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Stochastic H_∞ Control and Estimation of State-multiplicative Discrete-time Systems with Delay

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***Abstract*— Linear, state delayed, discrete-time systems with stochastic uncertainties in their state-space model are considered. The problems of both H_∞ state-feedback control and filtering are solved, for the stationary case, via an input-output approach by which the system is replaced by a nonretarded system with deterministic norm-bounded uncertainties. In both problems, a cost function is defined which is the expected value of the standard H_∞ performance index with respect to the uncertain parameters.**

Keywords : Stochastic H_∞ control, retarded discrete-time systems, input-output

I. INTRODUCTION

We address the problems of H_∞ state-feedback control and filtering of state-delayed, discrete-time, state-multiplicative linear systems via the *input-output* approach based on the stability and Bounded Real Lemma (BRL) of these systems, which are developed here. The multiplicative noise appears in the system model in both the delayed and the non delayed states of the system.

The analysis and design of controllers for systems with stochastic uncertainties have matured greatly in the last two decades (see [1] and the references therein). Numerous solutions to various stochastic control and filtering problems including those that ensure a worst case performance bound in the H_∞ sense, have been derived and solved for both: delay-free [1]- [8] and retarded linear stochastic systems [9]-[18].

Delay-free systems with parameter uncertainties that are modelled as white noise processes in a linear setting have been treated in [2] -[4], for the continuous-time case and in [5] -[8] for the discrete-time case. Such models of uncertainties are encountered in many areas of applications (see [1] and the references therein) such as: nuclear fission and heat transfer, population models and immunology. In control theory, such models are encountered in gain scheduling when the scheduling parameters are corrupted with measurement noise.

The stability and control of stochastic delayed systems of various types (i.e constant time-delay, slow and fast varying delay) have been a central issue in the theory of stochastic state-multiplicative systems over the last decade [9]-[17]. The results that have been obtained for the stability of deterministic retarded systems, since the 90's, have been extended also to the stochastic case, mainly for continuous-time systems [19] - [25]. In the continuous-time stochastic setting, for example, the Lyapunov-Krasovskii (L-K) approach is applied in [12] and [13], to systems with constant delays, and stability criteria are derived for cases with norm-bounded uncertainties. The H_∞ state-feedback control for systems with time-varying delay is treated in [11] for restricted LKFs that provide delay-independent, rate dependent results. Also [10] considers H_∞ control (both state and output feedback) and estimation of time delay systems.

In the discrete-time setting, the mean square exponential stability and the control and filtering problems of these systems were treated by several groups [16]-[18]. In [16], the state-feedback control problem solution is solved for norm-bounded uncertain systems, for the restrictive case where the same multiplicative noise sequence multiplies both the states and the input of the system. The solution there is delay-dependent.

To the best of our knowledge, the input-output approach has not been applied to the discrete-time stochastic setting. In [14],[15], this method was shown to achieve better results, in the continuous-time case, over other solution methods. Also this method allows for the solution of the latter problems for polytopic-type uncertain systems (in addition to norm-bounded uncertain systems).

The point of view taken in the present paper is similar to the one taken for the solution of both the continuous-time state-feedback control and filtering problems in [14],[15]. Here, we adopt the input-output approach of [23], [24] for deterministic systems, to delay-dependent solutions of the above discrete-time counter part stochastic problems. This approach is based on the representation of the system's delay action by linear operators, with no delay, which in turn allows one to replace the underlying system with an equivalent one which possesses a norm-bounded uncertainty, and therefore may be treated by the theory of norm bounded uncertain, non-retarded systems with state-multiplicative noise [1].

In our system we allow for a time-varying delay where the uncertain stochastic parameters multiply both the delayed and the non delayed states in the state space model of the system. Unlike the solution of [16], in our model the system is subjected to three different white noise sequences that multiply, the states, the delayed states and the system input.

This paper is organized as follows: Based on the input-output approach, the solution of the stability issue is achieved in Section III, followed by the solution of the BRL in Section IV. The state-feedback control problem is treated in Section V. The filtering problem is treated in Section VI, for a general-type filter, resulting in a single LMI that yields the filter parameters.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, $\mathcal{N}$ is the set of natural numbers and the notation $P > 0$, (respectively, $P \geq 0$) for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (respectively, semi-definite). We denote by $L^2(\Omega, \mathcal{R}^n)$ the space of square-integrable $\mathcal{R}^n-$ valued functions on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ algebra of a subset of Ω called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_k)_{k\in\mathcal{N}}$ we denote an increasing family of σ-algebras $\mathcal{F}_k \subset \mathcal{F}$. We also denote by $\tilde{l}^2(\mathcal{N}; \mathcal{R}^n)$ the n-dimensional space of nonanticipative stochastic processes $\{f_k\}_{k\in\mathcal{N}}$ with respect to

