2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Performance limitations for distributed systems via spatial-frequency Bode integrals *

Alain Sarlette **

Abstract— **The Bode integral expresses a standard performance limitation for (almost) any controller that asymptotically stabilizes a linear time-invariant system. For the control of distributed systems, spatial invariance allows to write one such 'Bode time-integral' per spatial frequency. The present paper inverses the roles of spatial and temporal independent variables in this latter viewpoint. By transposing the notions of controller, causality, and asymptotic stability to the spatial variable, we obtain and interpret 'Bode space-integrals', one per temporal frequency. The result directly connects to the notion of string stability.**

I. INTRODUCTION

Among the many problem types related to coordination control of multi-agent systems, one that has reached actual industrial implementation is the stabilization of large arrays of more or less identical subsystems distributed along some spatial dimension(s). Current applications include paper machines [1], segmented telescope mirrors [2], and vehicle chains on highways (projected in the near future, see e.g. [3], [4]).

Efficient controller design methods can be devised when such arrays feature (approximate) spatial invariance, see [5], [1] and many others. Linear temporally and spatially invariant systems are usually analyzed along the temporal axis, decoupled spatial frequency by spatial frequency. Our idea in this paper is to take the complementary viewpoint and make an analysis along the spatial axis, time frequency by time frequency. While it is standard to advertise that independent variables in dynamical systems can represent other things than time, the proposed viewpoint seems original in the context of performance limitations for distributed systems and leads to complementary results.

The crucial difference between the time and space axes is that time is fundamentally causal. A control signal at time t cannot depend on actual state/output values at time $t+\tau$ with $\tau > 0$, and differential equations in time have to be integrated from lower to higher times, with 'boundary' conditions at the lowest time value. Spatially, interaction structures can a priori be chosen where a control signal at point k depends on state/output values at points $k+\kappa$ with κ indifferently positive or negative; solutions can be indifferently parameterized by boundary conditions at lowest or highest value of k. However, some control paradigms, like the leader-follower strategy, deliberately let the control of k depend on the states of $j \leq k$ only. For such a "spatially causal" architecture, it appears that no distinction remains between the temporal and the spatial independent variables. Thus mathematical conclusions on time have exact analogs on space.

We will in particular focus on the Bode integral, a standard performance limitation criterion introduced in [6]. The temporal-frequency Bode integral has been applied to distributed systems e.g. in [3], [7], [8] (details below), studying in particular how disturbances at one point can be amplified as they propagate through a network, a problem called *string instability* and introduced by [9]. The idea in the present paper is to characterize and interpret how disturbances get amplified along the spatial axis by considering the latter one as the main variable, writing Bode integrals along the spatial dimension and interpreting the related concepts.

The paper is organized as follows. The setting is formalized in Section II. In Section III, we first recall a string instability result that uses a *time domain* Bode integral, probably more familiar to the reader. Then Section IV introduces the spatial Bode integral viewpoint and discusses it. Section V gives a concluding illustration.

II. MODEL FORMULATION

We consider a chain of identical linear time-invariant subsystems $\mathcal{S}_k$ in single input / single output form coupled to each other, and approximate it as infinite-length i.e. $k \in \mathbb{Z}$. We impose that for any $\alpha \in \mathbb{Z}$, the model is invariant under the transformation $\mathcal{S}_k \rightarrow \mathcal{S}_{k+\alpha}\ \forall\, k$. We call such a model *spatially invariant*, picturing the subsystems as being aligned along some spatial dimension. Linearity, space and time invariance allow to decouple the overall system in spatiotemporal frequency domain. We will work both with continuous and discrete time and space variables. Here continuous-space is motivated as a PDE approximation to the discrete chain. While it is well-known that there can be important differences between the continuous- and discrete-space situations [10], the critical cases where relevant discrepancies are likely to occur can be identified to a large extent, and by avoiding them one can still get useful insight from the PDE approximation before checking the discrete case numerically. Denoting ω the temporal frequency and ξ the spatial one, with associated Laplace variables s_ω, s_ξ and Z-variables z_ω, z_ξ, we get e.g.:

$$y(s_\omega, z_\xi) = H(s_\omega, z_\xi)\, u(s_\omega, z_\xi) \qquad (1)$$

*This paper presents research results of the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Program, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with its author.

