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Optimal placement of actuators and sensors for control of nonequilibrium dynamics

Subhrajit Sinha[1] and Umesh Vaidya[2] and Rajeev Rajaram[3]

Abstract— **In this paper, we provide a systematic convex programming-based approach for the optimal locations of static actuators and sensors for the control of nonequilibrium dynamics. The problem is motivated with regard to its application for control of nonequilibrium dynamics in the form of temperature in building systems and control of oil spill in oceanographic flow. The controlled evolution of a passive scalar field, modeling the temperature distribution or the density of oil dispersant, is governed by the linear advection partial differential equation (PDE) with spatially located actuators and sensors. Spatial locations of actuators and sensors are optimized to maximize the controllability and observability of the linear advection PDE. Linear transfer Perron-Frobenius and Koopman operators, associated with the advective velocity field, are used to provide analytical characterization for the controllable and observable spaces of the advection PDE. Set-oriented numerical methods are used for the finite dimensional approximation of the transfer operators and in the formulation of the optimization problem. Application of the framework is demonstrated for the optimal placement of actuators for the release of dispersant for oil spill control.**

I. Introduction

We study the problem of optimal placement of actuators and sensors for the control of nonequilibrium dynamics. The problem is motivated with regard to the control of nonequilibrium dynamics as seen in systems involving fluid flow. For example, in a building system application, the objective is to determine optimal locations of actuators in the form of vents/ducts for the control of temperature [1], [2]. For the control of contaminants from the oil spill in oceanographic flows, the goal is to determine the optimal location for the release of dispersant. The distribution of temperature in the room or the dispersant on the ocean surface is assumed to be modeled by a passive scalar density. The passive scalar is advected by the fluid flow velocity field. Under some simplifying assumptions of negligible buoyancy and diffusion, the evolution of the passive scalar density is modeled using the linear advection partial differential equation (PDE). For the purpose of control and observation, we assume that the actuators and sensors are spatially located.

The problem of control of temperature or oil spill is a challenging one because of the nonequilibrium nature of dynamics that are involved. The fluid flow velocity field involved in these applications are quite complex with dynamics consisting of multiple equilibrium points, periodic orbits, limit cycles, and chaotic attractor [3], [4]. The proposed advection PDE-based approach provides for a linear, albeit infinite dimensional, framework for the analysis of such complex dynamics. There is an extensive amount of literature on actuator and sensor placement for linear PDEs with [5] providing an excellent review of the results. Most of the methods for the optimization of sensor and actuator location involves finite dimensional approximation of the infinite dimensional system. In [6], sensor and actuator placement for the diffusive and heat type partial differential equations are discussed. The placement problem for flexible structures is studied in [7]. In [1], we derived an analytical expression for the finite time controllability and observability gramain for advection PDE. The selection criteria for the optimal location of actuators and sensors was proposed based on the maximization of gramians. In this paper, we take our approach from [1] one step forward. Specifically, the following are the main contributions of this paper. We provide a systematic approach for the computation of the infinite time controllability and observability regions for linear advection PDE. These regions are expressed in terms of the Perron Frobenius semigroup associated with the advection PDE. Set-oriented numerical methods are proposed for the finite dimensional approximation of the P-F semigroup and the controllability and observability gramians [8], [9]. The finite dimensional approximation of the P-F semigroup is used for the formulation of an optimization problem to determine the optimal location of actuators and sensors. Convex relaxation of the optimization problem is proposed. The application of the developed framework is demonstrated on the optimal location of actuators for the control of oil spill in a Double Gyre flow field.

The organization of the paper is as follows. In section II, we formulate the problem. Some of the preliminaries on PDE are provided in section III. The main results of this paper are presented in section IV followed by discussion on set-oriented numerical methods for finite dimensional approximation in section V. The convex optimization formulation for the optimal locations of sensors and actuators is proposed in section VI. Simulation results on proposed applications and conclusions are presented in section VII followed by conclusions in section VIII.

II. Problem formulation

We consider the problem of optimal locations for static actuators and sensors for the control of nonequilibrium dynamics.

