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# Spatio-temporal symmetries in linear control systems with an application to formation control.

Luca Consolini[1] and Mario Tosques[1]

*Abstract*— **We study a class of controlled linear systems characterized by a property of spatio-temporal symmetry. We study its stabilizability and detectability properties. We present a design technique to locally stabilize spatio-temporally symmetric orbits of equivariant systems, that has applications in the control of identical interconnected systems. As an example, we present the stabilization of a formation of unicycle robots in cyclic pursuit.**

## I. INTRODUCTION

Various biological and human made systems are composed of identical interconnected subsystems. In some of these, all components follow the same periodic behavior with a phase difference. Following the terminology used in [1], we say that the corresponding system trajectory has a spatio-temporal symmetry.

Some examples are the following:

1) Animal locomotion is controlled, in part, by a central pattern generator (CPG), which is an intraspinal network of neurons capable of generating a rhythmic output. Signals involved in the quadruped gaits: walk, trot and pace are characterized by spatio-temporal symmetries (see [2], [3]).

2) The heart is constituted by a set of interconnected muscle cells that produce a periodic output in which the electrical stimulation of one cell spreads to its neighbors. The voltage signals at the various cells are characterized by properties of spatio-temporal symmetry. The breaking of this symmetry can result in hearth rhythm disorders and potential heart failures (see for instance [4]).

3) In formation control, it may be desirable to create a platoon of vehicles in which each member follows the same trajectory and keeps a fixed distance from the previous one, this lead to a spatio-temporal symmetry in the formation (see for instance [5], [6]).

In a system with state variable $x$, composed of identical subsystems, if $\Gamma$ represents a permutation of the subsystems, a desired coordinated motion with spatio-temporal symmetry satisfies the property $\Gamma x(t) = x(t+\tau)$.

One relevant problem is to stabilize such a motion with a controller, in such a way that every subsystem uses the same feedback law. Following the previous examples, this has applications in the stabilization of periodic biological signals, such as the ones involved in animal locomotion and heart activity, or in the coordination of distributed systems, such as mobile vehicles formations.

In this paper, we show that if the system is equivariant with respect to the transformation $\Gamma$, the linearized system with respect to the reference trajectory satisfies suitable symmetry properties (see (2a)-(2d) below). We call the linear systems that possesses these properties "spatio-temporally symmetric".

We characterize the properties of this class of linear systems and present conditions for the synthesis of a stabilizing controller and of an asymptotic observer. In particular, we show that, after a change of coordinates, systems belonging to this class are equivalent to hybrid systems, represented by a periodic linear system with periodic state jumps (see (5) below).

The contribution of this paper is two-fold. First, it characterizes the stabilizability and detectability properties of this special class of linear time-varying systems (see definition 1 below). The definition of these systems has some analogies with the class of patterned system recently introduced and characterized in [7]. Second, it presents a design technique to locally stabilize spatio-temporally symmetric orbits of equivariant system, that has applications in the control of distributed systems. As an example, we present the stabilization of a formation of unicycle robots in cyclic pursuit.

### A. Notations

If $\tau > 0$, we set $\tau\mathbb{Z} = \{\tau i | i \in \mathbb{Z}\}$, $\mathbb{R}\backslash\tau\mathbb{Z} = \{t \in \mathbb{R} \,|\, t \notin \tau\mathbb{Z}\}$. If $\Omega \subset \mathbb{R}$ we denote by $\mathcal{C}(\Omega, \mathbb{R}^n)$ the set of continuous functions defined on $\Omega$ with values in $\mathbb{R}^n$ and by $\mathcal{C}^1(\Omega, \mathbb{R}^n)$ the $\mathcal{C}^1$ functions on $\Omega$ with values in $\mathbb{R}^n$.

## II. LINEAR SYSTEMS WITH SPATIO-TEMPORAL SYMMETRY

We consider the class of linear time-varying systems

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t), \end{aligned} \tag{1}$$

where $A \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n\times n})$, $B \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n\times m})$, $C \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{p\times n})$, $D \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{p\times m})$.

This class of systems is identified by the quadruple $(A,B,C,D)$. In particular we focus our attention on those systems that obey the following definition.

***Definition 1:*** We say that $(A,B,C,D)$ is $(\Gamma,\Theta,\Sigma,\tau)$-symmetric if there exist invertible matrices $\Gamma$, $\Theta$, $\Sigma$ and a positive real number $\tau$ such that the following conditions hold

$$\Gamma A(t) = A(t+\tau)\Gamma, \tag{2a}$$

$$\Gamma B(t) = B(t+\tau)\Theta, \tag{2b}$$

$$\Sigma C(t) = C(t+\tau)\Gamma, \tag{2c}$$

$$\Sigma D(t) = D(t+\tau)\Theta. \tag{2d}$$

and there exists $k > 0$, $k \in \mathbb{N}$ such that

$$\Gamma^k = I, \Theta^k = I, \Sigma^k = I, \tag{3}$$

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[1] Both the authors are with the Department of Information Engineering, University of Parma, Via Usberti 181/A, 43124 Parma, Italy `lucac@ce.unipr.it, mario.tosques@unipr.it`

where $I$ represents the identity matrix of the appropriate dimension.

Similarly, we say that $A$ is $(\Gamma,\tau)$-symmetric if (2a) is verified and that the set $(A,B)$ is $(\Gamma,\Theta,\tau)$-symmetric if (2a), (2b) hold. If this is the case, we also say that system (1) is $(\Gamma,\Theta,\Sigma,\tau)$-symmetric.

***Remark 1:*** A consequence of definition 1 is that $\Gamma^k A(t) = A(t+k\tau)\Gamma^k$, which, by (3), implies that $A$ is $k\tau$-periodic. Similarly, it follows that matrix functions $B$, $C$, $D$ are also $k\tau$-periodic.

***Remark 2:*** Systems obeying definition 1 can be thought as "spatio-temporally symmetric" systems. In section IV we will show that properties (2a)-(2d) characterize the linearization of equivariant nonlinear systems along a spatio-temporal symmetric solution.