**A. Sarlette is with the SYSTeMS lab, Faculty of Engineering and Architecture, Ghent University; address: Technologiepark Zwijnaarde 914, 9052 Zwijnaarde (Gent), Belgium. alain.sarlette@ugent.be

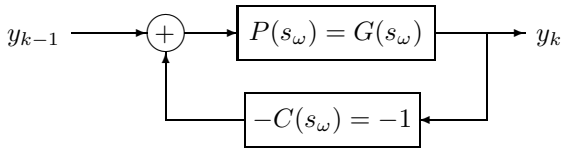

Fig. 1. Block-diagram of a general predecessor-following controller

where u, y are input and output, and H is a general two-argument transfer function. For instance, the standard consensus equation

$$\frac{d}{dt}y_k = \left(\frac{1}{\#\mathcal{N}} \sum_{l\in\mathcal{N}} y_{k+l}\right) - y_k \; + \; u_k \tag{2}$$

with neighborhood pattern $\mathcal{N} \subset \mathbb{Z} \setminus \{0\}$, corresponds to

$$H(s_\omega, z_\xi) \; = \; \frac{1}{s_\omega + 1 - \frac{1}{\#\mathcal{N}} \sum_{l\in\mathcal{N}} (z_\xi)^l} \; .$$

We will denote $i = \sqrt{-1}$ the imaginary unit.

Our general goal is, like in the preceding example, to counter disturbances and stabilize the system to the set $y_1 = y_2 = y_3 ...$ (or $y(k) =$ constant in the PDE approximation where k would be continuous). Section I gives related applications, possibly including a different reference position for each y_k (e.g. offsets to separate the cars in a vehicle chain). We are in particular interested in the behavior of the subsystems as k tends to infinity. This is related to the notion of *string instability* [9].

Definition 1: Denote by $H_{j\to k}(s_\omega)$ the transfer function from y_j to y_k in a chain of subsystems. The chain is said to feature *string instability* if $\max_\omega |H_{1\to k}(i\omega)|$ grows unbounded as k tends to infinity. If instead it remains bounded, the chain is said *string stable*. □

III. STRING INSTABILITY FROM STANDARD BODE INTEGRAL

A simple result on error propagation along a chain of subsystems follows from a time-domain Bode integral, in the case where each subsystem $\mathcal{S}_k$ takes the output of its predecessor $\mathcal{S}_{k-1}$ as a reference to track. We now rephrase this result from [3] to introduce the complementary contribution of Section IV. Spatial frequency domain is not used here, as spatial invariance is in fact not strictly necessary.

Consider any linear time-invariant predecessor-following controller

$$y_k(s_\omega) = G(s_\omega)\,[y_{k-1}(s_\omega) - y_k(s_\omega)] \quad \text{for } k = 1, 2, ..., N \tag{3}$$

where a guiding input $y_0(s_\omega)$ is provided to the first agent. The associated block-diagram for one subsystem is shown in Figure 1. The transfer function from y_{k-1} to y_k has the form of a complementary sensitivity function associated to the loop transfer function G:

$$y_k = \frac{G(s_\omega)}{1 + G(s_\omega)}\, y_{k-1} = T(s_\omega)\, y_{k-1} \, . \tag{4}$$

The following classical Bode result then restricts how the system can behave.

Lemma 1: Consider any rational transfer function $G(s)$ that leads to a stable causal closed-loop system (4).

$$\text{If} \qquad \lim_{s\to 0} \; sG(s) \;\text{ is unbounded} \, , \tag{5}$$

$$\text{then} \qquad \int_0^{+\infty} \frac{\log_e |T(i\omega)|}{\omega^2}\, d\omega \; = \; \pi \sum \frac{1}{z_j} \, , \tag{6}$$

where $\{z_j\}$ are the open right-half-plane zeros of $G(s)$. □

According to (6) and since practical $G(s_\omega)$ will decrease at high frequencies, any dynamics satisfying (5) will necessarily have $|T(i\omega)| > 1$ for some ω. At these frequencies a disturbance gets amplified as it passes from one subsystem to the next. Thus with the predecessor-following architecture, where trivially $H_{1\to k} = T^{k-1}$, any dynamics satisfying (5) leads to string instability. Extensions of this result, looking for stabilizing effects when $G(s_\omega)$ can depend on k, when each subsystem gets signals from a more general neighborhood or even from $\mathcal{S}_1$, or incorporating a "headway tolerance", are given in [7]. Assumption (5) may be realistic for vehicles, where typically force-control on flat terrain introduces a double integrator. But for general distributed systems, it seems less natural to have this condition at zero frequency (e.g. less natural than the high-frequency roll-off or cut-off required for the *direct* sensitivity Bode integral); a simple, obvious controller like (2) indeed does not satisfy the assumptions of the Bode limitation, and is string stable as recalled e.g. in [11].

