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Asynchronous Distributed Calibration of Camera Networks

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Abstract— **This work focuses on the problem of calibrating planar networks of cameras in a distributed fashion. The camera network is modeled by a graph, and along each edge a noisy relative angular measurement is available. The goal is to achieve the absolute orientation of each camera with respect to a fixed external reference frame, in order to be able to perform monitoring and patrolling tasks. The idea is to exploit the cycles in the graph, along which all relative measurements sum to zero, in order to eliminate the noise. We design a distributed algorithm for the cameras to autonomously calibrate and we adopt an asynchronous gossip-like communication protocol. The proposed algorithm is proved to converge, almost surely and in the mean square sense, to the set of angles with zero cycle error. Finally, numerical experiments are presented to compare the performance of the algorithm on different graph topologies.**

I. Introduction

In the last decades many efforts have been done to design distributed strategies to accomplish various tasks by means of peer interacting agents. A decentralized approach is more resilient to communication and agents failures, and there is a higher adaptivity of the network towards changes. It usually requires a smaller computational effort since it requires only local information, but as a drawback the convergence of a distributed algorithm can be slower.

In this paper, we deal with a camera network, in which each agent is equipped with a video device and the applications can include motion capture, monitoring, tracking and surveillance issues. In order to accomplish those tasks, the cameras have to share a common reference frame. The calibration problem consists in letting each camera understand which is its position and orientation with respect to the global reference frame, in order to be able to coordinate with neighbors. Calibrating a network of cameras in a decentralized way has recently aroused interest and challenged the research community, since it leads to a higher robustness and an easy periodic recalibration. In our setup, the cameras are supposed to lie in the same plane, and the input data that cameras have are the noisy relative orientations of neighboring cameras.

We assume the communication protocol to be random gossip-like (see [6], [8]), in which at each iteration only one link is updated gathering only the states of neighboring cameras. Convergence is considered in a probabilistic sense and performance is studied in terms of mean square convergence.

The algorithm is proved to converge in the mean square sense for general planar graphs, and the cycle-error vector has null second moment. If we focus on ring graphs, our algorithm converges exponentially fast almost surely, and the expected value of the limit random variable equals the optimal solution, written in closed form.

Related work. Suppose the input data are relative measurements among nodes, and the aim is to find the absolute pose (position and orientation) of each node. In vector spaces, the so-called localization problem (restricted to the position vector) has been deeply investigated in [2], [1], [3], in which a distributed algorithm is proposed, the optimality of the solution is shown and the scalability with respect to the number of nodes is characterized.

In [11], [12], [13] a consensus algorithm over $SO(2)$ is presented, based on the gradient flow of a cost function defined by means of the geodesic distance, while in [14] a similar cost function is considered, involving the chordal distance, and they face the more general problem of calibration in $SE(3)$. In both cases the cost function shows different local minima. An approach based on a non-convex optimization problem is considered in [4], [5]. The geodesic distance is used to define the cost function and an optimal estimate is provided by solving a hybrid optimization problem: first a convex region containing a global minimum of the cost is estimated, and then a convex problem is solved. A similar approach has also been exploited in [10] where estimation is carried on by projecting the relative measurements into the sub-manifold of vectors whose sum along the cycles of the graph is a multiple of 2π.

Paper contributions and outline of the work. In Section II we provide basic definitions concerning algebraic graph theory. The setup and the minimization problem we want to deal with are described in Section III, and we recall the deterministic algorithm in [10], adding some performance analysis. In Section IV, we propose an asynchronous gossip version of the one proposed in [10] and Section V contains the convergence properties for general planar graphs (Section V-A) and ring graphs (Section V-B). Numerical experiments validating our analysis are provided in Section VI, and the conclusions of the work are drawn in Section VII.

II. Preliminaries of algebraic graph theory

In this Section we recall some known facts from algebraic graph theory which are instrumental throughout the paper.

An undirected graph is a couple $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, N\}$ is the set of nodes, and $\mathcal{E}$ is a subset

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of unordered pairs of elements of $\mathcal{V}$ called edges, with $M := |\mathcal{E}|$. An *orientation* on $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a pair of maps $s : \mathcal{E} \to \mathcal{V}$ and $t : \mathcal{E} \to \mathcal{V}$ such that $e = \{s(e), t(e)\}$ for every $e \in \mathcal{E}$. According to this definition, $s(e)$ and $t(e)$ are called the source and terminal node of the edge e, respectively. Assume from now on that we have fixed an orientation (s, t) on $\mathcal{G}$. The *incidence matrix* $B \in \{\pm 1, 0\}^{\mathcal{E} \times \mathcal{V}}$ of $\mathcal{G}$ is defined by putting $B_{e,s(e)} = 1$, $B_{e,t(e)} = -1$, and $B_{e,v} = 0$ if $v \neq s(e), t(e)$. A path h of length n is an ordered sequence of nodes $h = (v_1, v_2, \cdots, v_{n+1})$ such that $\{v_i, v_{i+1}\} \in \mathcal{E}$ for all $i = 1, \ldots, n$; v_1 and v_N are called the extremes of the path. $\mathcal{G}$ is said to be connected if for all pairs v, w there exists a path in $\mathcal{G}$ having v and w as extremes. A path $h = (v_1, v_2, \cdots, v_{n+1})$ is said to be closed if $v_1 = v_{n+1}$. A closed path $h = (v_1, v_2, \cdots, v_n, v_1)$ is said to be a cycle if $n \geq 3$ and $v_i \neq v_j$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