***Definition 2:*** Set $\lfloor t \rfloor = \max\{i \in \mathbb{Z}, | i \le t\}$ as the integer part of $t$ and denote by $\pi : \mathbb{R} \to [0,\tau)$ the map defined by

$$\pi(t) = t - \lfloor \frac{t}{\tau} \rfloor \tau .$$

The following proposition shows that any $(\Gamma,\Theta,\Sigma,\tau)$-symmetric quadruple $(A,B,C,D)$ is uniquely determined by its value in the interval $[0,\tau)$.

***Proposition 1:*** The quadruple $(A,B,C,D)$ is $(\Gamma,\Theta,\Sigma,\tau)$-symmetric if and only if there exist matrices $\Gamma$, $\Theta$, $\Sigma$ and a positive real number $\tau$ such that, $\forall t \in \mathbb{R}$,

$$A(t) = \Gamma^{\lfloor \frac{t}{\tau} \rfloor} A(\pi(t)) \Gamma^{-\lfloor \frac{t}{\tau} \rfloor} \tag{4a}$$

$$B(t) = \Gamma^{\lfloor \frac{t}{\tau} \rfloor} B(\pi(t)) \Theta^{-\lfloor \frac{t}{\tau} \rfloor} \tag{4b}$$

$$C(t) = \Sigma^{\lfloor \frac{t}{\tau} \rfloor} C(\pi(t)) \Gamma^{-\lfloor \frac{t}{\tau} \rfloor} \tag{4c}$$

$$D(t) = \Sigma^{\lfloor \frac{t}{\tau} \rfloor} D(\pi(t)) \Theta^{-\lfloor \frac{t}{\tau} \rfloor} \tag{4d}$$

*Proof:* We prove the first of (4a), the others are analogous. Applying $\lfloor \frac{t}{\tau} \rfloor$ times (2a), it follows that

$$\Gamma^{\lfloor \frac{t}{\tau} \rfloor} A(\pi(t)) = \Gamma^{\lfloor \frac{t}{\tau} \rfloor} A(t - \lfloor \frac{t}{\tau} \rfloor \tau) = A(t)\Gamma^{\lfloor \frac{t}{\tau} \rfloor}, \forall t \in \mathbb{R} .$$

from which (4a) follows. Conversely, if (4a) holds,

$$\begin{aligned} \Gamma A(t) &= \Gamma^{\lfloor \frac{t}{\tau} \rfloor + 1} A(\pi(t)) \Gamma^{-\lfloor \frac{t}{\tau} \rfloor} \\ &= \Gamma^{\lfloor \frac{t}{\tau} \rfloor + 1} A(\pi(t)) \Gamma^{-\lfloor \frac{t}{\tau} \rfloor - 1} \Gamma = A(t+\tau)\Gamma, \forall t \in \mathbb{R}, \end{aligned}$$

□

The following proposition shows that if $(A,B,C,D)$ is $(\Gamma,\Theta,\Sigma,\tau)$-symmetric, system (1) is equivalent, after a change of variables, to a linear hybrid periodic system.

***Proposition 2:*** Suppose that $(A,B,C,D)$ is $(\Gamma,\Theta,\Sigma,\tau)$-symmetric. Then if $x$, $y$, $u$ satisfy (1), functions

$$\xi(t) = \Gamma^{-\lfloor \frac{t}{\tau} \rfloor} x(t), \eta(t) = \Sigma^{-\lfloor \frac{t}{\tau} \rfloor} y(t), v(t) = \theta^{-\lfloor \frac{t}{\tau} \rfloor} u(t) ,$$

solve system

$$\begin{cases} \dot{\xi}(t) = A(\pi(t))\xi(t) + B(\pi(t))v(t), \text{ if } t \in \mathbb{R}\backslash\tau\mathbb{Z} \\ \xi(t) = \lim_{s \to t^-} \Gamma^{-1}\xi(s), \text{ if } t \in \tau\mathbb{Z} \\ \eta(t) = C(\pi(t))\xi(t) + D(\pi(t))v(t) \, \forall t \in \mathbb{R} . \end{cases} \tag{5}$$

Conversely, if $(\xi,\eta,v)$ is a solution of (5), then $x(t) = \Gamma^{\lfloor \frac{t}{\tau} \rfloor}\xi(t)$, $y(t) = \Sigma^{\lfloor \frac{t}{\tau} \rfloor}\eta(t)$, $u(t) = \Theta^{\lfloor \frac{t}{\tau} \rfloor} v(t)$ is a solution of (1).

*Proof:* It follows from proposition 10 (see the Appendix) with $G(t) = B(t)u(t)$ and from (4c), (4d). □

### *A. Autonomous systems with spatio-temporal symmetries*

In this section we present some properties of the autonomous system

$$\dot{x}(t) = Ax(t)$$

where $A$ is $(\Gamma,\tau)$-symmetric.

In the following, we denote by $\Phi(t,t_0)$ the transition matrix of $A \in \mathcal{C}(\mathbb{R},\mathbb{R}^{n\times n})$, which is the solution $M \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^{n\times n})$ of

$$\begin{cases} \dot{M} = A(t)M \\ M(t_0) = I , \end{cases} \tag{6}$$

where $I$ is the identity matrix.

***Proposition 3:*** Suppose that $A$ is $(\Gamma,\tau)$-symmetric. Then system

$$\dot{x}(t) = A(t)x(t)$$

is asymptotically stable if and only if all the eigenvalues $\lambda$ of $\Gamma^{-1}\Phi(\tau,0)$ are such that $|\lambda| < 1$.

*Proof:* The thesis follows from propositions 10 and 12 (see the Appendix) and the fact that $\Gamma^{\lfloor \frac{t}{\tau} \rfloor}$ is bounded since $\Gamma^k = I$. □

### *B. Controlled linear systems with spatio-temporal symmetry*

Consider the controlled system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) . \tag{7}$$

The following proposition gives a condition under which the $(\Gamma,\Theta,\tau)$-symmetry of (7) is preserved after the application of the feedback law $u(t) = F(t)x(t)$.