Financial support from the Natinal Science Foundation award ECCS-1002053 is gratefully acknowledged. S.. Sinha and U. Vaidya are with the Department of Electrical & Computer Engineering, Iowa State University, Ames, IA 50011. R. Rajaram is with the Department of Mathematics, Kent State University, Ashtabula, OH 44004. Email : [1]ssinha@iastate.edu, [2]ugvaidya@iastate.edu, [3]rrajaram@kent.edu

The motivating applications for the nonequilibrium dynamics that we consider includes control of oil spill in fluid flow, control of temperature or detection of contaminant in building systems, and environmental monitoring in oceanographic flows. In all these applications, the system state can be conveniently modeled using a scalar density. For example, in the case of oil spill control, the spatial distribution of oil on the ocean surface can be modeled using a scalar density. We assume that the scalar density is advected by vector field described by following differential equation

$$\dot{x} = f(x) \tag{1}$$

where $x \in X \subset \mathbb{R}^n$ is the state. The state space X is assumed to be compact. Typically $n = 2, 3$ in all our proposed applications. The evolution of the scalar density $\rho(x,t)$ is governed by the following linear advection partial differential equation.

$$\frac{\partial \rho(x,t)}{\partial t} = \nabla \cdot (f(x)\rho(x,t)), \tag{2}$$

where $\rho(x,t) \in L^2(X)$. Let $A_k \subset X$ and $S_\ell \subset X$ be the locations of actuators and senors respectively for $k = 1, \ldots, p$ and $\ell = 1, \ldots, q$.

Assumption 1: We assume that, $0 < m(A_k) << m(X)$ and $0 < m(S_\ell) << m(X)$, for $k = 1, \ldots, p$ and $\ell = 1, \ldots, q$, where m is the Lebesgue measure.

Remark 2: The assumption of positive Lebesgue measure is technical in nature and is necessary for the application of linear partial differential equation theory [10]. The assumption that the Lebesgue measure of the actuator and sensor is much smaller than the measure of the state space X is reasonable and reflects the physical constraints of the problem.

The controlled evolution of the linear advection partial differential equation is written as follows:

$$\frac{\partial \rho(x,t)}{\partial t} = \nabla \cdot (f(x)\rho(x,t)) + \sum_{k=1}^{p} \chi_{A_k}(x) u_k(x,t), \tag{3}$$

where $\chi_{A_k}(x)$ is the indicator function for the set A_k and $u_k(x,t)$ is the control input. The control formulation of the advection PDE in Eq. (3) can be motivated as follows. Consider the application of oil spill control, where the control in the form of dispersant is released from multiple locations, say $A_k \subset X$, in the ocean. If the fluid flow field is modeled by the vector field $\dot{x} = f(x)$, then the evolution of the dispersant, $\rho(x,t)$, is modeled by Eq. (3). Similarly, the controlled evolution of temperature density in the building system application will be modeled using Eq. (3). For the building system application, the control input will consist of hot/cold air released from various vent locations in the room. There are a number of simplifying assumptions that are made in the derivation of control Eq. (3). In particular, following assumptions are made

1) The contribution of the diffusion term in the evolution of scalar density is much smaller than the advection term. This assumption is justified for the application under consideration where the diffusion term is an order of magnitude smaller than the advection term [1].
2) The advection velocity field, $f(x)$, is assumed to be steady i.e., time independent. If the velocity field is time-dependent, then a time independent velocity field can be obtained by taking time average over suitable time period.

Similarly, the system equation for the sensor placement problem are as follows:

$$\begin{aligned} \frac{\partial \rho(x,t)}{\partial t} &= \nabla \cdot (f(x)\rho(x,t)) \\ y_\ell(x,t) &= \chi_{S_\ell}(x)\rho(x,t), \quad \ell = 1, \ldots, q \end{aligned} \tag{4}$$

where $y_\ell(x,t)$ is the output of the ℓ^{th} sensor. Here again it is assumed that the sensor can measure the density in the small region, S_ℓ, of the state space X.

Our objective is to provide a systematic approach for the optimal locations of actuators, A_k, and sensors, S_ℓ, while minimizing a certain cost function. The cost function that we use for optimization is motivated from the controllability and observability analysis of the linear advection partial differential equation (PDE) as seen in [1]. In the following section, we review some of the preliminaries on linear PDE and provide a brief overview of key results from [1].