Other Bode integrals can be written, which do not necessarily have powerful string stability interpretations in time-domain, but will be useful along the spatial dimension. The most standard Bode integral is on (direct) sensitivity S:

Lemma 2: Consider any rational transfer function $F(s)$ that leads to a stable causal closed-loop system $y = 1/(1 + F(s))\, u = S(s)\, u$.

$$\text{If} \qquad \lim_{s\to\infty} \; sF(s) = 0 \, , \tag{7}$$

$$\text{then} \qquad \int_0^{+\infty} \log_e |S(i\omega)|\, d\omega \; = \; \pi \sum p_j \, , \tag{8}$$

where $\{p_j\}$ are the open right-half-plane poles of $F(s)$. □

The situation $y_k = S(s_\omega)\, y_{k-1}$ can be obtained with the system of Fig.1 by defining $F(s_\omega) = 1/G(s_\omega)$. The loop equation equivalent to (4) then writes $F(s_\omega) y_k = y_{k-1} - y_k$. With the same reasoning as before, since $H_{1\to k} = S^{k-1}$ Lemma 2 predicts string instability for our system if (7) holds. However, the latter makes $F(s_\omega)$ of integrative type, such that in the associated differential equation the highest derivatives on y_k and on y_{k-1} would have the same order. This is a somewhat critical situation where problematic behavior is not unexpected. We obtain its practical meaning by writing out

$$y_k(s_\omega) = S^k y_0 = y_0(s_\omega) - F(s_\omega) \left[\sum_{j=1}^{k} y_j(s_\omega) \right] \, .$$

Thus its practical meaning is that each subsystem in the chain is connected to all its predecessors in the same (integral controller) way, and the first subsystem's input y_0 is fed through. We here see the initial subsystem value prominently appear, a point to be commented in Section IV-A.

Discrete-time Bode integrals are proposed in [12]. Unfortunately it turns out that the one on T is poorly meaningful, as it combines both positive and negative terms related to the controller choice on the right-hand side of the analog of (6). The discrete-time analog of Lemma 2 however is clearer.

Lemma 3: Consider any transfer function of the form $F(z) = \frac{z^{-n}\, P_a(z^{-1})}{P_b(z^{-1})}$ where P_a, P_b are polynomials, $P_b(0) \neq 0$ and $n \geq 1$, that leads to a stable causal closed-loop system $y = 1/(1+F(z))\, u = S(z)\, u$.

$$\text{Then} \qquad \int_0^\pi \log_e |S(e^{i\omega})|\, d\omega \;=\; \pi \sum |p_j|\;, \qquad (9)$$

where $\{p_j\}$ are the poles of $F(s)$ with $|p_j| > 1$. □

The continuous-time restrictions (5),(7) are here replaced by the requirement $n \geq 1$, i.e. that the output related to $F(z)$ lags at least one step behind its input. The system $y_k = S(z_\omega)\, y_{k-1}$, for which $H_{1\to k} = S^{k-1}$, i.e. where Lemma 3 would allow to conclude string instability, writes

$$z_\omega^{-n}\, L(z_\omega)\, y_k(z_\omega) = y_{k-1}(z_\omega) - y_k(z_\omega)\;, \qquad (10)$$

where $L(z) = z^n F(z)$ is subject to no further restrictions than to be a rational function in z^{-1}. Analogously to the case of Lemma 2, the associated difference equation gets its latest components from the right-hand-side of (10), such that y_k at t directly depends on y_{k-1} at t (and not only on its values at $t' < t$). However, in a spatially causal interpretation, nothing forbids to let the clock of $\mathcal{S}_k$ run slightly late w.r.t. $\mathcal{S}_{k-1}$, such that implementing the scheme in practice makes sense. Admittedly, one could question the pertinence of being restricted to controllers of the particular type (10)

The authors of [8] write a Bode integral for general spatially invariant systems, which for SISO systems decouples spatial frequency by spatial frequency. A Bode-type result that integrates along the spatial frequency seems to be absent from the literature. We now introduce this complementary viewpoint and argue that it gives useful original insight on limitations for controlling subsystem chains, by changing assumptions and meanings of the conclusions.