***Proposition 4:*** Suppose that the couple $(A,B)$ is $(\Gamma,\Theta,\tau)$-symmetric, then the couple $(A+BF,B)$ has the same property if

$$\Theta F(t) = F(t+\tau)\Gamma, \forall t \in \mathbb{R} . \tag{8}$$

Conversely, if (8) holds and $B(t)$ is full rank for all $t \in \mathbb{R}$, then $(A+BF,B)$ is $(\Gamma,\Theta,\tau)$-symmetric.

*Proof:* (Sufficiency) Assume that (8) holds, then $\Gamma(A(t)+B(t)F(t)) = A(t+\tau)\Gamma + B(t+\tau)\Theta F(t) = (A(t+\tau) + B(t+\tau)F(t+\tau))\Gamma$.

(Necessity) If the closed-loop system is spatio-temporally symmetric, it follows that

$$\Gamma(A(t)+B(t)F(t)) = (A(t+\tau)+B(t+\tau)F(t+\tau))\Gamma .$$

Moreover,

$$\Gamma(A(t)+B(t)F(t)) = A(t+\tau)\Gamma + B(t+\tau)\Theta F(t) .$$

These two properties imply that

$$B(t+\tau)(\Theta F(t) - F(t+\tau)\Gamma) = 0 .$$

Since $B(t+\tau)$ is full rank for every $t \in \mathbb{R}$, equation (8) follows. □

The following discussion on stabilizability is an adaptation to systems with spatio-temporal symmetry of classical results for periodic system (see [8], [9], [10]). In particular, the following definition is an adaptation of the notion of $W$-stabilizability.

***Definition 3:*** The $(\Gamma,\Theta,\tau)$-symmetric couple $(A,B)$ is $(\Gamma,\Theta,\tau)$-stabilizable if there exists a matrix function $F : \mathbb{R} \to \mathbb{R}^{m\times n}$ satisfying (8), such that system

$$\dot{x}(t) = (A(t)+B(t)F(t))x(t)$$

is asymptotically stable.

***Definition** 4:* A complex number $\lambda$ is called an uncontrollable eigenvalue of the $(\Gamma,\Theta)$-symmetric system (5) if there exists $\eta \in \mathbb{C}^n$ such that

$$\begin{array}{l} \eta^T \Gamma^{-1} \Phi(\tau,0) = \lambda \eta^T \\ \eta^T (\Phi(0,\tau)\Gamma)^i \Phi(0,s) B(s) = 0, \forall s \in [0,\tau), \forall i = 0,\ldots,n-1 . \end{array} \tag{9}$$

The following theorem characterizes the stabilizability of linear systems with spatio-temporal symmetries.

***Theorem** 1:* System (7) is $(\Gamma,\Theta,\tau)$-stabilizable if and only if all its uncontrollable eigenvalues $\lambda$ are such that $|\lambda| < 1$. Moreover, given symmetric positive definite matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, the system is stabilized by the feedback matrix $F$

$$F(t) = -\Theta^{\lfloor \frac{t}{\tau} \rfloor} B(\pi(t)) R^{-1} B^T(\pi(t)) S(t) \Gamma^{-\lfloor \frac{t}{\tau} \rfloor},$$

where $S$ is the $\tau$-periodic matrix solution of the following hybrid Riccati equation

$$\left\{ \begin{array}{l} \dot{S} - SB(\pi(t))R^{-1}B^T(\pi(t))S + SA(\pi(t)) \\ \quad +A^T(\pi(t))S + Q = 0, \text{ if } t \in \mathbb{R}\backslash\tau\mathbb{Z} \\ S(t) = \lim_{s\to t^-} \Gamma^T S(s) \Gamma, \text{ if } t \in \tau\mathbb{Z} . \end{array} \right. \tag{10}$$

*Proof:* It follows from propositions 2 and 13 (see the appendix). □

## III. OBSERVABILITY AND DETECTABILITY

Observability and detectability properties of (1) are characterized on the basis of the dual system.

***Definition** 5:* The dual system of (1) is given by

$$\begin{array}{l} \dot{x}(t) = A_D(t)x(t) + B_D(t)u(t) \\ y(t) = C_D(t)x(t) + D_D(t)u(t) \end{array} \tag{11}$$

where

$$A_D(t) = A^T(-t), B_D(t) = C^T(-t),$$
$$C_D(t) = B^T(-t), D_D(t) = D^T(-t).$$

***Proposition** 5:* The dual system $(A_D, B_D, C_D, D_D)$ is $(\Gamma^T, \Sigma^T, \Theta^T, \tau)$-symmetric.

*Proof:* We prove condition (2a), conditions (2b), (2c), (2d) can be proven similarly,

$$\begin{aligned} \Gamma^T A_D(t) = \Gamma^T A^T(-t) &= (A(-t)\Gamma)^T = (\Gamma A(-t-\tau))^T \\ &= A(-t-\tau)^T \Gamma^T = A_D(t+\tau)\Gamma^T . \end{aligned}$$

□

With the aim of designing an asymptotic observer for system (1) that respects the symmetry properties (2a)-(2d), we consider an observer of the form

$$\begin{array}{l} \dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + K(t)(\hat{y}(t) - y(t))) \\ \hat{y}(t) = C(t)\hat{x}(t) + D(t)u(t) \end{array}$$

and require matrix $A(t) + K(t)C(t)$ to be $(\Gamma,\tau)$-symmetric. This symmetry property is characterized by the following proposition, analogous to proposition 4, whose proof is omitted.

***Proposition** 6:* If the couple $(A,C)$ is $(\Gamma,\Sigma,\tau)$-symmetric, then the couple $(A+KC,C)$ has the same property if

$$\Gamma K(t) = K(t+\tau)\Sigma, \forall t \in \mathbb{R} . \tag{12}$$

Conversely, if (12) holds and $C(\bar{t})$ is full rank for all $t \in \mathbb{R}$, then $(A+KC,C)$ is $(\Gamma,\Sigma,\tau)$-symmetric.