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$(\mathcal{F}_k)_{k\in\mathcal{N}}$ where $f_k \in L^2(\Omega, \mathcal{R}^n)$. On the latter space the following l^2-norm is defined:

$$||\{f_k\}||^2_{\tilde{l}_2} = E\{\sum_0^\infty ||f_k||^2\} = \sum_0^\infty E\{||f_k||^2\} < \infty, \quad \{f_k\} \in \tilde{l}_2(\mathcal{N}; \mathcal{R}^n), \tag{1}$$

where $||\cdot||$ is the standard Euclidean norm. We denote by $\mathrm{Tr}\{\cdot\}$ the trace of a matrix and by δ_{ij} the Kronecker delta function. Throughout the manuscript we refer to the notation of exponential l^2 stability, or internal stability, in the sense of [6] (see Definition 2.1, page 927, there).

II. Problem Formulation

We consider the following linear retarded system:

$$\begin{array}{l} x_{k+1} = (A_0 + D\nu_k)x_k + (A_1 + F\mu_k)x_{k-\tau(k)} \\ +B_1 w_k + (B_2 + G\zeta_k)u_k, \quad x_l = 0,\ l \le 0, \\ y_k = C_2 x_k + D_{21} n_k \end{array} \tag{2a,b}$$

with the objective vector

$$z_k = C_1 x_k + D_{12} u_k, \tag{3}$$

where $x_k \in \mathcal{R}^n$ is the system state vector, $w_k \in \mathcal{R}^q$ is the exogenous disturbance signal, $n_k \in \mathcal{R}^p$ is the the measurement noise signal, $u_k \in \mathcal{R}^\ell$ is the control input, $y_k \in \mathcal{R}^m$ is the measured output and $z_k \in \mathcal{R}^r$ is the state combination (objective function signal) to be regulated and where the time delay bound is denoted by h. The variables $\{\zeta_k\}$ $\{\mu_k\}$ and $\{\nu_k\}$ are zero-mean real scalar white-noise sequences that satisfy:

$$E\{\nu_k\nu_j\} = \delta_{kj},\ E\{\zeta_k\zeta_j\} = \delta_{kj},\ E\{\mu_k\mu_j\} = \delta_{kj}$$
$$E\{\zeta_k\mu_j\} = E\{\zeta_k\nu_j\} = E\{\mu_k\nu_j\} = 0,\ \ \forall k, j \quad \ge 0.$$

The matrices in (2a,b), (3) are constant matrices of appropriate dimensions.

We treat the following two problems:

i) H_∞ state-feedback control:
We consider the system of (2a) and (3) and the following performance index:

$$J_E \stackrel{\Delta}{=} ||z_k||^2_{\tilde{l}_2} - \gamma^2 ||w_k||^2_{\tilde{l}_2}. \tag{4}$$

Our objective is to find a state-feedback control law $u_k = Kx_k$ that achieves $J_E < 0$, for the worst-case of the process disturbance $w_k \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty); \mathcal{R}^q)$ and for the prescribed scalar $\gamma > 0$.

ii) H_∞ filtering:
We consider the system of (2a,b) and (3) where $B_2 = 0,\ G = 0,\ D_{12} = 0$ and consider the estimator of the following general form:

$$\begin{array}{l} \hat{x}_{k+1} = A_c\hat{x}_k + B_c y_k, \\ \hat{z}_k = C_c \hat{x}_k. \end{array} \tag{5a,b}$$

We denote

$$e_k = x_k - \hat{x}_k, \quad \text{and} \quad \bar{z}_k = z_k - \hat{z}_k, \tag{6}$$

and we consider the following cost function:

$$J_F \stackrel{\Delta}{=} ||\bar{z}_k||^2_{\tilde{l}_2} - \gamma^2[||w_k||^2_{\tilde{l}_2} + ||n_{k+1}||^2_{\tilde{l}_2}]. \tag{7}$$

Given $\gamma > 0$, we seek an estimate $C_c\hat{x}_k$ of $C_1 x_k$ over the infinite time horizon $[0,\infty)$ such that J_F given by (7) is negative for all nonzero w_k, n_k where $w_k \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty); \mathcal{R}^q)$, $n_k \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty]; \mathcal{R}^p)$.

III. Mean-square exponential stability

In order to solve the above two problems we start with the stability issue of the retarded discrete-time system. We first introduce the following scalar operators which are needed, in the sequel, for transforming the delayed system to a norm-bounded nominal one:

$$\Delta_1(g_k) = g_{k-h}, \quad \Delta_2(g_k) = \sum_{j=k-h}^{k-1} g_j. \tag{8a,b}$$

Denoting $\bar{y}_k = x_{k+1} - x_k$ and using the fact that $\Delta_2(\bar{y}_k) = x_k - x_{k-h}$, the following state space description of the system is obtained:

$$\begin{array}{l} x_{k+1} = (A_0 + D\nu_k + M)x_k + (A_1 - M + F\mu_k)\Delta_1(x_k) \\ -M\Delta_2(\bar{y}_k) + B_1 w_k + (B_2 + G\zeta_k)u_k, \quad x_l = 0,\ l \le 0, \\ y_k = C_2 x_k + D_{21} n_k, \\ z_k = C_1 x_k + D_{12} u_k, \end{array} \tag{9a-c}$$

where the matrix M is a free decision variable that will be determined later.