IV. THE SPATIAL BODE INTEGRAL VIEWPOINT

We first write a general interaction among subsystems in time-frequency domain,

$$y_k(s_\omega) = \textstyle\sum_j G_j(s_\omega)\,[y_{k-j}(s_\omega) - y_k(s_\omega)] + F_j(s_\omega)\, u_{k-j}(s) \qquad (11)$$

where u_k can be seen as a disturbance. The same equation obviously holds for a discrete-time system, just replacing s_ω by z_ω.

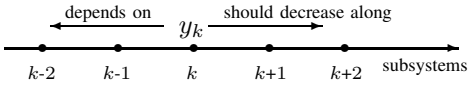

Fig. 2. Schematic illustration of spatial causality and stability; the analogy with time domain is apparent.

We now take the complementary viewpoint to Section III, insisting on propagation along the spatial axis and (for an instant) making abstraction of the time variable. The latter fact could be motivated by considering that we decompose the time-invariant system, temporal frequency by temporal frequency ($s_\omega = i\omega$ or $z_\omega = e^{i\omega}$), and require good behavior for all these temporal modes.

A. Causality and stability in space

Definition 2: A system (11) is *spatially causal* if it is understood as: y_k depends on $\{\, y_{k-j}, u_{k-j} \;:\; j \geq 0\}$. It is *asymptotically stable w.r.t. spatial initial condition* if and only if any y_0 leads to y_k that tend to 0 as k tends to $+\infty$. *Spatial BIBO stability* (bounded input ⇒ bounded output) means that any bounded $\{u_k : k = 0, 1, 2, ...\}$ implies a solution with bounded $\{y_k : k = 0, 1, 2, ...\}$. □

Figure 2 illustrates this definition. Spatial causality introduces the prominent restriction that the behavior of a given subsystem $\mathcal{S}_k$ in the chain must be viewed as a consequence, and never a cause, of the subsystems $\mathcal{S}_j$ with $j < k$; e.g. $y_{k+1} = y_k$ means "y_{k+1} copies the value of y_k; boundary condition is y_0." Couplings where a subsystem looks behind itself in the chain to set up its value, are thus forbidden; but the subsystem might look as far ahead as it wants.

Stability to initial conditions is obviously linked to string stability. For Section III, we would apply a spatially-oscillatory pattern at initial time, $y_k(t=0) = \int f(0,\xi) e^{i\xi k}\, d\xi$, and examine how $f(t,\xi)$ decreases as time goes on for each ξ; in the present complementary viewpoint, we apply a time-oscillatory signal at the initial subsystem, $y_0(t) = \int f(\omega, 0) e^{i\omega t}\, d\omega$, and examine how $f(\omega, k)$ decreases for each ω as we consider subsystems further down the chain.

The BIBO stability requirement might seem stronger than stability to initial conditions, since bounded disturbances are allowed not only at $k = 0$ but at all positions k. In fact, exactly like in the time domain, BIBO stability depends on the interpretation of information flow and, under spatial causality, BIBO and initial condition stability are equivalent. Of course the analog definitions hold in continuous-space.

At this point it might be relevant to make an important remark about spatiotemporal stability. The standard decoupling of linear time and spatially invariant (LTSI) systems, spatial frequency by spatial frequency, relies on the unrealistic assumption of exact spatial invariance, as much as a decoupling of LTI systems into temporal frequencies, looking only for $H(i\omega)$ to be bounded, unduly neglects unavoidable initial conditions. That is, any spatially invariant

plant (except torus-like ones, which is mostly not the intention when considering spatially invariant system tuning) is an approximation for which a spatially initial boundary condition must be taken into account. The following can thus happen.

Proposition 1: There exist systems that are stable in time for all spatial frequencies, but that are not stable in space for any temporal frequency, i.e. where any sinusoidal time-varying signal on the first subsystem is amplified along the chain. □

As a proof, consider the following example:

$$\frac{\partial^2}{\partial t^2}y(t,k) + \lambda\frac{\partial}{\partial t}y(t,k) - \alpha\frac{\partial^2}{\partial k^2}y(t,k) = u(t,k)\,. \quad (12)$$

Practical meanings can be sought behind this equation, although their spatially causal interpretation deserves a caveat![1] The associated transfer function writes

$$H(s_\omega, s_\xi) = \frac{1}{s_\omega^2 + \lambda\, s_\omega - \alpha\, s_\xi^2}\,.$$