***Proposition** 7:* If $(A,C)$ is $(\Gamma,\Sigma,\tau)$-symmetric and $K$ is a matrix function that satisfies (12), then system $\dot{x} = (A(t) + K(t)C(t))x(t)$ is asymptotically stable if and only if the dual system

$$\dot{x}(t) = (A_D(t) + B_D(t)F_D(t))x(t)$$

is asymptotically stable, where $F_D(t) = K^T(-t)$.

*Proof:* Let $\Phi_D$ be the solution of

$$\left\{ \begin{array}{l} \dot{\Phi}_D(t) = (A_D(t) + B_D(t)F_D(t))\Phi_D(t) \\ \Phi_D(0) = I, \end{array} \right. \tag{13}$$

and $\Phi$ of

$$\left\{ \begin{array}{l} \dot{\Phi}(t) = (A(t) + K(t)C(t))\Phi(t) \\ \Phi(0) = I. \end{array} \right. \tag{14}$$

Let $Z(t) = \Phi_D^T(-t)\Phi(t)$, then $Z(0) = I$ and $\dot{Z}(t) = -\Phi_D^T(-t)(A_D^T(-t) + F_D^T(-t)B_D^T(-t))\Phi(t) + \Phi_D^T(-t)(A(t) + K(t)C(t))\Phi(t) = 0$, since $A_D^T(-t) + F_D^T(-t)B_D^T(-t)) = (A(t) + K(t)C(t))$. This implies that $Z(t) = I$, $\forall t \in \mathbb{R}$ and $\Phi(t) = \Phi_D^{-T}(-t)$, $\forall t \in \mathbb{R}$. By the $k\tau$-periodicity of all matrix functions appearing in (13) and (14) (see remark 1), $\Phi_D^T(k\tau) = \Phi_D^{-T}(-k\tau)$, therefore $\Phi(k\tau) = \Phi_D^T(k\tau)$. which implies that $\Phi(k\tau)$ has all eigenvalues inside the unit circle if and only if $\Phi_D(k\tau)$ has the same property. □

***Remark** 3:* As a consequence of propositions 5 and 7, a stabilizing observer gain matrix $K(t)$ can be obtained by applying the method presented in theorem 1 to the dual system (11).

## IV. STABILIZATION OF PERIODIC TRAJECTORIES OF EQUIVARIANT SYSTEMS

In this section

***Definition** 6:* Consider the nonlinear control system

$$\left\{ \begin{array}{l} \dot{x} = f(x,u) \\ y = h(x,u), \end{array} \right. \tag{15}$$

where $f \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, $h \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^p)$. We say that system (15) is $(\Sigma,\Gamma,\Theta)$-equivariant if there exists three matrices $\Sigma \in \mathbb{R}^{p\times p}$, $\Gamma \in \mathbb{R}^{n\times n}$, $\Theta \in \mathbb{R}^{m\times m}$ such that $\forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m$

$$\begin{array}{l} \Gamma f(x,u) = f(\Gamma x, \Theta u) \\ \Sigma h(x,u) = h(\Gamma x, \Theta u) . \end{array} \tag{16}$$

***Remark** 4:* System (15) is $(\Sigma,\Gamma,\Theta)$-equivariant if and only if the following diagram is commutative

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\Gamma"] & \mathbb{R}^n \\ \mathbb{R}^n \times \mathbb{R}^m \arrow[u, "f"] \arrow[r, "{\Theta,\Gamma}"] \arrow[d, "h"] & \mathbb{R}^n \times \mathbb{R}^n \arrow[u, "f"] \arrow[d, "h"] \\ \mathbb{R}^p \arrow[r, "\Sigma"] & \mathbb{R}^p \end{tikzcd}$$

Let $\tilde{u} \in \mathcal{C}(\mathbb{R},\mathbb{R}^m)$, $\tilde{x} \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^n)$ be such that (15) is verified and there exists $\tau > 0$ such that the following $(\Gamma,\tau)$-spatio-temporal symmetry properties holds, $\forall t \geq 0$,

$$\begin{array}{l} \Gamma \tilde{x}(t) = \tilde{x}(t+\tau) \\ \Theta \tilde{u}(t) = \tilde{u}(t+\tau) . \end{array}$$

Our purpose is to design an observer-based controller to stabilize the system on the reference trajectory $\tilde{x}$:

***Problem** 1:* Design a controller of the form

$$\left\{ \begin{array}{l} \dot{\hat{x}}(t) = g(t,\hat{x}(t),y(t)) \\ u(t) = l(t,\hat{x}(t)) \end{array} \right. \tag{17}$$

such that local asymptotical exact tracking is achieved for the closed-loop system (15)+(17):

$$\lim_{t\to\infty}(\tilde{x}(t) - x(t)) = 0\,,$$

and such that the controller (17) has the following spatio-temporal property:

$$\begin{array}{l} \Gamma g(t,\hat{x}(t),y(t)) = g(t,\Gamma\hat{x}(t),\Theta y(t)) \\ \Sigma l(\hat{x}(t),t) = l(\Sigma\hat{x}(t),t)\,. \end{array}$$

To solve problem 1, we consider the linearization of (15) along the trajectory $\tilde{x}$, $\tilde{u}$:

$$\begin{array}{l} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t)\,, \end{array} \tag{18}$$

where

$$\begin{array}{l} A(t) = \partial_x f(x,u)|_{x=\tilde{x}(t),u=\tilde{u}(t)}\,, \\ B(t) = \partial_u f(x,u)|_{x=\tilde{x}(t),u=\tilde{u}(t)}\,, \\ C(t) = \partial_x h(x,u)|_{x=\tilde{x}(t),u=\tilde{u}(t)}\,, \\ D(t) = \partial_u h(x,u)|_{x=\tilde{x}(t),u=\tilde{u}(t)}\,. \end{array} \tag{19}$$

The following proposition shows that function $A(t)$,$B(t)$,$C(t)$,$D(t)$ define a spatio-temporally symmetric system.