We consider then the following auxiliary system where we take $B_2 = 0$ and $G = 0$:

$$\begin{array}{l} x_{k+1} = (A_0 + D\nu_k + M)x_k + (A_1 - M + F\mu_k)w_{1,k} \\ -Mw_{2,k} + B_1 w_k, \end{array} \tag{10}$$

with the feedback

$$w_{1,k} = \Delta_1(x_k), \quad w_{2,k} = \Delta_2(\bar{y}_k). \tag{11a,b}$$

We consider the system of (10) where $B_1 = 0$ and the following Lyapunov function:

$$V_k \stackrel{\Delta}{=} x_k^T Q x_k. \tag{12}$$

Taking expectation with respect to $\nu_k,\ \mu_k$ and solving for (10) we obtain:

$$\begin{array}{l} E\{V_{k+1}\} - V_k = E\{[x_k^T(A_0^T + M^T + D^T\nu_k) \\ +w_{1,k}^T(A_1^T - M^T + F^T\mu_k) - w_{2,k}^T M^T]Q \\ [(A_0 + M + D\nu_k)x_k + (A_1 - M + F\mu_k)w_{1,k} - Mw_{2,k}]\} \\ -x_k^T Q x_k. \end{array} \tag{13}$$

We thus arrive at the following condition for $E\{V_{k+1}\} - V_k < 0$.

Theorem 1: The exponential stability in the mean square sense of the system (2a) where $B_1 = 0,\ B_2 = 0$ and $G = 0$, is guaranteed if there exist matrices $Q > 0$, $R_1 > 0$ and $R_2 > 0$, and a $n \times n$ matrix M that satisfy the following inequality:

$$\begin{bmatrix} \hat{\Gamma}_{11} & \hat{\Gamma}_{12} & 0 & 0 & \hat{\Gamma}_{15} \\ * & -Q & \hat{\Gamma}_{23} & QM & 0 \\ * & * & \hat{\Gamma}_{33} & 0 & \hat{\Gamma}_{35} \\ * & * & * & -R_2 & -hM^T R_2 \\ * & * & * & * & -R_2 \end{bmatrix} < 0, \tag{14}$$

where

$$\begin{array}{l} \hat{\Gamma}_{11} = -Q + D^T(Q + h^2 R_2)D + R_1, \\ \hat{\Gamma}_{12} = (A_0 + M)^T Q, \\ \hat{\Gamma}_{15} = h(A_0^T + M^T)R_2 - R_2 h, \\ \hat{\Gamma}_{33} = -R_1 + F^T(Q + h^2 R_2)F, \\ \hat{\Gamma}_{23} = Q(A_1 - M), \\ \hat{\Gamma}_{35} = h(A_1^T - M^T)R_2. \end{array}$$

Proof: Define $\tilde{x}_{k+1} = x_{k+1} - D\nu_k x_k - F\mu_k w_{1,k}$ and $\tilde{y}_k = \bar{y}_k - D\nu_k x_k - F\mu_k w_{1,k}$ and denote

$$\eta_k = col\{x_k,\ \tilde{x}_{k+1},\ w_{1,k},\ w_{2,k},\ h\tilde{y}_k\}.$$

If (14) is satisfied for the appropriate Q, R_1, R_2, and M, then the following is satisfied:

$$\theta_k \triangleq \eta_k^T \Gamma \eta_k < 0. \quad (15)$$

Carrying out the multiplications in (15) we find that

$$\begin{array}{c}\theta_k = -x_k^T Q x_k + x_k^T D^T (Q + h^2 R_2) D x_k + x_k^T R_1 x_k \\ -\tilde{x}_{k+1}^T Q \tilde{x}_{k+1} + \bar{\eta}_k - w_{1,k}^T (R_1 - F^T (Q + R_2) F) w_{1,k} \\ +h^2 \tilde{y}_k^T R_2 \tilde{y}_k - w_{2,k}^T R_2 w_{2,k},\end{array}$$

where $\bar{\eta}_k = 2x_k^T (A_0^T + M^T) Q \tilde{x}_{k+1} - 2 w_2^T M^T Q \tilde{x}_{k+1} + 2\tilde{x}_{k+1}^T Q (A_1 - M) w_1 = 2\tilde{x}_{k+1}^T Q \tilde{x}_{k+1}$. It thus follows that

$$\begin{array}{c}\theta_k = \tilde{x}_{k+1}^T Q \tilde{x}_{k+1} - x_k^T Q x_k + x_k^T D^T (Q + h^2 R_2) D x_k \\ +[x_k^T R_1 x_k - w_{1,k}^T R_1 w_{1,k}] + [h^2 \tilde{y}_k^T R_2 \tilde{y}_k - w_{2,k}^T R_2 w_{2,k}] \\ +w_{1,k}^T F^T (Q + h^2 R_2) F w_{1,k}.\end{array}$$