For any fixed $s_\xi = i\xi$, it has poles $s_\omega^* = \frac{-\lambda}{2} \pm \sqrt{\lambda^2/4 - \alpha\xi^2}$, i.e. the system is always temporally stable for $\alpha > 0$ and unstable for $\alpha < 0$. Conversely, for a fixed $s_\omega = i\omega$, the poles are $s_\xi^* = \pm\sqrt{(\lambda i\omega - \omega^2)/\alpha}$, so one of them will always have a positive real part for $\omega \neq 0$; i.e. the system is always spatially unstable under an input oscillation at $k = 0$, even for $\alpha > 0$.

In discrete time and space, the system characterized by $H(z_\omega, z_\xi) = \dfrac{1}{(z_\xi + 2)(z_\omega + 0.5)}$ features a temporal pole $z_\omega = -0.5$ of magnitude < 1, testifying stability in time for any spatial frequency, and a pole $z_\xi = -2$ of magnitude > 2, testifying *instability* in space for any temporal frequency. The corresponding difference equation is $y(t+1,k+1) + 2y(t+1,k) + 0.5y(t,k+1) + y(t,k) = u(t,k)$. Thus the state of $\mathcal{S}_{k+1}$ at a given time depends on the state of $\mathcal{S}_k$ at the same time, as already encountered in Section III. The trick to have a well-defined system is to let the clock of $\mathcal{S}_{k+1}$ be somewhat late w.r.t. $\mathcal{S}_k$ (in fact, explicitly introducing such a time lag in $H(z_\omega, z_\xi)$, we would observe instability on z_ω). The author so far only found such particular situations to illustrate Proposition 1, so it could be that the danger is actually automatically avoided by a proper treatment of update times.

B. Bode integrals and interpretation

We have purposefully written Lemma 1, Lemma 2 and Lemma 3 without indices on the s and z variables, such that they apply without modification for integration along spatial frequency in a spatially invariant system, provided stability and causality are considered also along the spatial axis according to Definition 2. The following interpretations then express fundamental limitations on controlling a back-blind chain, i.e. where the behavior of subsystem $\mathcal{S}_k$ can only depend on y_j with $j \leq k$.

Analog of Section III: We translate the three observations of Section III to our complementary viewpoint. This will lead to formulations in the PDE-like continous-space approximation mostly; the limitations for this approximate model should give an indication of how difficulties can appear in the spatial discretization, much like continuous-time Bode limitations point to difficulties that persist in digital implementations.

The predecessor-following formulation of Figure 1 lets $y_k(t)$ get closer to $y_{k-1}(t)$ when moving from t to $t + dt$, with the explicit purpose to set up a tradeoff between (i_a) keeping differences between consecutive subsystems small – to avoid potentially catastrophic consequences (think of the vehicle chain example); and (ii_a) letting $y_k(t)$ converge to zero as k tends to infinity $\forall t$ while the initial subsystem $\mathcal{S}_1$ is subject to a time-dependent bounded disturbance – strong string stability thus.

The complementary formalism would then let $y(t,k)$ get closer to $y(t-1,k)$ when moving from k to $k+dk$, with the explicit purposes: (i_b) keep differences between consecutive times small $\forall k$ – a seemingly relevant safety constraint; and (ii_b) let $y_k(t)$ converge to zero as t tends to infinity $\forall k$, while the initial time is subject to a bounded disturbance function of k. In the latter we recognize the standard goal of asymptotic stability, such that the complementary setting indeed appears to make sense in distributed control applications with a first-order temporal evolution.

The first observation of Section III translates to:

Proposition 2: Let

$$y_t(s_\xi) = G(s_\xi)\,[y_{t-1}(s_\xi) - y_t(s_\xi)] \quad \text{for } t = 1, 2, \ldots, \quad (13)$$

with $G(s_\xi)$ reflecting a spatially causal coupling such that $\lim_{s_\xi \to 0} \; s_\xi G(s_\xi)$ is unbounded. Then we have $y_t = \frac{G}{1+G}\, y_{t-1} = T\, y_{t-1}$ and from Lemma 1, if the system must be spatially stable, then $|T(s_\xi)| > 1$ at some s_ξ such that some sinusoidal initial-time perturbations will blow up in time. □

The condition on G makes $F = 1/G$ of spatial derivative form, so (13) can be understood as $y_t = y_{t-1} - F(s_\xi)\, y_t$ where $F(s_\xi)$ somehow looks at the local shape of the chain. Applying F to y_t and not y_{t-1} might be argued as nonstandard.