***Proposition** 8:* The quadruple $(A,B,C,D)$ defined in (19) satisfies properties (2a)- (2d), therefore $(A,B,C,D)$ is $(\Gamma,\Theta,\Sigma,\tau)$-symmetric.

*Proof:*

$$\begin{aligned} \Gamma A(t) &= \Gamma\partial_x f(x,u)|_{x=\tilde{x}(t),u=\tilde{u}(t)} \\ &= \partial_x \Gamma f(x,u)|_{x=\tilde{x}(t),u=\tilde{u}(t)} = \partial_x f(\Gamma x,\Theta u)|_{x=\tilde{x}(t),u=\tilde{u}(t)} \\ &= \partial_x f(x,u)|_{x=\Gamma\tilde{x}(t),\Theta u=\tilde{u}(t)}\Gamma \\ &= \partial_x f(x,u)|_{x=\tilde{x}(t+\tau),u=\tilde{u}(t+\tau)}\Gamma = A(t+\tau)\Gamma, \end{aligned}$$

the proof for $B(t),C(t),D(t)$ is analogous. □

The following proposition follows from standard arguments on the local stability of nonlinear systems.

***Proposition** 9:* Consider the linear controller

$$\begin{array}{l} \dot{\hat{x}}(t) = A(t)\hat{x}(t) + K(t)(y(t) - C(t)\hat{x}(t) - D(t)u(t)) \\ u(t) = F(t)\hat{x}(t)\,, \end{array} \tag{20}$$

if the closed-loop system (18)+(20) is exponentially stable and if conditions (8) and (12) are verified, then the controller

$$\begin{aligned} \dot{e}(t) =& A(t)e(t) \\ &+ K(t)(y(t) - \tilde{y}(t) - C(t)e(t) - D(t)(u(t) - \tilde{u}(t)) \\ u(t) =& \tilde{u}(t) + F(t)e(t)\,, \end{aligned} \tag{21}$$

solves problem 1.

## V. APPLICATION TO THE CONTROL OF A CYCLIC FORMATION OF MOBILE ROBOTS

In this section, we consider a cyclic formation of $n$ nonholonomic vehicles described by the following system, for $0 = 1,\ldots,n-1$

$$\left\{ \begin{array}{l} \dot{z}_i = v_i\cos\theta_i \\ \dot{w}_i = v_i\sin\theta_i \\ \dot{\theta}_i = \omega_i\,. \end{array} \right. \tag{22}$$

Vector $(z_i,w_i) \in \mathbb{R}^2$ is the position of the $i$-th robot and $\theta_i \in S^1$ is its direction. The linear and angular velocities $v_i$ and $\omega_i$ are the control inputs. It is assumed that the vehicles have a constant speed of 1 (i.e $v_i = 1$, $i = 0,\ldots,n-1$). Let $x_i = (z_i,w_i,\theta_i)$ be the state of the $i$-th robot and $x = (x_1,x_2,\ldots,x_n)$ the state of the formation. The output function is given by the couple $y(x)$, $q(x) = (q_1(x),q_2(x),\ldots,q_n(x))$ where

$$y(x) = \frac{1}{n}\sum_{i=0}^{n-1}\left(\begin{array}{c} x_i \\ y_i \\ \theta_i \end{array}\right)$$

is the average of the positions and the angles of the robots and

$$q_i(x) = \left\| \left(\begin{array}{c} z_i \\ w_i \end{array}\right) - \left(\begin{array}{c} z_{i+1} \\ w_{i+1} \end{array}\right) \right\|,$$

is the distance between the $i$-th and the $i+1$-th robot, where the indexes are computed modulo $n$ (i.e. if $i = n-1$, $i+1 = 0$).

Let $P$ be the cyclic permutation matrix, defined as

$$P = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix},$$

and define $\Gamma = P\otimes I_3$, $\Sigma = P$, $\Theta =$ blkdiag $(I_3,P)$, where $I_3$ denotes the 3 by 3 identity matrix and the term "blkdiag " denotes a block diagonal matrix.

Then system (22) with the output $y,q$ is $(\Sigma,\Gamma,\Theta)$-equivariant. Let $\gamma \in \mathcal{C}^2(\mathbb{R},\mathbb{R}^2)$ be a $L$-periodic function that represents a closed curve in $\mathbb{R}^2$ such that $\|\dot{\gamma}(t)\| = 1$, $\forall t \in \mathbb{R}$. Set $x_r : \mathbb{R} \to \mathbb{R}^2 \times S^1$ such that $x_r(t) = (\gamma(t), \arg\dot{\gamma}(t))$ and $\omega_r : \mathbb{R} \to \mathbb{R}$ such that $\omega_r(t) = \frac{d}{dt}\arg\dot{\gamma}(t)$. Set $\tilde{x}(t) = (x_r(t), x_r(t+L/n),\ldots,\tilde{x}_r(t+L\frac{n-1}{n}))$, $\tilde{\omega}(t) = (\omega_r(t),\omega_r(t+L/n),\ldots,\omega_r(t+L\frac{n-1}{n}))$. Then $\tilde{x}$, with control $\tilde{\omega}$ is a solution of (22). By construction $\Gamma\tilde{x}(t) = \tilde{x}(t+\tau)\Gamma$ and $\Sigma\tilde{\omega}(t) = \tilde{\omega}(t+\tau)\Sigma$, therefore, by proposition 8, the linearized system (18) is $(\Gamma,\Sigma,\Theta,L/n)$-symmetric. For instance, for $n = 4$, if $\gamma$ is the image of the curve

$$\gamma(s) = \left(\begin{array}{c} 3\cos(s/3) \\ \sin(s) \end{array}\right),$$

the initial configuration of the vehicles $\tilde{x}(0)$ is represented in figure 1.