Consider now $\mathbb{Z}^{\mathcal{E}}$, the $\mathbb{Z}$-module of $\mathbb{Z}$-valued row vectors whose components are labelled by $\mathcal{E}$. We now associate with every path h, an element $\boldsymbol{r}_h \in \mathbb{Z}^{\mathcal{E}}$ as follows. First, if $h = (v_1, v_2)$, we put $\boldsymbol{r}_h(e) = B_{ev_1}$, if $e = \{v_1, v_2\}$, and $\boldsymbol{r}_h(e) = 0$ otherwise. For paths $h = (v_1, v_2, \cdots, v_{n+1})$ with non-repeating edges, $\boldsymbol{r}_h$ is built by assigning $\boldsymbol{r}_h(e) = \pm 1$ if the edge e appears in h with a coherent (resp. incoherent) orientation, and $\boldsymbol{r}_h(e) = 0$ otherwise. Denote now by Γ the $\mathbb{Z}$-submodule of $\mathbb{Z}^{\mathcal{E}}$ generated by all the vectors $\boldsymbol{r}_h$ as h varies in the set of closed paths. It holds true that Γ has dimension equal to $M - N + 1$ (see [4], [5]).

In this paper we will exclusively focus on *planar* graphs, namely graphs whose nodes and edges can be embedded in the Euclidean plane without intersection between edges but in the nodes (see [7]). Given a planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, its *dual graph* $\tilde{\mathcal{G}} = (\mathcal{F}, \tilde{\mathcal{E}})$ is defined by putting $\{c, c'\} \in \tilde{\mathcal{E}}$ iff c and c' are adjacent minimal cycles. The complementary part of a planar graph in the plane is a union of disconnected bounded domains (called faces) and one unbounded one. In this case, a basis of Γ can be easily obtained considering all the vectors $\boldsymbol{r}_h$ as h varies in the set $\mathcal{F}$ of closed paths which are the boundaries of the (bounded) faces all run in a clockwise fashion. We will refer to the closed paths in $\mathcal{F}$ as to the *minimal cycles* of $\mathcal{G}$. Given a minimal cycle $c \in \mathcal{F}$, let $|c|$ denote its length, and d_c the number of *adjacent* minimal cycles, namely the elements in $\mathcal{F}$ that share with c one edge. Clearly $d_c \leq |c|$. We define a *border minimal cycle*, any elements in $\mathcal{F}$ such that $d_c < |c|$. A *border edge* will consequently be any edge in the support of only one minimal cycle in $\mathcal{F}$. We now consider the so-called *cycle matrix* $R \in \mathbb{Z}^{M-N+1 \times M}$ having as rows the chosen basis of Γ, thus $\text{rank}(R) = M - N + 1$. Define the *essential cycle matrix* as the square $(M - N + 1)$-dimensional matrix $C := RR^T$. Given a symmetric matrix $P \in \mathbb{R}^{N \times N}$ we define the *graph associated with* P as $\mathcal{G}_P = (\mathcal{V}, \mathcal{E}_P)$ where $\mathcal{E}_P := \{\{i, j\} \mid P_{ij} = P_{ji} \neq 0\}$. P is called *stochastic* if $P_{ij} \geq 0$ for all i and j, and $\sum_j P_{ij} = 1$ for all i. If we denote by $\mathbf{1}$ the vector with all components equal to 1, the last row condition on P can simply be rephrased as $P\mathbf{1} = \mathbf{1}$. The matrix P is called *sub-stochastic* if $(P\mathbf{1})_i \leq 1$ for all i, and there exists at least one index i_0 for which the inequality is strict. A classical result is that if P is a sub-stochastic matrix with $\mathcal{G}_P$ connected, then $P^t \to 0$ for $t \to +\infty$.

III. Problem Setup

The calibration of a camera network can be stated as the following estimation problem. The camera network is modeled by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with orientation (s, t), $N = |\mathcal{V}|$ nodes, and $M = |\mathcal{E}|$ edges. A camera is placed at each node, and has its own reference frame defined as a couple $(R_i, T_i) \in SE(3), i = 1, \ldots, N$, where $R_i \in SO(3)$ is a rotation matrix, and $T_i \in \mathbb{R}^3$ is a translation vector with respect to a common external reference frame. We focus on the planar case, namely $R_i \in SO(2)$, and we concentrate on the estimation of the rotational part, neglecting the translation part. Note that each 2D rotation matrix $R_i, i \in \mathcal{V}$ can be associated with a unique $\bar{\theta}_i \in [-\pi, \pi)$ that denotes the true absolute orientation of node i. Each camera can measure its noisy relative orientation with respect to its neighbors. In order to formalize the model, let us define $(\cdot)_{2\pi} : \mathbb{R} \to [-\pi, \pi)$, $(x)_{2\pi} := x - 2\pi q_{2\pi}(x)$ with $q_{2\pi}(x) := \left\lfloor \frac{x+\pi}{2\pi} \right\rfloor$. Notice that the function $q_{2\pi}(x)$ is a quantizer such that $x \in [-\pi, \pi)$ if and only if $q_{2\pi}(x) = 0$, thus $(x)_{2\pi} = x$. Formally, denote the relative noisy orientation along every edge $e \in \mathcal{E}$ as $\eta_e = (\bar{\theta}_{s(e)} - \bar{\theta}_{t(e)} + \epsilon_e)_{2\pi}$, where $\epsilon_e \in \mathbb{R}$ is a bounded noise. More precisely, there exists $\bar{\epsilon} \in \mathbb{R}_{>0}$, such that $|\epsilon_e| \leq \bar{\epsilon}$, for each $e \in \mathcal{E}$ and for each realization. Define $\bar{\psi}_e := (B\boldsymbol{\eta})_e = \bar{\theta}_{s(e)} - \bar{\theta}_{t(e)}$, where B is the incidence matrix. Let $\boldsymbol{\eta} \in [-\pi, \pi)^{\mathcal{E}}, \bar{\boldsymbol{\theta}} \in [-\pi, \pi)^{\mathcal{V}}, \bar{\boldsymbol{\psi}} \in [-\pi, \pi)^{\mathcal{E}}$, and $\boldsymbol{\epsilon} \in [-\bar{\epsilon}, \bar{\epsilon}]^{\mathcal{E}}$ denote the vectors of all $\eta_e, \bar{\theta}_i, \bar{\psi}_e$ and ϵ_e respectively. The goal of the paper is to find a distributed algorithm that, given the relative measurements $\boldsymbol{\eta}$, estimates the absolute poses of the cameras, up to integer multiples of 2π. In other words, our distributed algorithm finds an estimate of the true relative orientation $\bar{\psi}$, and from the latter, the vector of the true absolute orientations $\bar{\theta}$ can be easily recovered by means of known distributed algorithms. We fix an anchor node, called *root* (say node 1), that knows exactly its orientation with respect to the external reference frame, and we assume $\bar{\theta}_1 = 0$. Since $\bar{\boldsymbol{\theta}}$ can be found up to integer multiples of 2π, we do not affect generality if we fix an anchor node, called *root* (say node 1), that knows exactly its orientation with respect to the external reference frame. We assume $\bar{\theta}_1 = 0$ and we find the relative orientations of any other node with respect to node 1.