Translating the second time-domain observation leads to a setting where spatial derivatives of y_t and y_{t-1} are compared:

Proposition 3: Let

$$y_t(s_\xi) = G(s_\xi)\,[y_{t-1}(s_\xi) - y_t s_\xi] \quad \text{for } t = 1, 2, \ldots, \quad (14)$$

with $G(s_\xi) = 1/F(s_\xi)$ reflecting a spatially causal coupling such that $\lim_{s \to \infty} \; s_\xi\, F(s_\xi) = 0$. Then we have $y_t = \frac{1}{1+F}\, y_{t-1} = S\, y_{t-1}$ and from Lemma 2, if the system must be spatially stable, then $|S(s_\xi)| > 1$ at some s_ξ such that

[1] For instance, forgetting causality for a moment: y could be the position of a mechanical system subject to damping, external disturbances u, and a force proportional to the local curvature. For a violin string, we have $\alpha > 0$ in the restoring force, d would be the bow; in a fluid tube, $\alpha > 0$ represents a local pressure gradient; or approximating a chain of vehicles, $\alpha < 0$ could represent the effect that a vehicle close ahead gives you wind shadow while a vehicle close behind would suck you back. Of course one can also always assume a causal control law behind this model, although how plausible this control is could be questioned.

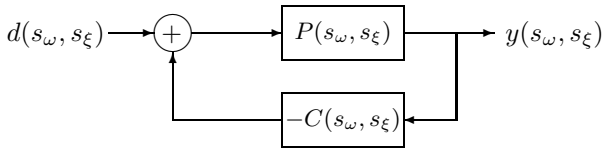

Fig. 3. Block-diagram of the standard disturbance rejection situation

some sinusoidal initial-time perturbations will blow up in time. □

We can write out (14) as

$$y_t(s_\xi) = y_0(s_\xi) - F(s_\xi)\left[\sum_{\tau=1}^{k} y_\tau(s_\xi)\right],$$

showing that the control depends on a time-integral of deviations $y(s_\xi)$, which further pass through a spatially integrating filter. The latter viewpoint is arguably less standard than (14), where $G(s_\xi)$ is of spatial derivative type.

For a discrete chain of subsystems, we get by translating the third time-domain observation:

Proposition 4: Let

$$y_t(z_\xi) = y_{t-1}(z_\xi) - \frac{z_\xi^{-n}\, P_a(z_\xi^{-1})}{P_b(z_\xi^{-1})}\, y_t(z_\xi) \;=\; y_{t-1} - F\, y_t \tag{15}$$

for $t = 1, 2, \ldots,$ with P_a, P_b polynomials, $P_b(0) \neq 0$ and $n \geq 1$. Then we have $y_t = \frac{1}{1+F}\, y_{t-1} = S\, y_{t-1}$ and from Lemma 3, if the system must be spatially stable, then $|S(z_\xi)| > 1$ at some z_ξ such that some sinusoidal initial-time perturbations will blow up in time. □

The way (15) is written suggests that y_t is obtained by updating y_{t-1} by some value that depends on some comparison, modeled by F, of y_t at different k values. Having y_t instead of y_{t-1} in the latter may appear counterintuitive, but it is again a matter of interpreting the model. Indeed, $n > 1$ introduces a spatial delay such that the update of subsystem $\mathcal{S}_k$ depends only on a comparison of strictly preceding subsystems $\mathcal{S}_j$ with $j < k$. Thus again, if the clock of $\mathcal{S}_k$ runs slightly late w.r.t. $\mathcal{S}_{k-1}$, interpreting the first equality in (15) as an assignment makes perfect sense. However, Proposition 4 shows that such a strategy would unavoidably lead to instability.

Limitation on distributed disturbance rejection: Consider the setting illustrated on Fig. 3, where a disturbance d on a plant P must be rejected through controller C. The only difference here with respect to the most standard setting is that the system is spatially invariant and expressed in spatiotemporal frequency domain. Here as well, instead of looking at the system spatial frequency by spatial frequency, we can look at it temporal frequency by temporal frequency. The following result is a direct adaptation of Lemma 2.