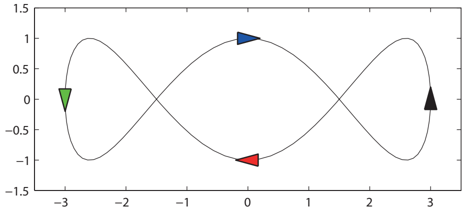


Fig. 1. The curve $\gamma$ and the initial reference configuration $\tilde{x}(0)$.

By proposition 9, we can locally stabilize trajectory $\tilde{x}$ with controller (21) if the matrix functions $F$, $K$ stabilize the linearized system (18)+ (20). To find these functions, we use the method presented in section II. In particular $K$ and $F$ are obtained using the Riccati equation (10), with matrices $Q$ and $R$ chosen as the identity. The vehicles' trajectories, together with the the norm of the tracking error and the observer error are reported in Figures 2, 3(a), 3(b).

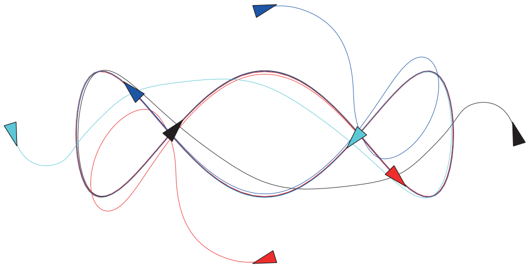
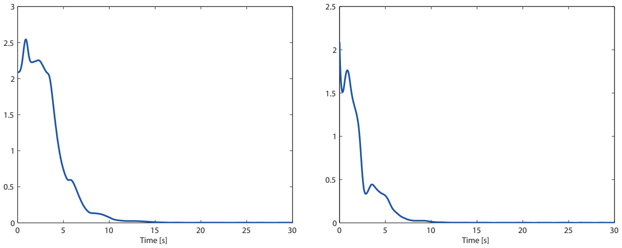


Fig. 2. The closed-loop trajectories $x(t)$



(a) Tracking error norm

(b) Observer error norm

Fig. 3. Plot (a): the tracking error norm $\|x(t)-\tilde{x}(t)\|$. Plot (b): the observer error norm $\|x(t)-\tilde{x}(t)-e(t)\|$ for example 1.

## VI. CONCLUSIONS

In this paper, we have studied a class of controlled linear systems characterized by spatio-temporal symmetry. We have characterized its stabilizability and detectability properties and have presented a method for feedback synthesis based on a hybrid Riccati equation. We have presented a design technique to locally stabilize spatio-temporally symmetric orbits of equivariant system, that has applications in the control of identical interconnected systems. As an example, we have presented the stabilization of a formation of unicycle robots in cyclic pursuit.

## APPENDIX

In this appendix, we report some technical results that have been used in the proofs of the paper. We denote by $\mathcal{C}_p(\mathbb{R},\mathbb{R}^n)$ the set of piecewise continuous maps defined on $\mathbb{R}$ with values in $\mathbb{R}^n$ and by $\mathcal{C}_+(\mathbb{R},\mathbb{R}^n)$ the set of right continuous maps defined on $\mathbb{R}$ with values in $\mathbb{R}^n$.

***Proposition** 10:* If $A$ is $(\Gamma,\tau)$-symmetric and $G \in \mathcal{C}_p(\mathbb{R},\mathbb{R}^n)$, then

a) If $x \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^n)$ is a solution of

$$\dot{x}(t) = A(t)x(t) + G(t), \forall t \in \mathbb{R} \tag{23}$$

then the map $\xi(t) = \Gamma^{-\lfloor \frac{t}{\tau} \rfloor}x(t)$ is such that $\xi \in \mathcal{C}^1(\mathbb{R}\backslash\tau\mathbb{Z},\mathbb{R}^n) \cap \mathcal{C}_{\tau+}(\mathbb{R},\mathbb{R}^n)$ and

$$\begin{array}{l} \dot{\xi}(t) = A(\pi(t))\xi(t) + \Gamma^{-\lfloor \frac{t}{\tau} \rfloor}G(t), \forall t \in \mathbb{R}\backslash\tau\mathbb{Z} \\ \xi(t) = \lim_{s\to t^-}\Gamma^{-1}\xi(s), \text{if } t \in \tau\mathbb{Z}\,. \end{array} \tag{24}$$

b) Conversely if $\xi \in \mathcal{C}^1(\mathbb{R}\backslash\tau\mathbb{Z},\mathbb{R}^n) \cap \mathcal{C}_+(\mathbb{R},\mathbb{R}^n)$ verifies (24) then

$$x(t) = \Gamma^{\lfloor \frac{t}{\tau} \rfloor}\xi(t), \forall \mathbb{R}, \tag{25}$$

is such that $x \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^n)$ and is a solution of system (23).

*Proof:* a) Since $\lfloor \frac{t}{\tau} \rfloor$ is locally constant on the open subsets of $\mathbb{R}\backslash\tau\mathbb{Z}$, by property (4a) it follows that, $\forall t \in \mathbb{R}\backslash\tau\mathbb{Z}$

$$\begin{aligned} \dot{\xi}(t) &= \Gamma^{-\lfloor \frac{t}{\tau} \rfloor}\dot{x}(t) = \Gamma^{-\lfloor \frac{t}{\tau} \rfloor}A(t)x(t) + \Gamma^{-\lfloor \frac{t}{\tau} \rfloor}G(t) \\ &= A(t - \lfloor \frac{t}{\tau} \rfloor \tau)\Gamma^{-\lfloor \frac{t}{\tau} \rfloor}x(t) + \Gamma^{-\lfloor \frac{t}{\tau} \rfloor}G(t) \\ &= A(\pi(t))\xi(t) + \Gamma^{-\lfloor \frac{t}{\tau} \rfloor}G(t)\,. \end{aligned}$$