In the noiseless case, an efficient way to reconstruct (exactly) $\bar{\theta}$ from η can be achieved considering any spanning tree $\mathcal{T}$ of the graph. For every node $i \in \mathcal{V}$, $\bar{\theta}_i$ can be obtained starting from the root value $\bar{\theta}_1$ and adding the relative orientations along edges in the shortest path connecting the root to node i. In principle this can also be done in the noisy case; nevertheless, in this way we do not take advantage of all the redundant measurements corresponding to edges in $\mathcal{E}\backslash\mathcal{E}_{\mathcal{T}}$. In [10], the following estimation approach is proposed: first η is projected onto the manifold of vectors having the property that sums along closed paths are 0 and then the estimation is obtained through a spanning tree as explained above. Our proposed algorithm is the randomized asynchronous version of this one. The estimation problem is split in two parts. Step 1 corresponds to solve the following non-convex and non-linear problem.

Problem 1 ***(Planar orientation localization problem)*** *The noisy relative orientations are given, i.e.* $\boldsymbol{\eta} = (\bar{\boldsymbol{\psi}} + \boldsymbol{\epsilon})_{2\pi}$*, where* $\bar{\boldsymbol{\psi}} = B\bar{\boldsymbol{\theta}}$ *are the correct relative orientations and* $\boldsymbol{\epsilon}$ *the noise vector. Find an estimate of the relative orientation* $\hat{\boldsymbol{\psi}} \in [-\pi, \pi)^{\mathcal{E}}$ *such that*

$$\begin{cases} \|(\hat{\boldsymbol{\psi}} - \boldsymbol{\eta})_{2\pi}\|_2 = \min_{\boldsymbol{\psi}} \|(\boldsymbol{\psi} - \boldsymbol{\eta})_{2\pi}\|_2\,, \\ r_c\hat{\boldsymbol{\psi}} = 0 \mod 2\pi, \text{for every cycle } c. \end{cases} \tag{1}$$

In [10], the authors solves Problem 1 in distributed fashion, with a synchronous algorithm, here recalled. In the next we use the following definitions. First, define the *cycle-error vector* at iteration $t \in \mathbb{N}$ as $\hat{\boldsymbol{\epsilon}}(t) := R\hat{\boldsymbol{\psi}}(t) \in \mathbb{R}^{M-N+1}$, therefore $\hat{\boldsymbol{\epsilon}}(0) := R\boldsymbol{\eta}$ is the initial error on cycles. Recall that R is the cycle matrix defined in Section II. Second, the *projected cycle-error vector* at time $t \in \mathbb{N}$ is $\boldsymbol{\epsilon}(t) := (R\hat{\boldsymbol{\psi}}(t))_{2\pi} \in [-\pi, \pi)^{M-N+1}$. For each iteration $t \in \mathbb{N}$, the following update is performed

$$\begin{cases} \hat{\boldsymbol{\psi}}(0) = \boldsymbol{\eta}, \\ \hat{\boldsymbol{\psi}}(t+1) = \hat{\boldsymbol{\psi}}(t) - kR^T(R\hat{\boldsymbol{\psi}}(t))_{2\pi}. \end{cases} \tag{2}$$

Step 2 consists in finding the estimate $\hat{\boldsymbol{\theta}}$ of the absolute poses $\bar{\boldsymbol{\theta}}$ starting from the anchor value $\bar{\theta}_1$ and adding the corresponding estimated relative orientations $\hat{\psi}$ along the the minimum path in the spanning tree $\mathcal{T}$ connecting each node to the anchor. This final step is straightforward and can be performed in a distributed way, thus our focus is on Step 1.

Concerning Step 1, the authors in [10] provide convergence results, stating that $\hat{\psi}$ solves Problem 1. We now provide a corollary of that result (see [10, Theorem 12]) concerning the convergence rate and the error of the estimate provided by their algorithm, depending on the topology of the underlying graph.