III. Preliminaries

We provide some preliminaries on semigroup theory of partial differential equations. Consider the following ordinary differential equation (ODE):

$$\dot{x} = f(x), \quad x(0) = x_0, \tag{5}$$

where $x \in X \subset \mathbb{R}^N$ a compact set. We denote by $\phi_t(x)$ the solution of ODE (5) starting from the initial condition x. ODE (5) is used to define two linear infinitesimal operators, $\mathcal{A}_K : L^2(X) \to L^2(X)$ and $\mathcal{A}_{PF} : L^2(X) \to L^2(X)$ defined as follows:

$$\mathcal{A}_K \rho = f \cdot \nabla \rho, \quad \mathcal{A}_{PF}\rho = -\nabla \cdot (f\rho).$$

The domains of the above operators are given as follows:

$$D(\mathcal{A}_K) = \{\rho \in H^1(X) : \rho|_{\Gamma_o} = 0\},$$

$$D(\mathcal{A}_{PF}) = \{\rho \in H^1(X) : \rho|_{\Gamma_i} = 0\},$$

where Γ_o and Γ_i are the outflow and inflow portions of the boundary ∂X defined as follows:

$$\Gamma_o = \{x \in \partial X : f \cdot \eta > 0\}, \quad \Gamma_i = \{x \in \partial X : f \cdot \eta < 0\},$$

where η is the outward normal to the boundary ∂X. The semigroups corresponding to the $\mathcal{A}_K$ and $\mathcal{A}_{PF}$ are called as Koopman ($\mathbb{U}_t$) and Perron-Frobenius ($\mathbb{P}_t$) operators respectively. These operators are defined as follows:

$$\mathbb{U}_t : L^2(X) \to L^2(X), \;\; (\mathbb{U}_t\rho)(x) = \rho(\phi_t(x)),$$

$$\mathbb{P}_t : L^2(X) \to L^2(X), \;\; (\mathbb{P}_t\rho)(x) = \rho(\phi_{-t}(x)) \left| \frac{\partial \phi_t(x)}{\partial x} \right|^{-1},$$

where $|\cdot|$ denotes the determinant. These semigroups can be shown to satisfy the following partial differential equations [11]:

$$\frac{\partial \rho}{\partial t} - \mathcal{A}_K \rho = 0, \rho|_{\Gamma_o} = 0; \quad \frac{\partial \rho}{\partial t} - \mathcal{A}_{PF} \rho = 0, \rho|_{\Gamma_i} = 0.$$

The Koopman and Perron-Frobenius semigroup operators and their infinitesimal generators are adjoint to each other i.e.,

$$\int_X (\mathbb{P}_t \rho_1)(x) \rho_2(x) dx = \int_X \rho_1(x) (\mathbb{U}_t \rho_2)(x) dx,$$

for all $\rho_1, \rho_2 \in L^2(X)$.

IV. Controllability and observation of advection PDE

A. Controllability of advection PDE

We write the controlled advection PDE as follows:

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla \cdot (f(x)\rho) = \sum_{k=1}^{p} \chi_{A_k}(x) u_k(x,t); \\ &\rho|_{\Gamma_i} = 0; \;\; \rho(x,0) = \rho_0(x). \end{aligned} \tag{6}$$

The solution formula for PDE (6) is written as follows:

$$\rho(x,t) = \mathbb{P}_t \rho_0(x) + \sum_{k=1}^{p} \int_0^t \mathbb{P}_{t-s}(\chi_{A_k}(x) u_k(x,s)) ds.$$

We have the following theorem on the controllability property of the PDE (6).

Theorem 3: Let $\mathcal{R}^\tau = \cup_{k=1}^{p} \cup_{t=0}^{\tau} \phi_t(A_k)$. The PDE (6) is exactly controllable in a given time $\tau > 0$ for all initial and terminal states in the space $L^2(\mathcal{R}^\tau)$ i.e. given initial and terminal states $\bar{\rho}_0(x)$ and $\bar{\rho}_\tau(x)$ in $\mathcal{R}^\tau$, there exists a control $u(x,t) \in L^2([0,\tau] : L^2(B))$ such that $\rho(x,0) = \bar{\rho}_0(x)$, and $\rho(x,\tau) = \bar{\rho}_\tau(x)$, where $\rho(x,t)$ is the solution of (6).

Refer to [1] for the proof. We also have following theorem connecting the controllability set and P-F semigroup.