Moreover, if $t \in \tau\mathbb{Z}$, there exists $i \in \mathbb{Z}$ such that $t = i\tau$, therefore

$$\xi(t) = \Gamma^{-i}x(i\tau) = \Gamma^{-i}\lim_{s\to i\tau^{\pm}} x(s)\,,$$

which implies that

$$\xi(t) = \lim_{s\to i\tau^+}\Gamma^{-i}x(s) = \lim_{s\to t^+}\xi(s)\,,$$

that is $\xi$ is right continuous and

$$\xi(t) = \Gamma^{-1}\lim_{s\to t^-}\Gamma^{-(i-1)}x(s) = \Gamma^{-1}\lim_{s\to t^-}\xi(s)\,.$$

b) Conversely, since $\lfloor \frac{t}{\tau} \rfloor$ is locally constant on $\mathbb{R}\backslash\tau\mathbb{Z}$, $x(t)$ given by (25) is $\mathcal{C}^1$ on $\mathbb{R}\backslash\tau\mathbb{Z}$ and $\forall t \in \mathbb{R}\backslash\tau\mathbb{Z}$

$$\dot{x}(t) = \Gamma^{\lfloor \frac{t}{\tau} \rfloor}\dot{\xi}(t) = \Gamma^{\lfloor \frac{t}{\tau} \rfloor}A(\pi(t))\xi(t) + \Gamma^{\lfloor \frac{t}{\tau} \rfloor}G(t)\,.$$

From (4a) it follows that

$$\Gamma^{\lfloor \frac{t}{\tau} \rfloor}A(\pi(t)) = \Gamma^{\lfloor \frac{t}{\tau} \rfloor}A(t - \lfloor \frac{t}{\tau} \rfloor \tau) = A(t)\Gamma^{\lfloor \frac{t}{\tau} \rfloor}\,.$$

Therefore, $\forall t \in \mathbb{R}\backslash\tau\mathbb{Z}$,

$$\dot{x}(t) = A(t)\Gamma^{\lfloor \frac{t}{\tau} \rfloor}\xi(t) + G(t) = A(t)x(t) + G(t)\,.$$

Moreover, $x \in \mathcal{C}(\mathbb{R},\mathbb{R}^n)$, in fact, if $t \in \tau\mathbb{Z}$, there exists $i \in \mathbb{Z}$ such that $t = i\tau$ and

$$\begin{aligned} \lim_{s\to t^-} x(s) &= \lim_{s\to i\tau^-}\Gamma^{i-1}\xi(s) = \Gamma^{i-1}\lim_{s\to i\tau^-}\xi(s) \\ &= \Gamma^{i}\lim_{s\to i\tau^-}\Gamma^{-1}\xi(s) = \Gamma^{i}\xi(i\tau) = x(t)\,, \end{aligned}$$

$$\begin{aligned} \lim_{s\to t^+} x(s) &= \lim_{s\to i\tau^+}\Gamma^{i}\xi(s) = \Gamma^{i}\lim_{s\to i\tau^+}\xi(s) \\ &= \Gamma^{i}\xi(i\tau) = x(t)\,, \end{aligned}$$

being $\xi$ right continuous. Therefore $x \in \mathcal{C}(\mathbb{R},\mathbb{R}^n) \cap \mathcal{C}^1(\mathbb{R}\backslash\tau\mathbb{Z},\mathbb{R}^n)$ which verifies the system on $\mathbb{R}\backslash\tau\mathbb{Z}$. Being $A$ and $G$ continuous, $x$ belongs to $\mathcal{C}^1(\mathbb{R},\mathbb{R}^n)$ satisfies system (23) on $\mathbb{R}$. ☐

***Proposition** 11:* Suppose that $A$ is $(\Gamma,\tau)$-symmetric, then the matrix solution of

$$\left\{ \begin{array}{l} \dot{\Psi}(t) = A(\pi(t))\Psi(t), \text{ if } t \in \mathbb{R}\backslash\tau\mathbb{Z} \\ \Psi(t) = \lim_{s\to t^-}\Gamma^{-1}\Psi(s), \text{ if } t \in \tau\mathbb{Z} \\ \Psi(0) = I\,, \end{array} \right. \tag{26}$$

is given by

$$\Psi(t) = \Phi(\pi(t),0)(\Gamma^{-1}\Phi(\tau,0))^{\lfloor \frac{t}{\tau} \rfloor}, \forall t \in \mathbb{R}\,. \tag{27}$$

*Proof:* Clearly $\Phi(0) = \Phi(0,0)(\Gamma^{-1}\Phi(\tau,0)^{\lfloor 0 \rfloor} = I$ and if $t \in \tau\mathbb{Z}$, there exists $i \in \mathbb{Z}$ such that $t = i\tau$, then

$$\begin{aligned} \Psi(t) &= \Phi(0,0)(\Gamma^{-1}\Phi(\tau,0))^{i} \\ &= \Gamma^{-1} \lim_{s \to i\tau^-} \Phi(s,0)(\Gamma^{-1}\Phi(\tau,0))^{i-1} \\ &= \Gamma^{-1} \lim_{s \to i\tau^-} \Phi(\pi(s),0)(\Gamma^{-1}\Phi(\tau,0))^{i-1} = \lim_{s \to i\tau} \Gamma^{-1}\Psi(s) . \end{aligned}$$

Moreover, being $\lfloor \frac{t}{\tau} \rfloor$ locally constant on $\mathbb{R}\backslash\tau\mathbb{Z}$, we have that, $\forall t \in \mathbb{R}\backslash\tau\mathbb{Z}$

$$\begin{aligned} \dot{\Psi}(t) &= \frac{d}{dt}\Phi(\pi(t))(\Gamma^{-1}\Phi(\tau,0))^{\lfloor \frac{t}{\tau} \rfloor} \\ &= A(\pi(t))\Phi(\pi(t))(\Gamma^{-1}\Phi(\tau,0))^{\lfloor \frac{t}{\tau} \rfloor} = A(\pi(t))\Psi(t) . \end{aligned}$$