Theorem 1 ***(Error characterization for the synchronous algorithm)*** *Given* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$*, and the noisy relative measurements* $\boldsymbol{\eta} \in [-\pi, \pi)^{\mathcal{E}}$*, let* $\hat{\boldsymbol{\psi}} \in [-\pi, \pi)^{\mathcal{E}}$ *be the estimate provided by Algorithm in [10] in Eq.* (2) *with stepsize* $0 < k < 2/(1 + \lambda_{max}(C))$*, then*

$$\|\hat{\boldsymbol{\psi}} - \boldsymbol{\eta}\|_2 \leq \frac{\sqrt{\lambda_{max}(C)}}{2-k}\|(R\boldsymbol{\eta})_{2\pi}\|_2 \tag{3}$$

where $\lambda_{max}(C)$ *denotes the largest eigenvalue of the essential cycle matrix* C*. Moreover*

(i) *Ring graph:* $\lambda_{max}(C) = N$*;*
(ii) *Grid graph:* $\lambda_{max}(C) \leq 8$*;*
(iii) *Complete graph:* $\lambda_{max}(C) \leq 6$*;*
(iv) *If* $\mathcal{G}$ *is such that each edge belongs to* s *minimal cycles, and the minimal cycle length is* l*:* $\lambda_{max}(C) \leq l \cdot s$*. Notice that if* $\mathcal{G}$ *is planar* $s = 2$*.*

Proof: In order to prove Eq. (3), note that

$$\|\hat{\psi}_e(t) - \eta_e\|_2 = \|\sum_{s=0}^{t-1} \hat{\psi}_e(s+1) - \hat{\psi}_e(s)\|_2 \leq \sum_{s=0}^{t-1} k\|R^T\hat{\boldsymbol{\psi}}(s)\|_2.$$

Hence, passing to the limit as time goes to infinity, and defining $\hat{\boldsymbol{\psi}} := \hat{\boldsymbol{\psi}}(\infty) = \lim_{t\to\infty} \hat{\boldsymbol{\psi}}(t)$, it holds

$$\begin{aligned} \|\hat{\boldsymbol{\psi}} - \boldsymbol{\eta}\|_2 &\leq \sum_{s=0}^{\infty} k\|R^T\epsilon(s)\|_2 \leq \frac{k}{1-\rho}\|R^T\|_2\|\boldsymbol{\epsilon}(0)\|_2 \\ &= \frac{\|R^T\|_2\|\boldsymbol{\epsilon}(0)\|_2}{2-k} \leq \frac{\lambda_{max}(C)^{1/2}}{2-k}\|(R\boldsymbol{\eta})_{2\pi}\|_2. \end{aligned}$$

We used the fact that $\|\boldsymbol{\epsilon}(t)\|_2^2 \leq \rho^t \|\boldsymbol{\epsilon}(0)\|_2^2$, $\rho = (1-k)^2$ (see [10, Theorem 12]) and $\left\|R^T\right\|_2 = \|C\|_2^{1/2} = \sqrt{\lambda_{max}(C)}$. Then, to achieves the subsequent estimates of the largest eigenvalue of C, it suffices to apply Gershgorin Theorem and to observe that $C_{c_i,c_j} = \boldsymbol{r}_{c_i}\boldsymbol{r}_{c_j}^T$. ■

IV. Description of the Proposed Algorithm

Given a fixed camera network, the algorithm we proposed is a randomized asynchronous version of the one proposed in [10], it solves Step 1 stated in Problem 1, and it is formally described in Algorithm 1.

Algorithm 1 Asynchronous-Gossip-Algorithm

(**Input variables**)
$\eta_1, \dots, \eta_M$ noisy relative orientations;
$\boldsymbol{r}_{c_1}, \dots, \boldsymbol{r}_{c_{M-N+1}}$ minimal cycles vectors;
τ time horizon;
k stepsize;

(**Step A: initialization**)
$\hat{\boldsymbol{\psi}}(0) = \boldsymbol{\eta}$;

(**Step B: estimate** $(B\bar{\boldsymbol{\theta}})_{2\pi}$)
for $t = 1, 2, 3, \dots, \tau$ **do**
Choose randomly $e \in \mathcal{E}$;
$\hat{\psi}_e^+ = \hat{\psi}_e - k\sum_{i=1}^{M-N+1} r_{c_i}(e)(\boldsymbol{r}_{c_i} \cdot \hat{\boldsymbol{\psi}})_{2\pi}$;
Send ψ_e^+ to all $f \in \mathcal{E}$ such that $f \in c_i$ and $r_{c_i}(e) \neq 0$.
end for

In the last step the updated value information is locally spread along the graph as follows: if $e(t)$ is the selected edge at iteration t, for every cycle c such that $r_c(e(t)) \neq 0$, send the updated state $\hat{\psi}_e(t)$ to all $f \in \mathcal{E}$ with $r_c(f) \neq 0$.

The update can be rewritten as

$$\hat{\psi}_e(t+1) - \hat{\psi}_e(t) = \begin{cases} -k \sum_{c \in \mathcal{F}} r_c(e)\epsilon_c(t), & \text{if } e(t) = e, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

If we define $I_e = (i_{kl}) \in \{0,1\}^{M \times M}$ as the null matrix except for $i_{e,e} = 1$, Eq. (4) takes the form

$$\hat{\boldsymbol{\psi}}(t+1) = \hat{\boldsymbol{\psi}}(t) - kI_{e(t)}R^T\boldsymbol{\epsilon}(t). \tag{5}$$

In what follows, we take the initial error ϵ as fixed (not random). Randomness is thus completely coded in the sequence of chosen edges $e(t)$ which we assume to be independent and uniformly distributed in the set of all edges. Note that $\boldsymbol{\epsilon}(t)$ and $\hat{\boldsymbol{\psi}(t)}$ are random variables. $\mathbb{E}$ will denote expectation with respect to the choice of the sequence $e(t)$. Also, for a generic random vector, we will use the notation $\|f\|_{L^2}$ to denote the mean square norm: $\|f\|_{L^2} := (\mathbb{E}\|f\|^2)^{1/2}$, where $\|\cdot\|$ is the usual Euclidean norm of a vector.