Theorem 4: The set over which the PDE (6) is controllable, $\mathcal{R}^\tau = \cup_{k=1}^{p} \cup_{t=0}^{\tau} \phi_t(A_k)$, is equal to the support of following function

$$\rho_\tau(x) = \left(\int_0^\tau \mathbb{P}_t \sum_{k=1}^{p} \chi_{A_k}(x) dt \right). \tag{7}$$

We refer the readers to [1] for the proof.

B. Observability of advection PDE

For the observability problem, we consider the advection partial differential equation with output measurement as follows:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nabla \cdot (f\rho), \quad \rho|_{\Gamma_i} = 0, \quad \rho(x,0) = \rho_0(x) \\ y_l(x,t) &= \chi_{S_l}(x) \rho(x,t), l = 1, ..., q, \end{aligned} \tag{8}$$

where $y_\ell(x,t)$ is the output of ℓ^{th} sensor. A counterpart of the controllability theorem on the observability of linear advection PDE (8) holds true. We refer the readers to [1] for the theorem and its proof.

C. Lyaunov density for infinite time controllability and observability

To extend the definition of finite time controllability and observability gramian from [1] over infinite time, we need to introduce *Lyapunov density*. Lyapunov density and Lyapunov measure were introduced in [12], [13] for almost everywhere stability verification of continuous and discrete time dynamical systems respectively. In this section, we provide an alternate interpretation of Lyapunov density for the characterization of the controllable and observable spaces over an infinite time interval. First, we provide brief introduction to Lyapunov density.

Definition 5 (Almost everywhere uniform stability): For the continuous time dynamical system (5), the equilibrium point $x = 0$ is said to be almost everywhere uniformly stable if for any given ϵ, there exists $T(\epsilon)$ such that

$$\int_T^\infty m(A_t) dt < \epsilon, \quad A_t = \{x \in X : \phi_t(x) \in A\}$$

for all measurable sets $A \subset X \setminus B_\delta$, where B_δ is the δ neighborhood of $x = 0$ and m is Lebesgue measure.

Almost everywhere uniform stability can be verified using Lyapunov density. In particular, the existence of Lyapunov density is shown to be both necessary and sufficient condition for almost everywhere uniform stability. For more details on Lyapunov density, we refer the interested readers to [12]. In the context of this paper, we are particularly interested in the following result from [12]. For almost everywhere stable system, the Lyapunov density can be computed as follows:

$$\rho(x) = \int_0^\infty \mathbb{P}_t^1 \rho_0(x) dt \tag{9}$$

where $\mathbb{P}_t^1$ is the restriction of P-F semigroup outside the equilibrium point $x = 0$ and $\rho_0(x) > 0$ is a positive function in $L^2(X)$. Roughly speaking, almost everywhere uniform stability requires that almost every initial condition in X converge into the equilibrium point $x = 0$ fast enough so that a certain integrability condition is satisfied i.e., Lyapunov density exists. We notice that the controllability over infinite time interval is well defined only when the advective vector field, $\dot{x} = f(x)$, is almost everywhere uniformly stable. However, for the fluid flow application that we consider in this paper, the vector field will not be almost everywhere uniformly stable. Hence, we need an alternate approach for the computation of infinite time controllability set defined as follows:

$$\mathcal{R}^\infty = \cup_{k=1}^{p} \cup_{t=0}^{\infty} \phi_t(A_k). \tag{10}$$

In the following we provide an alternate approach for the computation of infinite time controllability without making the stability assumption on the vector field.

Since $\mathbb{P}_t$ is a strongly continuous semigroup, semigroup theory guarantees the existence of $\omega > 0$ such that

$$\parallel \mathbb{P}_t \parallel_2 \leq M e^{\omega t}.$$

Hence, one can always choose a β such that $\beta > \omega$ and the following function over infinite time interval is well defined

$$\rho_{\infty,\beta}(x) = \int_0^\infty e^{-\beta t}\mathbb{P}_t\left[\sum_{k=1}^{p}\chi_{A_k}(x)\right]dt. \quad (11)$$

We have following theorem relating the infinite and finite time controllability sets.