Since $E\{\bar{y}_k^T R_2 \bar{y}_k\} = E\{\tilde{y}_k^T R_2 \tilde{y}_k\} + E\{x_k^T D^T R_2 D x_k\} + E\{w_{1,k}^T F^T R_2 F w_{1,k}\}$ and
$E\{x_{k+1}^T Q x_{k+1}\} = E\{\tilde{x}_{k+1}^T Q \tilde{x}_{k+1}\} + E\{x_k^T D^T Q D^T x_k\} + E\{w_{1,k}^T F^T Q F w_{1,k}\}$,
we find that if (14) is satisfied then:

$$\begin{array}{l}E\{\theta_k\} = E\{x_{k+1}^T Q x_{k+1} - x_k^T Q x_k\} \\ +E\{\bar{y}_k^T h^2 R_2 \bar{y}_k - w_{2,k}^T R_2 w_{2,k}\} \\ +E\{x_k^T R_1 x_k - w_{1,k}^T R_1 w_{1,k}\} < 0.\end{array} \quad (16)$$

Since for all sequences $\{r_k\}$ in R^n

$$\|\Delta_1 r_k\|^2 \leq \|r_k\|^2, \quad \text{and} \quad \|\Delta_2 r_k\|^2 \leq h^2 \|r_k\|^2,$$

we have that $E\{h^2 \bar{y}_k^T R_2 \bar{y}_k - w_{2,k}^T R_2 w_{2,k}\} > 0$ and $E\{x_k^T R_1 x_k - w_{1,k}^T R_1 w_{1,k}\} > 0$.
Thus, (16) implies that $E\{V_{k+1}\} - V_k = E\{x_{k+1}^T Q x_{k+1}\} - x_k^T Q x_k < 0$ and the stability is guaranteed.

We note that inequality (14) is bilinear due to the terms QM and $R_2 M$. Similar bilinearity is observed in the continuous-time counterpart of the stability problem. There exist few algorithms that may solve bilinear matrix inequalities, however they do not always converge to a global minimum and they may require considerable computational effort. In order to remain in the linear domain, we choose $R_2 = \epsilon Q$ where ϵ is a positive tuning scalar. Defining $Q_M = QM$ we obtain the following result:

Corollary 1: The exponential stability in the mean square sense of the system (2a) where $B_1 = 0$, $B_2 = 0$ and $G = 0$, is guaranteed if there exist $n \times n$ matrices $Q > 0$, $R_1 > 0$ and a $n \times n$ matrix Q_M and a tuning scalar $\epsilon > 0$ that satisfy the following inequality:

$$\bar{\Gamma} \triangleq \begin{bmatrix} \tilde{\Gamma}_{11} & A_0^T Q + Q_M^T & 0 & 0 & \tilde{\Gamma}_{15} \\ * & -Q & QA_1 - Q_M & Q_M & 0 \\ * & * & \tilde{\Gamma}_{33} & 0 & \tilde{\Gamma}_{35} \\ * & * & * & -\epsilon Q & -h\epsilon Q_M^T \\ * & * & * & * & -\epsilon Q \end{bmatrix} < 0, \quad (17)$$

where

$$\begin{array}{l}\tilde{\Gamma}_{11} = -Q + D^T Q[1 + \epsilon h^2] D + R_1, \\ \tilde{\Gamma}_{15} = \epsilon h[A_0^T Q + Q_M^T] - \epsilon h Q, \\ \tilde{\Gamma}_{33} = -R_1 + (1+\epsilon h^2) F^T Q F, \\ \tilde{\Gamma}_{35} = \epsilon h[A_1^T Q - Q_M^T].\end{array}$$

