_\infty)$ are connected with a path, the related limits coincide: $c_j = c_k$. This proves ii) and the sufficiency part in iii). The necessity is immediate from Proposition 2. ■

D. Proof of Theorem 2

Consider a solution $X(t) = col[x_1(t), \ldots, x_N(t)]$ of (4) and put $a_{jk}(t) := \alpha_{jk}(t, X(t))$ if $(j,k) \in E(t)$ and $a_{jk}(t) := 0$ otherwise. Then $X(t)$ obeys (1) on the domain of definition of $X(\cdot)$. Hence Proposition 1 applied to $X(\cdot)$ yields that the convex hull $V(t) := co\{x_1(t), \ldots, x_N(t)\}$ shrinks as time goes by, in particular $x_j(t) \in V(0) \subset D$ for any $t \geq 0$, which proves 1) in Theorem 2. To prove 2), it suffices to show, by virtue of Theorem 1, that $a_{jk}(t)$ satisfy (3). Indeed, if $(j,k) \in E(t)$, then $(k,j) \in E(t)$ since the graph $\Gamma(t)$ is undirected, and applying (6) to the compact set $\mathfrak{K} := V(0)^N = V(0) \times V(0) \ldots \times V(0)$ results in (3) with $K := \frac{\Psi(\mathfrak{K})}{\psi(\mathfrak{K})}$, otherwise $a_{jk}(t) = a_{kj}(t) = 0$.

Now we observe that the graph Γ_∞ corresponding to the matrix (a_{jk}) coincides with Γ_{ess}. Indeed, due to (6), we have $\psi(\mathfrak{K})\mu \leq \int_0^\infty a_{jk}(t)dt \leq \Psi(\mathfrak{K})\mu$, where $\mu := \text{mes}\{t \geq 0 : (j,k) \in E(t)\}$. Thus $\int_0^\infty a_{jk}(t)dt = \infty$ if and only if $\mu = \infty$. In the light of this, claims 3) and 4) follow from Theorem 1. ■

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