Proposition 5: Consider the situation of Fig. 3, where the plant and controller are both spatially causal, $\lim_{s\to\infty}\; s_\xi\, P(s_\omega, s_\xi)\, C(s_\omega, s_\xi) = 0$ for all s_ω, and we require the system to be spatially stable. Then, for each ω, the disturbance $d_\omega(s_\xi)$ shall be multiplied w.r.t. the open-loop situation by a factor

$$S_\omega(s_\xi) \;=\; \frac{1}{P_\omega(s_\xi) C_\omega(s_\xi)} \text{ which has to be larger than 1 over some spatial frequencies,}$$

since $\int_0^{+\infty} \log_e |S_\omega(i\xi)|\, d\xi = \pi \sum p_{j,\omega}$ where $\{p_{j,\omega}\} \ni s_\xi$ are the open right-half-plane poles of $P_\omega(s_\xi) C_\omega$. □

Thus loosely speaking, if the coupling is spatially causal (strong assumption), the system must be spatially stable (reasonable), and too spatially disrupted disturbances are filtered out either by the plant or by the controller (reasonable), then *at each temporal frequency*, the control designer faces a tradeoff between spatial scales, some at which chain deformations have to be rejected and others where amplification must be tolerated w.r.t. the open-loop situation.

V. ILLUSTRATION

We finally illustrate, on an example, the role of spatial causality for the Bode-induced limitations. Consider a system of the form (15) with $n = 1$, $P_b(z^{-1}) = 1 + b\, z^{-1}$ and $P_a(z^{-1}) = \beta + \gamma z^{-1}$. This yields

$$y_t \;=\; \frac{z_\xi + b}{z_\xi + (b+\beta) + \gamma\, z_\xi^{-1}}\, y_{t-1} = S(z_\xi)\, y_{t-1}\,, \tag{16}$$

with positive parameters b, β, γ without loss of generality. For a spatially causal system, transfer function $S(z_\xi)$ is spatially stable if $\gamma < 1$ and $b + \beta < 2\sqrt{\gamma}$. According to Proposition 4, the system should be temporally unstable, i.e. $|S(e^{i\xi})| > 1$ for some ξ, under these conditions. Writing

$$\begin{aligned} N_\xi &= e^{i\xi} + b\,, \\ D_\xi &= e^{i\xi} + b + \beta + \gamma e^{-i\xi}\,, \end{aligned}$$

we can indeed show that $|S(e^{i\xi})| = |N_\xi|/|D_\xi| > 1$ for some ξ, for all b, β, γ that guarantee spatial stability. For this, we note that $2\sqrt{\gamma} \leq 1 + \gamma$ for $\gamma \in [0, 1]$, such that spatial stability implies: there exists $\psi < \pi/2$ such that $(1+\gamma)\sin(\psi) = b + \beta$. Then we geometrically see that $|D_{\pi/2+\psi}| = (1-\gamma)\cos(\psi)$ and $|N_{\pi/2+\psi}| \geq \cos(\psi)$, so $|S(e^{i\xi})| > 1$ for $\xi = \pi/2 + \psi$ and the system is indeed temporally unstable.

The difference equation associated to (16) is

$$y_{t,k+1} = y_{t-1,k+1} + b y_{t-1,k} - (b+\beta) y_{t,k} - \gamma y_{t,k-1}\,. \tag{17}$$

To further highlight the role of spatial causality, we will rewrite (17) as an assignment to some variable at a time $t+1$.

• For a spatially causal system, $\mathcal{S}_{k+1}$ at $t+1$ depends on its predecessors, i.e. we write:

$$y_{t+1,k+1} = y_{t-1,k+1} + b y_{t-1,k} - (b+\beta) y_{t,k} - \gamma y_{t,k-1}$$

for $k \geq 1$, dropping the undefined terms for $k = 0$ and $k = 1$. We can now write the state transition matrix in

time, for a state vector of dimension $2N$ containing the state $x_k(t) = (\, y_k(t),\, y_k(t-1)\,)$ of N subsystems. The associated state transition matrix has exactly N eigenvalues 1 and N eigenvalues -1. The system is still unstable, but only through its nontrivial Jordan form. (This is easily shown by index reordering and taking linear combinations of rows of the characteristic matrix to get a simple characteristic equation.)