IV. The Bounded real lemma

In the case where $w_k \neq 0$, (15) implies that $E\{[\tilde{x}_{k+1} - B_1 w_k]^T Q [\tilde{x}_{k+1} - B_1 w_k]\} - x_k^T Q x_k < 0$. Denoting

$$J_B = E\{x_{k+1}^T Q x_{k+1}\} - x_k^T Q x_k + z_k^T z_k - \gamma^2 w_k^T w_k, \quad (18)$$

where $z_k = C_1 x_k$, we add $B_1 w_k$ to the previously defined $\tilde{x}_{k+1}$ and readily find that $J_B < 0, \forall\, w_k \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty); \mathcal{R}^q)$ if

$$\hat{\eta}_k^T \hat{\Gamma} \hat{\eta}_k < 0, \quad (19)$$

where $\hat{\eta}_k = col\{x_k,\ \tilde{x}_{k+1},\ w_{1,k},\ w_{2,k},\ h\tilde{y}_k,\ w_k,\ z_k\}$ and where $\hat{\Gamma} =$

$$\begin{bmatrix} \bar{\Gamma}_{11} & \bar{\Gamma}_{12} & 0 & 0 & \bar{\Gamma}_{15} & 0 & C_1^T \\ * & -Q & \bar{\Gamma}_{23} & QM & 0 & QB_1 & 0 \\ * & * & \bar{\Gamma}_{33} & 0 & & 0 & 0 \\ * & * & * & -R_2 & -hM^T R_2 & 0 & 0 \\ * & * & * & * & -R_2 & hR_2 B_1 & 0 \\ * & * & * & * & * & -\gamma^2 I_q & 0 \\ * & * & * & * & * & * & -I_r \end{bmatrix}, \quad (20)$$

where

$$\begin{array}{l}\bar{\Gamma}_{11} = -Q + D^T (Q + h^2 R_2) D + R_1, \\ \bar{\Gamma}_{12} = (A_0 + M) TQ, \\ \bar{\Gamma}_{23} = Q(A_1 - M), \\ \bar{\Gamma}_{33} = -R_1 + F^T (Q + h^2 R_2) F, \\ \bar{\Gamma}_{35} = h(A_1^T - M^T) R_2 . \bar{\Gamma}_{12} = (A_0 + M)^T Q, \\ \bar{\Gamma}_{15} = h(A_0^T + M^T) R_2 - R_2 h.\end{array}$$

Note that when carrying out the multiplications in (19) the product $\tilde{x}_{k+1}^T \hat{\Gamma}_2 \hat{\eta}_k$ is zero, where $\hat{\Gamma}_2$ denotes the second row block of $\hat{\Gamma}$. We also note that the matrix in the 5th row and the 6th column blocks in the latter inequality stems from the fact that the expression for $\tilde{y}_k$ includes now the additional term $B_1 w_k$.

Similarly to the stability result of Corollary 1, the following result is readily obtained:

Theorem 2 Consider the system (2a) and (3) with $B_2 = 0$, $G = 0$ and $D_{12} = 0$. The system is exponentially stable in the mean square sense and, for a prescribed scalar $\gamma > 0$ and a given scalar tuning parameter $\epsilon_b > 0$, the requirement of $J_B < 0$ is achieved for all nonzero $w \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty); \mathcal{R}^q)$, if there exist $n \times n$ matrices $Q > 0$, $R_1 > 0$ and a $n \times n$ matrix Q_m that satisfy the following LMI:

$$\begin{bmatrix} \tilde{\Gamma}_{11} & \tilde{\Gamma}_{12} & 0 & 0 & \tilde{\Gamma}_{15} & 0 & C_1^T \\ * & -Q & \tilde{\Gamma}_{23} & Q_m & 0 & QB_1 & 0 \\ * & * & \tilde{\Gamma}_{33} & 0 & \tilde{\Gamma}_{35} & 0 & 0 \\ * & * & * & -\epsilon_b Q & -h\epsilon_b Q_m^T & 0 & 0 \\ * & * & * & * & -\epsilon_b Q & \epsilon_b h Q B_1 & 0 \\ * & * & * & * & * & -\gamma^2 I_q & 0 \\ * & * & * & * & * & * & -I_r \end{bmatrix} < 0, \quad (21)$$

where

$$\begin{array}{l}\tilde{\Gamma}_{11} = -Q + D^T Q[1 + \epsilon_b h^2] D + R_1, \\ \tilde{\Gamma}_{12} = A_0^T Q + Q_m^T, \\ \tilde{\Gamma}_{15} = \epsilon_b h[A_0^T Q + Q_m^T] - \epsilon_b h Q, \\ \tilde{\Gamma}_{23} = QA_1 - Q_m, \\ \tilde{\Gamma}_{33} = -R_1 + (1 + \epsilon_b h^2) F^T Q F, \\ \tilde{\Gamma}_{35} = \epsilon_b h[A_1^T Q - Q_m^T].\end{array}$$

V. State-feedback control

In this section we address the problem of finding the following state-feedback control law

$$u_k = K x_k, \tag{22}$$

that stabilizes the system and achieves a prescribed level of attenuation. We consider the system of (2a) and (3) and we apply the control law of (22), where A_0 is replaced by $(A_0 + B_2 K)$ and C_1 is replaced by $C_1 + D_{12}K$. We obtain the following inequality:

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & 0 & 0 & \Upsilon_{15} & 0 & \Upsilon_{17} \\ * & -Q & \Upsilon_{23} & Q_M & 0 & QB_1 & 0 \\ * & * & \Upsilon_{33} & 0 & \Upsilon_{35} & 0 & 0 \\ * & * & * & -\epsilon_b Q & \Upsilon_{45} & 0 & 0 \\ * & * & * & * & -\epsilon_b Q & h\epsilon_b Q B_1 & 0 \\ * & * & * & * & * & -\gamma^2 I_q & 0 \\ * & * & * & * & * & * & -I_r \end{bmatrix} < 0, \tag{23}$$

where

$$\begin{aligned} &\Upsilon_{11} = -Q + R_1 + \Upsilon_{11,a}, \\ &\Upsilon_{11,a} = D^T Q[1+\epsilon_b h^2]D + K^T G^T Q[1+\epsilon_b h^2]GK, \\ &\Upsilon_{12} = [A_0 + B_2 K]^T Q + Q_M^T, \\ &\Upsilon_{15} = \epsilon_b h[(A_0 + B_2 K)^T Q + Q_M^T] - \epsilon_b h Q, \\ &\Upsilon_{17} = [C_1 + D_{12}K]^T, \\ &\Upsilon_{23} = QA_1 - Q_M, \\ &\Upsilon_{33} = -R_1 + (1+\epsilon_b h^2) F^T Q F, \\ &\Upsilon_{35} = \epsilon_b h[A_1^T Q - Q_M^T], \\ &\Upsilon_{45} = -h\epsilon_b Q_M^T. \end{aligned}$$

Multiplying the above inequality by

$$diag\{Q^{-1}, Q^{-1}, Q^{-1}, Q^{-1}, Q^{-1}, I_q, I_r\}$$

, from the left and the right and denoting , $\bar{R}_1 = Q^{-1} R_1 Q^{-1},\ P \triangleq Q^{-1},\ M_P = MP,\ K_P = KP$, we obtain the following LMI:

$$\begin{bmatrix} -P + \bar{R}_1 & \bar{\Upsilon}_{12} & 0 & 0 & \bar{\Upsilon}_{15} & 0 & \bar{\Upsilon}_{17} & \bar{\epsilon} P D^T & 0 & \bar{\epsilon} K_P^T G^T \\ * & -P & A_1 P - M_P & M_P & 0 & B_1 & 0 & 0 & 0 & 0 \\ * & * & -\bar{R}_1 & 0 & \bar{\Upsilon}_{35} & 0 & 0 & 0 & \bar{\epsilon} P F^T & 0 \\ * & * & * & -\epsilon_b P & -h\epsilon_b M_P^T & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_b P & h\epsilon_b B_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -I_r & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -P & 0 & 0 \\ * & * & * & * & * & * & * & * & -P & 0 \\ * & * & * & * & * & * & * & * & * & -P \end{bmatrix} < 0. \tag{24}$$

where

$$\begin{aligned} &\bar{\Upsilon}_{12} = P A_0^T + K_P^T B_2^T + M_P^T, \\ &\bar{\Upsilon}_{15} = \epsilon_b h[P A_0^T + K_P^T B_2^T + M_P^T] - \epsilon_b h P, \\ &\bar{\Upsilon}_{17} = P C_1^T + K_P^T D_{12}^T, \\ &\bar{\Upsilon}_{35} = \epsilon_b h[P A_1^T - M_P^T], \\ &\bar{\epsilon}^2 = 1 + \epsilon_b h^2. \end{aligned}$$

We thus arrive at the following theorem:

Theorem 3 Consider the system (2a) and (3). For a prescribed scalar $\gamma > 0$, and positive tuning scalar $\epsilon_b > 0$, there exists a state-feedback gain that achieves negative J_E for all nonzero $w \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$, if there exist $n \times n$ matrices $P > 0,\ \bar{R}_1 > 0$, $n \times n$ matrix M_P and a $l \times n$ matrix K_P that satisfy the LMI of (24). In the latter case the state-feedback gain is given by:

$$K = K_P P^{-1}. \tag{25}$$

VI. Delayed Filtering

In this section we address the filtering problem of the delayed state-multiplicative noisy system. We consider the system of (2a,b) and (3) with $B_2 = 0,\ G = 0,\ D_{12} = 0$ and the general type filter of (5). Denoting $\xi_k^T \triangleq [x_k^T \ \ \hat{x}_k^T],\ \bar{w}_k^T \triangleq [w_k^T \ \ n_k^T]$ we obtain the following augmented system:

$$\begin{aligned} &\xi_{k+1} = \tilde{A}_0 \xi_k + \tilde{B}\bar{w}_k + \tilde{A}_1 \xi_{k-\tau(k)} + \tilde{D}\xi_k \nu_k + \tilde{F}\xi_{k-\tau(k)}\mu_k, \\ &\tilde{z}_k = \tilde{C}\xi_k, \quad \xi_l = 0,\ l \le 0 \end{aligned} \tag{26a,b}$$