V. PERFOMANCE ANALYSIS

In this section we analyze the behavior of the proposed gossip algorithm for the special class of planar graphs.

A. General graphs

From Eq. (5), we can determine the dynamics of the cycle-error vector as $\boldsymbol{\epsilon}(t+1) = (U_{e(t)}\boldsymbol{\epsilon}(t))_{2\pi}$, where, we recall, $e(t)$ is the randomly selected edge at iteration t, and U_e is a square $(M-N+1)$-dimensional matrix defined as

$$U_e := I - kRI_eR^T. \tag{6}$$

Note that U_e is symmetric, positive semidefinite, for every $e \in \mathcal{E}$. Moreover,

$$(U_e)_{cc'} = \begin{cases} 1, & \text{if } c = c', e \notin \text{supp}(\boldsymbol{c}) \\ 1-k, & \text{if } c = c', e \in \text{supp}(\boldsymbol{c}) \\ k, & \text{if } c \neq c', e \in \text{supp}(\boldsymbol{c}) \cap \text{supp}(\boldsymbol{c}') \\ 0, & \text{otherwise}. \end{cases}$$

It turns out that U_e is stochastic for any non-border edge e, while is sub-stochastic if e is a border edge. Since U_e is a stochastic or sub-stochastic matrix, it follows that if $\boldsymbol{x} \in [-\pi,\pi)^{N-M+1}$, then also $U_e\boldsymbol{x} \in [-\pi,\pi)^{N-M+1}$. This has an important consequence: since $\boldsymbol{\epsilon}(0) \in [-\pi,\pi)^{M-N+1}$, it follows that the dynamics of the $\boldsymbol{\epsilon}(t)$ actually satisfies the simpler dynamics

$$\boldsymbol{\epsilon}(t+1) = U_{e(t)}\boldsymbol{\epsilon}(t). \tag{7}$$

We can now state the following convergence result.

Theorem 2 ***(Convergence and estimate characterization for general graphs)*** *Given a connected planar graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$, the noisy relative measurements $\boldsymbol{\eta} \in [-\pi,\pi)^{\mathcal{E}}$ and the stepsize $0 < k < 1$, then there exist a number $\rho \in [0,1)$, and a r.v. $\hat{\boldsymbol{\psi}}(\infty)$ taking values in $\mathbb{R}^{\mathcal{E}}$ such that*

(i) *$\hat{\boldsymbol{\psi}}(t)$ converges to $\hat{\boldsymbol{\psi}}(\infty)$ almost surely and exponentially fast in mean square sense, that is*

$$\|\hat{\boldsymbol{\psi}}(t) - \hat{\boldsymbol{\psi}}(\infty)\|_{L^2} \leq \frac{k\|RR^T\|^{1/2}}{M(1-\rho)}\rho^t\|\boldsymbol{\epsilon}(0)\|_{L^2}; \tag{8}$$

(ii)

$$\|\hat{\boldsymbol{\psi}}(\infty) - \boldsymbol{\eta}\|_{L^2} \leq \frac{k\|RR^T\|^{1/2}}{M(1-\rho)}\|\boldsymbol{\epsilon}(0)\|_{L^2}; \tag{9}$$

(iii) *the limit cycle-error vector is null, that is*

$$\boldsymbol{\epsilon}(\infty) := (R\hat{\boldsymbol{\psi}}(\infty))_{2\pi} = 0.$$

Proof: (i) It follows from (7) that, for every $t \geq 0$, $\boldsymbol{\epsilon}(t) = U_{e(t)}\dots U_{e(1)}\boldsymbol{\epsilon}(0)$. Fix now a border edge e^* and notice that we can always build a sequence of edges $e_{j_1},\dots,e_{j_s}$ such that $e_{j_1} = e^*$, e_{j_l} and $e_{j_{l+1}}$ belong to the same minimal cycle, and for every minimal cycle $c \in \mathcal{F}$ there exists e_{j_l} with $r_c(e_{j_l}) \neq 0$. This simply follows from the fact that the dual graph of $\mathcal{G}$ is connected and by considering a path in this graph starting from the minimal cycle which e^* belongs to and touching all other minimal cycles. Consider now $Q = U_{e_{j_s}}\cdots U_{e_{j_1}}$. It is sub-stochastic since $U_{e_{j_1}}$ is so and the others are stochastic or sub-stochastic. Moreover, a simple induction argument shows that no row of Q sums to 1. Put $\alpha := \|Q\|_\infty = \max_c(Q\mathbf{1})_c \in (0,1)$. The sequence of edges $e_{j_1},\dots,e_{j_s}$ is chosen from left to right with a positive probability p (equal to $p = M^{-s}$), therefore, by Chernoff bound, for any $r = 1,\dots t^*$ and some $\beta > 0$, it holds

$$\mathbb{P}(\|\boldsymbol{\epsilon}(ns+r)\|_\infty \leq \alpha^{np/2}\|\boldsymbol{\epsilon}(r)\|_\infty) \geq 1 - e^{-n\beta}.$$