Theorem 6: We have following relationship among the various controllability sets

$$\mathcal{R}^\tau \subseteq \mathcal{R}^\infty = \mathcal{R}^{\infty,\beta}, \quad (12)$$

where, $\mathcal{R}^{\infty,\beta}$ is the support of $\rho_{\infty,\beta}$ from Eq. (11) and $\mathcal{R}^\tau$, $\mathcal{R}^\infty$ are defined as follows:

$$\mathcal{R}^\tau = \cup_{k=1}^{p}\cup_{t=0}^{\tau}\phi_t(A_k), \quad \mathcal{R}^\infty = \cup_{k=1}^{p}\cup_{t=0}^{\infty}\phi_t(A_k)$$

Proof: It was shown in [12] that $\mathcal{R}^\tau$ is the support of ρ_τ, and also the finite time controllability set. The fact that $\mathcal{R}^\tau \subseteq \mathcal{R}^\infty$ follows by definition of the two sets $\mathcal{R}^\tau$ and $\mathcal{R}^\infty$. The only thing left is to show that $\mathcal{R}^\infty = \mathcal{R}^{\infty,\beta}$. Since $m(A_k) > 0\ \forall k = 1, ..., p$, and A_k evolves into $\phi_t(A_k)$ in time t, for every $x \in \mathcal{R}^\infty$, there exist times $0 \le t_1^k(x) < t_2^k(x) \le \tau$ such that $x \in \phi_t(A_k)\ \forall t \in [t_1^k(x), t_2^k(x)]$. Hence, by the positivity of $\mathbb{P}_t$ we have that $e^{-\beta t}\mathbb{P}_t(\chi_{A_k}) > 0\ \forall t \in [t_1^k(x), t_2^k(x)] \subseteq [0,\tau]$. Hence we have the following:

$$\int_0^\infty e^{-\beta t}\mathbb{P}_t\chi_{A_k}(x)dt \ge \int_{t_1^k(x)}^{t_2^k(x)} e^{-\beta t}\mathbb{P}_t\chi_{A_k}(x)dt > 0$$
$$\forall x \in \mathcal{R}^\infty, k = 1, ..., p.$$

Hence, we have that $\cup_{t=0}^{\infty}\phi_t(A_k)$ is the support of $\int_0^\infty e^{-\beta t}\mathbb{P}_t\chi_{A_k}(x)dt$ for the case of a single actuator A_k. If we have p such actuators, then $\int_0^\infty e^{-\beta t}\mathbb{P}_t\left[\sum_{k=1}^{p}\chi_{A_k}(x)\right]dt$ is positive on sets where at least one of $\int_0^\infty e^{-\beta t}\mathbb{P}_t\chi_{A_k}(x)dt$ is positive. This set is nothing but $\cup_{k=1}^{p}\cup_{t=0}^{\infty}\phi_t(A_k)$ by definition of union of sets. Hence, $\mathcal{R}^\infty = \mathcal{R}^{\infty,\beta} = \cup_{k=1}^{p}\cup_{t=0}^{\infty}\phi_t(A_k)$. This proves the claim. ■

V. Finite dimensional approximation

In this section, we use set-oriented numerical methods for the finite dimensional approximation of the the linear Perron Frobenius (P-F) semigroup. The finite dimensional approximation of the P-F operator is used for the approximation of the controllable and observable sets of the linear advection PDE. For the finite dimensional approximation of the P-F operator, we consider the following partition of the state space X

$$\mathcal{X}_N = \{D_1, \dots, D_N\},$$

where $D_i \cap D_j = \emptyset$ and $\cup_{i=1}^{N} D_i = X$.

The finite dimensional approximation of the P-F semigroup on the partition $\mathcal{X}_N$ is an $N \times N$ matrix, P, as follows:

$$[P]_{ij} = \frac{m(\phi_{\delta t}(D_i) \cap D_j)}{m(D_i)} \quad (13)$$

where m is the Lebesgue measure and δt is discretization time-step and $\phi_{\delta t}$ is the solution of the system $\dot{x} = f(x)$. $[P]_{ij}$ is the $(i,j)^{th}$ entry of the matrix P. Computationally several short-term trajectories are used to compute the individual entries of matrix. The short time simulated solution of the differential equation, $\dot{x} = f(x)$, is used to propagate, say L, initial conditions chosen to be uniformly distributed over the set D_i. The entry $[P]_{ij}$ of the matrix P is then obtained by determining the fraction of initial conditions that makes transition from set D_i to D_j in δt time. In this finite dimensional section, we prove results only for the controllability, as the results for observability follows from duality argument. Since the Koopman and P-F operators are dual to each other, the finite dimensional approximation of the Koopman operator, U, is transpose of P-F operator, P, i.e., $U = P^T$.