where

$$\tilde{A}_0 = \begin{bmatrix} A_0 & 0 \\ B_c C_2 & A_c \end{bmatrix},\ \tilde{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_c D_{21} \end{bmatrix},$$
$$\tilde{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{D} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{C}^T = \begin{bmatrix} C_1^T \\ -C_c^T \end{bmatrix} \tag{27a-f}$$

and $\tilde{F} = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$. Using the BRL result of Section IV we obtain the following inequality condition:

$$\begin{bmatrix} \tilde{\Upsilon}_{11} & \tilde{\Upsilon}_{12} & 0 & 0 & \tilde{\Upsilon}_{15} & 0 & \tilde{C}^T & \tilde{\Upsilon}_{18} \\ * & -\tilde{Q} & \tilde{\Upsilon}_{23} & \tilde{Q}_M & 0 & \tilde{Q}\tilde{B} & 0 & 0 \\ * & * & \tilde{\Upsilon}_{33} & 0 & \tilde{\Upsilon}_{35} & 0 & 0 & 0 \\ * & * & * & -\epsilon_f \tilde{Q} & \tilde{\Upsilon}_{45} & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_f \tilde{Q} & \tilde{\Upsilon}_{56} & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & * & * & -I_r & 0 \\ * & * & * & * & * & * & * & -\tilde{Q} \end{bmatrix} < 0, \tag{28}$$

where

$$\begin{aligned} &\tilde{\Upsilon}_{11} = -\tilde{Q} + \tilde{R}_1,\ \tilde{\Upsilon}_{12} = \tilde{A}_0^T \tilde{Q} + \tilde{Q}_M^T, \\ &\tilde{\Upsilon}_{15} = \epsilon_f h[\tilde{A}_0^T \tilde{Q} + \tilde{Q}_M^T] - \epsilon_f h\tilde{Q}, \\ &\tilde{\Upsilon}_{18} = \tilde{D}^T \tilde{Q}\sqrt{1+\epsilon_f h^2},\ \tilde{\Upsilon}_{23} = \tilde{Q}\tilde{A}_1 - \tilde{Q}_M, \\ &\tilde{\Upsilon}_{33} = -\tilde{R}_1 + (1+\epsilon_f h^2)\tilde{F}^T \tilde{Q}\tilde{F}, \\ &\tilde{\Upsilon}_{35} = \epsilon_f h[\tilde{A}_1^T \tilde{Q} - \tilde{Q}_M^T],\ \tilde{\Upsilon}_{45} = -h\epsilon_f \tilde{Q}_M^T, \\ &\tilde{\Upsilon}_{56} = h\epsilon_f \tilde{Q}\tilde{B}. \end{aligned}$$

Defining $\tilde{P} = \tilde{Q}^{-1}$, denoting the following partitions $\tilde{P} = \begin{bmatrix} X & M^T \\ M & T \end{bmatrix},\ \tilde{Q} = \begin{bmatrix} Y & N^T \\ N & W \end{bmatrix}$, and

$J = \begin{bmatrix} X^{-1} & Y \\ 0 & N \end{bmatrix}$, we multiply (28) by $\hat{J} = diag\{\tilde{P}J, \tilde{P}J, \tilde{P}J, \tilde{P}J, \tilde{P}J, I, I, \tilde{P}J\}$ from the right and by $\hat{J}^T$, from the left. Denoting $\bar{R}_p = J^T \tilde{P}\tilde{R}_1\tilde{P}J$, $\bar{X} = X^{-1}$,

$$\bar{X}_y = \begin{bmatrix} \bar{X} & \bar{X} \\ \bar{X} & Y \end{bmatrix}, \ \tilde{P}_M = J^T \tilde{P}\tilde{Q}_M^T \tilde{P}J,$$

and carrying out the various multiplications and further denoting $K_0 = N^T A_c M\bar{X}$, $U = N^T B_c$ and $Z = C_c M\bar{X}$, the following result is obtained:

$$\begin{bmatrix} -\bar{X}_y + \bar{R}_p & \tilde{\Psi}_{12} & 0 & 0 & \tilde{\Psi}_{15} & 0 & \tilde{\Psi}_{17} & \tilde{\Psi}_{18} & 0 \\ * & -\bar{X}_y & \tilde{\Psi}_{23} & \tilde{P}_M & 0 & \tilde{\Psi}_{26} & 0 & 0 & 0 \\ * & * & -\bar{R}_p & 0 & \tilde{\Psi}_{35} & 0 & 0 & 0 & \tilde{\Psi}_{39} \\ * & * & * & -\epsilon_f \bar{X}_y & -h\epsilon_f \tilde{P}_M & 0 & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_f \bar{X}_y & \tilde{\Psi}_{56} & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I_{q+p} & 0 & 0 & 0 \\ * & * & * & * & * & * & -I_r & 0 & 0 \\ * & * & * & * & * & * & * & -\bar{X}_y & 0 \\ * & * & * & * & * & * & * & * & -\bar{X}_y \end{bmatrix} < 0, \qquad (29)$$