An application of Borel-Cantelli Lemma now yields that $\boldsymbol{\epsilon}(t)$ is summable almost surely. From (5) it follows that $\hat{\psi}(t)$ is a Cauchy sequence almost surely, so that it converges, almost surely, to a measurable r.v. $\hat{\psi}(\infty)$. We now investigate convergence in mean square sense. Notice that

$$\begin{aligned}\mathbb{E}[\|\boldsymbol{\epsilon}(t+1)\|^2] &= \mathbb{E}[\mathbb{E}[\boldsymbol{\epsilon}^T(t)U^2_{e(t)}\boldsymbol{\epsilon}(t) \mid e(0),\dots,e(t-1)]] \\ &= \mathbb{E}[\boldsymbol{\epsilon}^T(t)\mathbb{E}(U^2_{e(t)})\boldsymbol{\epsilon}(t)],\end{aligned}$$

and $\mathbb{E}(U^2_{e(t)})$ is a symmetric matrix. If we denote by ρ^2 its spectral radius (largest in modulo eigenvalue), we can estimate $\mathbb{E}[\|\boldsymbol{\epsilon}(t+1)\|^2] \leq \rho^2\mathbb{E}\|\boldsymbol{\epsilon}(t)\|^2$. This yields, $\forall t \geq 0$,

$$\|\boldsymbol{\epsilon}(t+1)\|_{L^2} \leq \rho^t\|\boldsymbol{\epsilon}(0)\|_{L^2}. \tag{10}$$

Let us now verify that $\rho \in [0,1)$. Indeed, since U_e^2 inherits the property of U_e, namely it is stochastic (resp. sub-stochastic) if U_e is so, and since U_e is sub-stochastic with positive probability, it turns out that the average $\mathbb{E}(U^2_{e(t)})$ is sub-stochastic. Moreover, it is easy to see that is also irreducible since the corresponding graph coincides with the dual of $\mathcal{G}$. This implies the thesis on ρ. From (5) it now

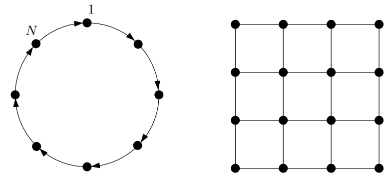

Fig. 1. This figure shows a ring graph (left), and a grid graph (left) as an example of a planar graph. Algorithm 1 is analyzed on these different communication networks in Section V-A and V-B respectively.

immediately follows that $\hat{\psi}(t)$ is a Cauchy sequence in L^2 sense, so that it must converge to $\hat{\psi}(\infty)$ in mean square sense too. Moreover we can estimate $\|\hat{\boldsymbol{\psi}}(\infty) - \hat{\boldsymbol{\psi}}(t)\|_{L^2} \leq k \sum_{s=t}^{+\infty} \|R^T I_{e(s)} \boldsymbol{\epsilon}(s)\|_{L^2}$ and thus

$$\|\hat{\boldsymbol{\psi}}(\infty) - \hat{\boldsymbol{\psi}}(t)\|_{L^2} \leq \frac{k}{M} \sum_{s=t}^{+\infty} \|RR^T\|^{1/2} \rho^s \|\boldsymbol{\epsilon}(0)\|_{L^2}.$$

This immediately yields (8).

(ii) The thesis follows immediately from previous relation by taking $t = 0$.

(iii) Suppose by contradiction $\boldsymbol{\epsilon}(\infty) := (R\hat{\boldsymbol{\psi}}(\infty))_{2\pi} \neq 0$, then it would exist a cycle c, numbers $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $\mathbb{P}(R\hat{\boldsymbol{\psi}}(\infty)_c \in]n^{-1}, 2\pi - n^{-1}] + 2k\pi) > 0$. Since $R\hat{\boldsymbol{\psi}}(t) \to R\hat{\boldsymbol{\psi}}(\infty)$, it follows that

$$\mathbb{P}(\exists t_0 : \; R\hat{\boldsymbol{\psi}}(t)_c \in]n^{-1}, 2\pi - n^{-1}] + 2k\pi, \; \forall t \geq t_0) > 0$$

which yields $\mathbb{P}(\exists t_0 : \; |((R\hat{\boldsymbol{\psi}}(t))_{2\pi})_c| > n^{-1}, \; \forall t \geq t_0) > 0$. This contradicts the fact that $\boldsymbol{\epsilon}(t) = (R\hat{\boldsymbol{\psi}}(t))_{2\pi} \to 0$ almost surely. The proof is thus complete. ■

Remark 3 ***(Convergence rate characterization)*** *From the proof of Theorem 2, a very simple characterization for the rate of convergence ρ can be obtained. Indeed it coincides with the spectral radius of the sub-stochastic matrix $\mathbb{E}[U_e^2]$. An analysis of this matrix for specific classes of graphs will be carried on elsewhere.*

Note that Theorem 1 and Theorem 2 (ii) provide bounds on the distance between the initial noisy orientations $\boldsymbol{\eta}$ and the final estimates obtained by means of the synchronous and asynchronous gossip-like algorithm respectively. They both depend on the stepsize k and the graph topology, embedded in the matrix R (see Section II). Of course, a fundamental fact missing in the performance analysis of Theorem 2, is an estimation of the displacement with respect to the optimal solution $\|\hat{\boldsymbol{\psi}}(\infty) - \boldsymbol{\psi}^*\|_{L^2}$. In this paper we will limit this analysis to the ring graphs.