For a spatially noncausal system, where each subsystem moves by reacting to its predecessor and to its follower, we rather adapt (17) toward:

$$y_{t+1,k} = \frac{y_{t-1,k+1} - y_{t,k+1} + by_{t-1,k} - \gamma y_{t,k-1}}{b+\beta}$$

for $1 < k < N$, dropping undefined terms for $k = 0$ and $k = 1$. One can check that with appropriate conditions on the parameters, this system is spatially stable (note that this requires the unit circle to be between the two poles). But now due to the noncausal coupling, the limitations expressed by Proposition 4 do not apply and we can hope to be temporally stable as well. By computations similar to the spatially causal case, we find that the eigenvalues of the state transition matrix in time are now N times $\sqrt{\frac{b}{b+\beta}}$ and N times $-\sqrt{\frac{b}{b+\beta}}$. So the system is indeed also temporally stable for $\beta > 0$.

VI. CONCLUSIONS

This paper draws attention to the fact that for distributed systems that are spatially invariant, meaningful Bode integrals can be written by integrating sensitivity functions along the *spatial* frequency instead of the standard temporal frequency. The associated limitations apply to couplings that are spatially causal, i.e. each system only looks at its predecessors and not at its followers, and require string stability. For such systems, we state and interpret the Bode integral results both in the temporal and in the spatial frequency formulations. In essence, both formulations indicate that being at the same time spatially and temporally stable is impossible, but they focus on different controller structures: one insisting on time coupling and the other on spatial coupling. The use of the framework is illustrated directly on stability issues, although for that case the coupling conditions to apply the theorems are not the most common ones, often featuring a relative degree zero. A more standard point is obtained by exploiting the viewpoint for disturbance rejection.

Developing the spatial frequency viewpoint in the future, it is not impossible that new observations could be applicable to classical (temporal) linear control. For instance, according to [7] and references cited therein, reverse communication i.e. letting u_k depend on x_{k-1} in addition to x_{k+1} seems to have little effect on string stability. Taking the complementary viewpoint, one might then ask (out of curiosity?) if the possibility to use temporally noncausal controllers would have little effect on asymptotic stability of a linear time-invariant system.

REFERENCES

[1] G. Stewart, D. Gorinevsky, and G. Dumont, "Feedback controller design for a spatially distributed system: the paper machine problem," *IEEE Trans.Cont.Syst.Techn.*, vol. 11, no. 5, pp. 612–628, 2003.

[2] C. Bastin, A. Sarlette, and R. Sepulchre, "E-ELT Programme, M1 control strategies study, final report," University of Liège / European Southern Observatory, Tech. Rep. E-TRE-ULG-0449-0006, 2009.

[3] P. Seiler, A. Pant, and K. Hedrick, "Disturbance propagation in vehicle strings," *IEEE Trans.Aut.Cont.*, vol. 49, no. 10, pp. 1835–1841, 2004.

[4] T. Robinson, E. Chan, and E. Coelingh, "Operating platoons on public motorways: an introduction to the SARTRE platooning programme," *Proc. 17th World Congress on Intelligent Transport Systems*, 2010.

[5] B. Bamieh, F. Paganini, and M. Dahleh, "Distributed control of spatially invariant systems," *IEEE Trans.Aut.Cont.*, vol. 47, no. 7, pp. 1091–1107, 2002.

[6] H. Bode, *Network analysis and feedback amplifier design*. D. Van Nostrand, Princeton NJ, 1945.

[7] R. Middleton and J. Braslavsky, "String instability in classes of linear time invariant formation control with limited communication range," *IEEE Trans.Aut.Cont.*, vol. 55, no. 7, pp. 1519–1530, 2010.

[8] P. Padmasola and N. Elia, "Bode integral limitations of spatially invariant multi-agent systems," *Proc. 45th IEEE Conf. Decision and Control*, pp. 4327–4332, 2006.

[9] L. Peppard, "String stability of relative-motion pid vehicle control systems," *IEEE Trans.Aut.Cont.*, vol. 19, no. 5, pp. 579–581, 1974.

[10] R. Curtain, O. Iftime, and H. Zwart, "A comparison between LQR control for a long string of SISO systems and LQR control of the infinite spatially invariant version," *Automatica*, vol. 46, no. 10, pp. 1604–1615, 2010.

[11] A. Sarlette and R. Sepulchre, "A PDE viewpoint on basic properties of coordination algorithms with symmetries," *48th IEEE Conf. Decision and Control*, pp. 5139–5144, 2009.

[12] C. Mohtadi, "Bode's integral theorem for discrete-time systems," *IEE Proceedings*, vol. 137, no. 2, pp. 57–66, 1990.