where

$$\tilde{\Psi}_{12} = \begin{bmatrix} A_0^T \bar{X} & A_0^T Y + C_2^T U^T + K_0^T \\ A_0^T \bar{X} & A_0^T Y + C_2^T U^T \end{bmatrix} + \tilde{P}_M,$$

$$\tilde{\Psi}_{15} = \epsilon_f h \begin{bmatrix} A_0^T \bar{X} & A_0^T Y + C_2^T U^T + K_0^T \\ A_0^T \bar{X} & A_0^T Y + C_2^T U^T \end{bmatrix}$$

$$+\epsilon_f h \tilde{P}_M - \epsilon_f h \begin{bmatrix} \bar{X} & \bar{X} \\ \bar{X} & Y \end{bmatrix}, \tilde{\Psi}_{17} = \begin{bmatrix} C_1^T - Z^T \\ C_1^T \end{bmatrix},$$

$$\tilde{\Psi}_{18} = \begin{bmatrix} D^T \bar{X} & D^T Y \\ D^T \bar{X} & D^T Y \end{bmatrix}, \ \tilde{\Psi}_{23} = \begin{bmatrix} \bar{X}A_1 & \bar{X}A_1 \\ YA_1 & YA_1 \end{bmatrix} - \tilde{P}_M,$$

$$\tilde{\Psi}_{26} = \begin{bmatrix} \bar{X}B_1 & 0 \\ YB_1 & UD_{21} \end{bmatrix}, \ \tilde{\Psi}_{39} = \begin{bmatrix} F^T \bar{X} & F^T Y \\ F^T \bar{X} & F^T Y \end{bmatrix},$$

$$\tilde{\Psi}_{35} = \epsilon_f h \begin{bmatrix} A_1^T \bar{X} & A_1^T Y \\ A_1^T \bar{X} & A_1^T Y \end{bmatrix} - \epsilon_f h \tilde{P}_M,$$

$$\tilde{\Psi}_{56} = h\epsilon_f \begin{bmatrix} \bar{X}B_1 & 0 \\ YB_1 & UD_{21} \end{bmatrix}, \ \bar{\epsilon}^2 = 1 + \epsilon_f h^2.$$

We thus arrive at the following theorem:

Theorem 4 Consider the system of (2a,b) and (3) with $B_2 = 0$, $G = 0$ and $D_{12} = 0$. For a prescribed scalar $\gamma > 0$ and a positive tuning scalar ϵ_f, there exists a filter of the structure (5) that achieves $J_F < 0$, where J_F is given in (7), for all nonzero $w \in \tilde{l}^2([0,\infty); \mathcal{R}^q)$, $n \in \tilde{l}^2([0,\infty); \mathcal{R}^p)$, if there exist $n \times n$ matrices $\bar{X} > 0$, $Y > 0$, $2n \times 2n$ matrix $\bar{R}_p > 0$, $n \times n$ matrices K_0 and U, $2n \times 2n$ matrix $\tilde{P}_M$ and a $n \times l$ matrix Z, that satisfy (29). In the latter case the filter parameters can be extracted using the following equations:

$$A_c = N^{-T} K_0 \bar{X} M^{-1}, \ B_c = N^{-T} U, \ C_c = Z\bar{X}M^{-1}. \qquad (30a\text{-}c)$$

Noting that $XY - M^T N = I$, the filter matrix parameters A_c, B_c, and C_c can be readily found, without any loss of generality, by a singular value decomposition of $I - XY$.

VII. Conclusions

In this paper the theory of linear H_∞ state-feedback control and filtering of state-multiplicative noisy systems is developed for discrete-time delayed systems, where the stochastic uncertainties are encountered in both the delayed and the non delayed states in the state space model of the system. The delay is assumed to be unknown and time-varying where only the bound on its size is given. Delay dependent analysis and synthesis methods are developed which are based on the input-output approach, in accordance with the approach taken for the solution of the state-feedback and filtering problems, in the continuous-time counterpart. This approach transforms the delayed system to a nonretarded system with norm-bounded operators. Sufficient conditions are thus derived for the stability of the system and the existence of a solution to the corresponding BRL. Based on the BRL derivation, the state-feedback control and filtering problems are formulated and solved. An inherent overdesign is admitted to our solution due to the use of the bounded operators which enable us to transform the retarded system to a norm-bounded one. Some additional overdesign is also admitted in our solution due to the special structure imposed on R_2.

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