With no loss of generality, we make following assumption on the location of the actuator sets.

Assumption 7:

$$A_i = D_{k_i}, \quad i = 1, \dots, p \ \text{ where } \ k_i \in N_a.$$

where, $N_a \subseteq \{1, \dots, N\}$ is the admissible set and it consists of the admissible locations where the actuators can be placed. For example, in the building system application, the actuators and sensors can be placed only on the boundary of the room along the walls and the ceilings.

A. Characterization of Controllable Space

The finite dimensional approximation of the P-F semigroup can be used to provide a characterization of the controllable sets. The finite dimensional approximation of the P-F operator introduces a further weaker notion of controllability which we refer to as *coarse controllability*. Coarse controllability essentially implies controllability modulo the size of the partition used in the finite dimensional approximation. Due to space constraints, we refer the readers to [14] for more details on coarse controllability and the associated theorems.

We have the following asymptotic characterization of the controllable set in terms of the finite dimensional approximation of the P-F operator. The following theorem can be viewed as a finite dimensional counterpart of Theorem 6 and is used for the purpose of computation of the controllable set.

Theorem 8: Let

$$\bar{\mu}_N = \sum_{i=1}^{p} e_{k_i}^T \sum_{\ell=0}^{N} P^\ell, \quad \bar{\mu}_M = \sum_{i=1}^{p} e_{k_i}^T \sum_{\ell=0}^{M} P^\ell,$$
$$\bar{\mu} = \sum_{i=1}^{p} e_{k_i}^T (I - \alpha P)^{-1}, \quad (14)$$

where, $M \ge N$, e is N dimensional column vector which is all zero except for ones at the k_i^{th} locations with $k_i \in N_a$, and α is any number less than one. Then,

$$\mathrm{supp}(\bar{\mu}_N) = \mathrm{supp}(\bar{\mu}_M) = \mathrm{supp}(\bar{\mu}).$$

We omit the proof here due to space constraints and refer the readers to [14] for the proof of this theorem.

Remark 9: It can be proved that the minimum number of actuators or sensors that are needed for coarse controllability and observability is closely connected to the spectrum the the finite dimensional approximation of the PF operator. In particular, the number of unit eigenvalues of the PF matrix will determine the minimum number of actuators and sensors for controllability and observability respectively [14]. It is know that eigenvector with eigenvalue one of the PF matrix corresponds the invariant measure of dynamical systems [15].

VI. Convex Optimization formulation for optimal placement

In this section, we formulate the optimization problem for the optimal location of the actuators. The optimization problem for the optimal locations of sensors can be formulated along same lines. There are various ways to formulate the optimization problem where the objective in all cases is to maximize controllability. In particular, with reference to results from Theorem 8 we would like to make

$$\bar{\mu}_k > 0, \quad k = 1, \ldots, N.$$

In the following optimization formulation, the objective is to achieve controllability over the entire state space X while keeping the number of actuators equal to p.

$$\begin{aligned} &\text{minimize} \;\; (-\textstyle\sum_{k \in N_a} \log(z^T (I - \alpha P)^{-1} e_k)) \\ &\text{subject to} \;\; 1^T z = p, \\ &z_i \in \{0,1\} \; i \in N_a \end{aligned} \tag{15}$$

where e_k is the column vector of all zeros except for one at k^{th} location, and 1 is N vector of all ones. The log barrier cost function ensures that every component of the vector $\bar{\mu}$ is positive, where $\bar{\mu} = z^T(I - \alpha P)^{-1}$. Note that the vector z of ones and zeros is an optimization parameter. The non-zero entries of vector z correspond to optimal location of actuators and there are p of them as ensured by the constraints. The optimization problem can be used to determine the minimum number of actuators required to achieve controllability. In particular, the cost function will be finite only when the system is controllable over the entire state space else the cost function will be infinite. Hence one can potentially start with smaller value of p and slowly increase p until the cost is finite. This optimization problem is Boolean-convex problem since the objective is a convex function of z for $z_i \geq 0$, the constraint $1^T z$ is linear but the constraint on z are Boolean. The relaxation for the Boolean-convex problem can be obtained by restricting $z_i \in [0,1]$. We obtain the following convex relaxation [16].