The uniqueness is proved by induction on intervals $[i\tau, \tau + i\tau]$ □

***Remark** 5:* By defining $\Psi(\tau,s) = \Psi(t)\Psi^{-1}(s)$, $\forall t,s \in \mathbb{R}$ it follows that

$$\Psi(t,s) = \Psi(\pi(t),0)(\Gamma^{-1}\Phi(\tau,0))^{\lfloor \frac{t}{\tau} \rfloor - \lfloor \frac{s}{\tau} \rfloor}\Phi(0,\pi(s)) .$$

In this way, the solution of (24) with initial condition $\xi(\bar{t}) = \bar{\xi}$ is given by

$$\xi(t) = \Psi(t,\bar{t})\bar{\xi} + \int_{\bar{t}}^{t} \Psi(t,s)\Gamma^{-\lfloor \frac{t}{\tau} \rfloor}G(s)ds. \tag{28}$$

***Proposition** 12:* System

$$\left\{ \begin{array}{l} \dot{\xi}(t) = A(\pi(t))\xi(t), \text{ if } t \in \mathbb{R}\backslash\tau\mathbb{Z} \\ \xi(t) = \lim_{s \to t^-} \Gamma^{-1}\xi(t), \forall t \in \tau\mathbb{Z} \\ \Psi(0) = I, \end{array} \right. \tag{29}$$

is asymptotically stable if and only if all the eigenvalues $\lambda$ of $\Psi(\tau,0) = \Gamma^{-1}\Phi(\tau,0)$ are such that $|\lambda| < 1$.

*Proof:* Assume that all eigenvalues $\lambda$ of $\Psi(\tau,0)$ are such that $|\lambda| < 1$. Since $\xi(i\tau) = \Psi(\tau,0)^{i}\bar{\xi}$, by stability properties of discrete-time systems it follows that

$$\lim_{i \to +\infty} \|\xi(i\tau)\| = 0 .$$

Moreover

$$\|\xi(t)\| = \|\Psi(t,0)\bar{\xi}\| \leq \|\Phi(\pi(t))\| \|\xi(\lfloor \frac{t}{\tau} \rfloor \tau)\|, \forall t \in \mathbb{R} ,$$

being $\|\Psi(\pi(t))\|$ bounded (since $\|A(\pi(t)\|$ is bounded on $\mathbb{R}$) it follows that

$$\lim_{t \to \infty} \|\xi(t)\| = 0 .$$

Conversely, if there exists an eigenvalue $\lambda$ of $\Psi(\tau,0)$ with $|\lambda| > 1$ with eigenvector $\bar{\xi}$, it follows that

$$\|\xi(i\tau)\| \geq \|\bar{\xi}\|, \forall i \geq 0 ,$$

which implies that (29) is not asymptotically stable. □

***Proposition** 13:* There exists a $\tau$-periodic feedback $F(t)$ such that the system

$$\left\{ \begin{array}{l} \dot{\xi}(t) = A(\pi(t))\xi(t) + B(\pi(t))v(t), \text{ if } t \in \mathbb{R}\backslash\tau\mathbb{Z} \\ \xi(t) = \lim_{s \to t^-} \Gamma^{-1}\xi(s), \text{ if } t \in \tau\mathbb{Z} \end{array} \right. \tag{30}$$

is asymptotically stable if and only if any $\lambda$ with the property that there exists $\eta \in \mathcal{C}^{n}$ such that

$$\begin{array}{l} \eta^{T}\Gamma^{-1}\Phi(\tau,0) = \lambda\eta^{T} \\ \eta^{T}(\Phi(0,\tau)\Gamma)^{i}\Phi(0,s)B(s) = 0, \forall s \in [0,\tau), \forall i = 0,\dots,n-1 , \end{array} \tag{31}$$

satisfies condition $|\lambda| < 1$. Moreover the stabilizing feedback is given by

$$F(t) = -B(\pi(t))R^{1}B^{T}(\pi(t))S(t)\xi(t) ,$$

where $S$ is a $\tau$-periodic solution of (10), and $R$, $Q$ are assigned positive definite matrices.

*Proof:* (necessity) Suppose that there exists an uncontrollable eigenvalue $\lambda$ with $|\lambda| \geq 1$. Then there exists $\eta$ that verifies (9). Let $\xi(t)$ be the solution of the following system

$$\left\{ \begin{array}{l} \dot{\xi}(t) = (A(\pi(t)) + B(\pi(t))F(\pi(t)))\xi(t), \forall t \in \mathbb{R}\backslash\tau\mathbb{Z} \\ \xi(t) = \lim_{s \to t^-} \Gamma^{-1}\xi(s), \, t \in \tau\mathbb{Z} \\ \xi(0) = \eta , \end{array} \right.$$

which is given by (see (28))

$$\xi(t) = \Psi(t,0)\left[\eta + \int_{0}^{t} \Psi(0,s)B(\pi(s))F(\pi(s))ds\right], \forall t \in \mathbb{R} .$$

By the definition of $\Psi(t,s)$ and the Cayley-Hamilton Theorem, conditions (31) are equivalent to

$$\eta^{T}\Psi(\tau,0) = \lambda\eta^{T}, \eta^{T}\Psi(0,s)B(\pi(s)) = 0, \forall s \in \mathbb{R} .$$

Therefore, left-multiplying by $\eta^{T}$, it follows that

$$\eta^{T}\xi(t) = \eta^{T}\Psi(t,0)\eta, \forall t \in \mathbb{R},$$

which implies, if $t = i\tau$, $i \in \mathbb{Z}$, that

$$\eta^{T}\xi(i\tau) = \eta^{T}(\Gamma^{-1}\Phi(\tau,0))^{i}\eta = \lambda^{i}\|\eta\|^{2}, \forall i \in \mathbb{Z} .$$

Therefore, since $|\lambda| > 1$,

$$|\eta^{T}\xi(i\tau)| \geq \|\eta\| > 0, \forall i \in \mathbb{Z} ,$$

which implies that (30) is not asymptotically stable.

(Sufficiency) This part of the proof is more involved and is omitted due to space limitations. It will be presented in a future journal paper.

□

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