B. Ring graphs

Consider a ring graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = N$. There is just one minimal cycle corresponding to $\boldsymbol{r} = \mathbf{1}$, see Fig. 1 (left). Note that Theorem 2 applies also in this case, but we can characterize additional properties of the estimate $\hat{\boldsymbol{\psi}}$.

If e is the sampled edge, the only updated component is

$$\psi_e(t+1) = \psi_e(t) - k \; (\mathbf{1}\, \boldsymbol{\psi}(t))_{2\pi},$$

that can be casted in vector form as

$$\begin{cases} \hat{\boldsymbol{\psi}}(t+1) = \hat{\boldsymbol{\psi}}(t) - k\boldsymbol{b}_{e(t)}\epsilon(t) \\ \hat{\boldsymbol{\psi}}(0) = \boldsymbol{\eta} \end{cases} \tag{11}$$

where $\boldsymbol{b}_{e(t)} \in \{0,1\}^N$ is defined as 1 in the component $e(t)$, and 0 otherwise. First, let us notice that the dynamics on the (scalar) cycle-error becomes deterministic, i.e. Eq. (7) becomes $\epsilon(t+1) = ((1-k)\epsilon(t))_{2\pi} = (1-k)\epsilon(t)$. This has an immediate consequence: $\hat{\boldsymbol{\psi}(t)}$, which is still a sequence of random variables, always converges to $\hat{\boldsymbol{\psi}(t)}$ (and not just almost surely). Moreover, equation (11) immediately yields

$$\hat{\boldsymbol{\psi}}(\infty) = \boldsymbol{\eta} - k\left(\sum_{s=0}^{\infty} (1-k)^s \epsilon(0) \boldsymbol{b}_{e(s)}\right). \tag{12}$$

In [9], it is proved that an optimal solution of Problem 1 for a ring graph is

$$\boldsymbol{\psi}^* = \boldsymbol{\eta} - \frac{1}{N}\boldsymbol{r}^T(\boldsymbol{r}\boldsymbol{\eta})_{2\pi}. \tag{13}$$

If we run the Algorithm described in Eq. (2) with $k = 1/N$, we obtain $\boldsymbol{\psi}^*$ in one time step (see [10]). From Eq. (12) and (13) we further obtain

$$\|\hat{\boldsymbol{\psi}}(\infty) - \boldsymbol{\psi}^*\|_{L^2} = \|k\sum_{s=0}^{+\infty}(1-k)^s\epsilon(0)[b_{e(s)} - N^{-1}\mathbf{1}]\|_{L^2}$$
$$\leq k\sum_{s=0}^{+\infty}(1-k)^s|\epsilon(0)|(1-N^{-2})^{1/2} \leq |\epsilon(0)|.$$

The latter difference remains bounded, it does not depend on the number of nodes N, and it implies

$$N^{-1}\|\hat{\boldsymbol{\psi}}(\infty) - \boldsymbol{\psi}^*\|_{L^2} = O(N^{-1}). \tag{14}$$

This can be rephrased also saying that the difference between the estimate in the two algorithms per edge goes to 0 as N is asymptotically large.

VI. Numerical examples

In this section we validate our distributed asynchronous Algorithm 1 through a numerical study.

RD:05 First, we consider a ring graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = N = 20$ and $M = |\mathcal{E}| = 20$ as depicted in Fig. 1 (left). Recall that $\bar{\boldsymbol{\psi}}$ denotes a solution to Problem 1, and $\boldsymbol{\psi}^*$ is the optimal estimate given by Algorithm in Eq. (2). We sample 50 different sequences of edges, and we average over these samples the following quantities

$$\begin{aligned} E_1(t) &:= \mathbb{E}[\|(\hat{\boldsymbol{\psi}}(t) - \bar{\boldsymbol{\psi}})_{2\pi}\|_2]/M, \\ E_2(t) &:= \mathbb{E}[\|\boldsymbol{\epsilon}(t)\|_2]/(M-N+1), \\ E_3(t) &:= \mathbb{E}[\|(\hat{\boldsymbol{\psi}}(t) - \boldsymbol{\psi}^*)_{2\pi}\|_2]/M. \end{aligned} \tag{15}$$

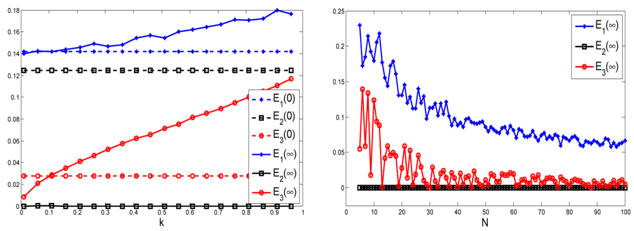

Fig. 2. Given a ring graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $N = 20$, the time horizon $\tau = 300$, and the number of samples $n_s = 50$, the left figure shows the initial and asymptotic values of E_1, E_2, E_3, while we vary the stepsize $k \in (0, 1)$. The projection error E_2 goes to zero for every k, while E_1, E_3 decrease only for k smaller than a certain threshold related to $\lambda_{\max}(C)$ (cfr. Theorem 1 for the synchronous algorithm threshold). The right plot shows $E_i(\infty), i = 1, 2, 3$, with $k = 0.3, \tau = 250$ and $N \in [5, 100]$.