Convex Relaxation

$$\begin{aligned} &\text{minimize} \;\; (-\textstyle\sum_{k \in N_a} \log(z^T (I - \alpha P)^{-1} e_k)) \\ &\text{subject to} \;\; 1^T z = p, \\ &z_i \in [0,1] \; i \in N_a \end{aligned} \tag{16}$$

The relaxed problem is convex and hence can be solved using standard optimization routines. The relaxed optimization problem will provide for a sub-optimal solution. The optimal solution z_i^* for the relaxed problem will not be binary. The solution z^* can be used to provide a sub-optimal solution to the actuator selection problem and there are various ways to do that. In particular, one can choose the indices corresponding to the first p largest values from the vector z^*. If $z_{k_1}^*, z_{k_2}^* \ldots, z_{k_p}^*$ are the entries of vector z^* rearranged in descending order, then the sub-optimal sensor selection is given as follows:

$$A_\ell = D_{k_\ell}, \quad \ell = 1, \ldots, p.$$

VII. Applications and simulation results

In this section, we demonstrate the application of our proposed framework for the optimal placement of actuators for the control of contaminants in a Double Gyre flow field. We have used `cvx` package of MATLAB to solve the convex optimization problem. The finite dimensional approximation of the P-F semigroup for the double gyre vector field has two eigenvalues equal to one and hence, with reference to remark 9 we choose to place two actuators.

A. Control of contaminant in a Double Gyre flow field

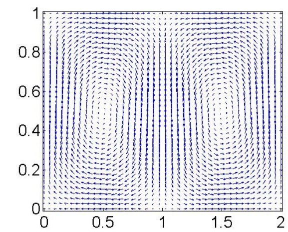

Fig. 1. Velocity field for double gyre fluid flow

The Double Gyre flow field [17] that we have used is described by the following equation

$$\dot{x} = -\pi \sin(\pi x) \cos(\pi y), \quad \dot{y} = \pi \sin(\pi y) \cos(\pi x),$$

where $(x, y) \in X = [0,2] \times [0,1]$. In Fig. 1, we show the plot of a Double Gyre velocity field. From the velocity plot in Fig. 1, we notice that there are two distinct regions in space, and the regions are separated by a ridge, known in literature as Lagrangian Coherent Structures(LCS) [18]. The LCS are also the extrema of the finite-time Lyapunov exponent field [18]. LCS have been shown to play an important role in the

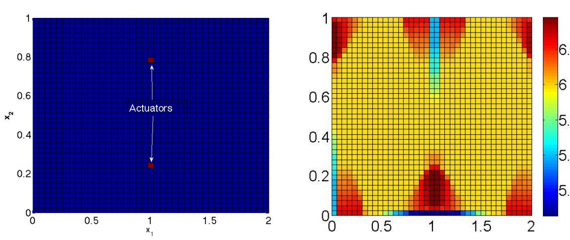

Fig. 2. a) Actuator placement; b) Optimal cost for actuator placement.

evolution of the contaminants in oceanographic flow [19]. For the finite dimensional approximation of the P-F operator we used a 40×40 partition of the state space, X. No constraints were placed on the location of the actuators, i.e., $N_a = \{1, \ldots, N\}$ (refer to Assumption 7). We use relaxed optimization formulation as described in Eq. (16) of Section VI for the placement of the actuator. The parameter α is chosen to be equal to 0.5. In Figs. 2(a) and 2(b), we show the optimal location of the two actuators and the corresponding plot for the optimal cost respectively. It is interesting to observe that the optimal location of the actuators lies on the LCS. While the importance of coherent structures for the prediction of contaminant flow in oceanographic flow is already proved, our result highlights the importance of these structure for the purpose of control.

VIII. Conclusions

We presented a systematic convex programming-based approach for the optimal placement of actuators and sensors for the control of nonequilibrium dynamics. We use the maximization of the controllability and observability region in the state space as the criteria for the optimal placement of actuators and sensors respectively. We provide an analytical characterization of the controllable and observable region over infinite time in terms of the solution of a linear PDE. Set-oriented numerical methods are used for the finite dimensional approximation of the controllable and observability region. The application of the developed framework is demonstrated on the problem of optimal placement of dispersant release for the control of contaminants in Double Gyre fluid flow field.

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