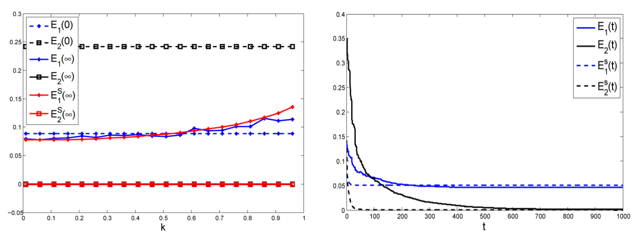

Fig. 3. In the left figure, we consider a grid $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $N = n^2$ nodes, $n = 5$, $\tau = 10^3$ and the number of samples $n_s = 50$. The figure shows the asymptotic values of $E_1(\infty), E_2(\infty)$, while we vary $k \in (0, 1)$. If we run the synchronous deterministic algorithm in [10], we obtain the corresponding asymptotic values $E_1^S(\infty), E_2^S(\infty)$. In the right plot we consider a non-planar graph (Cayley grid, [7]) with $N = 25$, and both the (a)synchronous algorithms are proved to converge in time.

As described in Section III, the noise on each each is sampled from a uniform distribution, namely $\epsilon_e \sim \text{Unif}([-\bar{\epsilon}, \bar{\epsilon}])$, for every $e \in \mathcal{E}$, with maximum noise $\bar{\epsilon} = \pi/3$. We consider the normalized final error $E_1(\infty)$ of the estimate $\hat{\psi}$ with respect to a true solution $\bar{\psi}$ of Problem 1, and the normalized final distance $E_3(\infty)$ between our estimate and the optimal estimate ψ^* given by the deterministic algorithm proposed in [10]. Both these quantities approach zero in the 2-norm as time goes to infinity, for any stepsize k under a certain threshold depending on N (see Fig. 2), while the normalized asymptotic projected cycle-error $\epsilon(\infty)$ goes to zero for any k (see the index E_2). Note that, as k increases, the precision of the estimate gets worse. On the other hand, for tiny stepsizes, the time required to guarantee a certain threshold precision becomes large. The estimate provided by the synchronous deterministic algorithm proposed in [10], with stepsize $k = 1/N$, leads to $E_1^S(\infty) = 2.6216$ (it is not shown in Fig. 2 for the sake of space). Therefore our randomized algorithm has a better performance according to E_1. Moreover, we vary the number of nodes N (see Fig. 2 right), in order to validate Eq. (14). Second, we fix a planar grid graph, and Fig. 3 shows the asymptotic values of E_1, E_2 of the estimate provided by Algorithm 1 and the asymptotic values of E_1^S, E_2^S of the synchronous deterministic version of our algorithm (see [10]) are comparable. Finally, we test the two algorithms on a non planar Cayley grid (Fig. 3 right), and they both converge.

VII. Conclusions

In this paper we provide a distributed and asynchronous algorithm that solves the calibration problem for a network of cameras. We cast the latter calibration issue as a constrained minimization problem. In the network, modeled by a graph, each edge represents an agent. At each iteration, only one agent is activated randomly, sampled from a uniform distribution, and it updates the corresponding relative orientation. First, we state characterizing properties of the deterministic synchronous version of our algorithm (see [10]). Second, we provide analytical results on the performance properties of the proposed algorithm, for general topologies. We then focus on ring graphs and deeply investigate the optimality of our estimate. Finally, numerical simulations are run on different graphs to show the effectiveness of the procedure.

References

[1] P. Barooah, N. M. da Silva, J. P. Hespanha, Distributed Optimal Estimation from Relative Measurements for Localization and Time Synchronization, Distributed Computing in Sensor Systems, Springer, 2006, vol. 4026, Lect. Notes in Comput. Science, pp. 266–281.

[2] P. Barooah and J. P. Hespanha, Distributed Estimation from Relative Measurements in Sensor Networks, Proceedings of the 2nd International Conference on Intelligent Sensing and Information Processing, Dec. 2005.

[3] P. Barooah and J. P. Hespanha, Estimation from relative measurements: Algorithms and scaling laws, CSM, 2007, vol. 27, n. 4, pp. 57–74.

[4] D. Borra, E. Lovisari, R. Carli, F. Fagnani and S. Zampieri, Autonomous Calibration Algorithms for Networks of Cameras, Proceedings of the Americal Control Conference, ACC'12, July 2012.

[5] D. Borra, E. Lovisari, R. Carli, F. Fagnani and S. Zampieri, Autonomous Calibration Algorithms for Networks of Cameras, 2012, in IFAC Automatica. (Submitted)

[6] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, Randomized gossip algorithms, IEEE Transactions on Information Theory, 52(6):2508-2530, 2006.

[7] R. Diestel, Graph Theory, Springer. Graduate Texts in Mathematics, 2005.

[8] F. Fagnani, S. Zampieri, Randomized consensus algorithms over large scale networks, IEEE Journal on Selected Areas in Communications, vol. 26, pp. 634–649, 2008.

[9] S. Kaczmarz, Approximate solution of systems of linear equations, International Journal of Control, vol. 57, n. 6, pp. 1269–1271, 1993.

[10] G. Piovan, I. Shames, B. Fidan, F. Bullo, B. D. O. Anderson, On Frame and Orientation Localization for Relative Sensing Networks, Automatica, 2011.

[11] A. Sarlette, R. Sepulchre, N. E. Leonard, Discrete-time synchronization on the N-torus, MTNS, June 2006, Kyoto, Japan.

[12] A. Sarlette and R. Sepulchre, Consensus Optimization on Manifolds, SICON, 2009, vol. 48, n. 1, pp. 56–76.

[13] A. Sarlette, Geometry and Symmetries in Coordination Control, University of Liège, Belgium, January 2009.

[14] R. Tron and R. Vidal, Distributed Image-Based 3-D Localization of Camera Sensor Networks, CDC, December 2009, Shanghai, China, pp. 901–908.