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State Estimation in Linear Time-Invariant Systems With Unknown Impulsive Inputs

Per Mattsson and Alexander Medvedev

Abstract— **This paper deals with state estimation in linear time-invariant systems subject to unknown impulsive input signals. A solution based on a linear impulsive observer and a finite-memory convolution operator is suggested. The problem arises e.g. in the context of systems with intrinsic pulse-modulated feedback that have recently been applied to mathematical modeling of endocrine systems with pulsatile hormone secretion. Simulation results illustrating the performance of the proposed method are provided.**

I. INTRODUCTION

Mathematical models of dynamical systems with impulsive input signals appear often in mechanics, power electronics and biology, see e.g. [1], mostly in the applications where impulses represent external momentaneous interaction. In engineered systems, impulsive control signals are typically known and therefore can be taken into account in a similar manner as non-impulsive ones. For instance, observer design for this kind of systems is not much different from that in a system with any other kind of input signal.

However, in biological applications, the impulsive control signal is often unknown. A characteristic example of this class of systems is presented by the pulsatile endocrine feedback [2]. In [3], a mathematical model for the non-basal testosterone regulation in the male, based on the concept of pulse-modulated feedback, is proposed and investigated.

A challenge in devising observers for a system with unknown impulsive input signal is that the state variables are reset after a finite time. Thus, most classical asymptotic observers, such as the Luenberger observer [4], lose track of the state vector after an impulse. A similar phenomenon occurs with respect to state estimation in switching system, when the switching time is unknown, as discussed in e.g. [5]. Impulsive observers can be utilized to deal with the state reset problem in impulsive systems.

An impulsive observer that finds the true state in finite time when the input is known is presented in [6], and a similar approach is used to implement observer-based control for systems with persistently acting impulsive input in [7]. In [8], observers for impulsive systems with linear continuous-time dynamics and a linear resetting law are considered. Further, in [9] and [10], a static gain observer for linear continuous systems under intrinsic impulsive feedback is studied, and conditions for local stability of the observer under periodic solutions in the plant are proved.

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P. Mattsson and A. Medvedev are with Information Technology, Uppsala University, Uppsala, Sweden.

Unfortunately, most of the existing approaches for designing an impulsive observer fail when there is no information about the input impulses. To fill this gap, an approach to the estimation of impulse times and weights by means of continuous least-squares observers is suggested in [11].

The present paper considers impulsive systems with linear continuous dynamics and unknown input impulses. As a solution to the state estimation problem, a method based on a linear observer coupled with an impulse estimation algorithm, similar to the one in [11], is considered.

The paper is composed as follows. First the equations governing the class of systems in hand are summarized. Then a general outline of the observer equations and impulse estimation algorithm is provided, followed by a more detailed explanation of each step. Finally, a numerical example illustrating the behaviour of the observer is given.

II. SYSTEM EQUATIONS AND ASSUMPTIONS

Consider the (hybrid) dynamical system with state resets

$$\frac{dx}{dt} = Ax\,, \qquad t \notin \mathcal{T} \tag{1}$$

$$\Delta x(t) = g_k B\,, \qquad t = t_k \in \mathcal{T} \tag{2}$$

$$y(t) = Cx(t), \tag{3}$$

where

$$\Delta x(t) = x(t^+) - x(t),$$
$$x(t^+) = \lim_{\epsilon \to 0^+} x(t+\epsilon) = x(t) + g_k B,$$

and $\mathcal{T}$ is a countable subset of $[0,\infty)$, where t_k are assumed to be ordered so that $t_1 < t_2 < t_3 < \ldots$.

The instants t_k are called impulse times, and to each impulse time, there is a corresponding impulse weight $g_k \in \mathbb{R}$. Notice that negative weights are allowed in this formulation. Equations (1)-(2) can be equivalently rewritten as

$$\dot{x}(t) = Ax + B\xi(t),$$

where

$$\xi(t) = \sum_{t_k \in \mathcal{T}} g_k \delta(t - t_k),$$

and $\delta(\cdot)$ is the Dirac delta function. This reformulation clarifies why t_k, g_k are called impulse time and weight, respectively.

It is assumed that the system has a minimum dwell time Φ, i.e.

$$0 < \Phi \leq t_{k+1} - t_k$$

for all k. This is a standard assumption in pulse-modulated systems. In biological systems, it is typically motivated by the time required for an organ or cell to recuperate.

Furthermore, it is assumed that A is a real $n \times n$ matrix, C is a real row vector, B is a real column vector and the matrix pair (A, C) is observable. It can be noted that the observer presented in this paper can easily be extended to also handle known inputs and more than one output.

III. THE OBSERVER

In order to estimate the state vector of (1), the impulsive observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + K(y - \hat{y})\,, \qquad t \notin \hat{\mathcal{T}} \tag{4}$$

$$\Delta\hat{x}(t) = \hat{g}_l B\,, \qquad t = \hat{t}_l \in \hat{\mathcal{T}} \tag{5}$$

$$\hat{y}(t) = C\hat{x}(t) \tag{6}$$

will be used, where $\hat{t}_l, \hat{g}_l$ are refereed to as the observer impulse times and weights respectively. Let

$$D = A - KC.$$

It is assumed that the observer gain K is chosen so that D is Hurwitz stable with distinct eigenvalues. Notice that, since (A, C) is observable, it follows that (D, C) is observable too.

The state estimation error $\varepsilon(t) = x(t) - \hat{x}(t)$ is governed by the equations

$$\frac{d\varepsilon}{dt} = D\varepsilon\,, \qquad t \notin \mathcal{T} \cup \hat{\mathcal{T}}, \tag{7}$$

$$\Delta\varepsilon(t) = g_k B\,, \qquad t = t_k \in \mathcal{T}, \tag{8}$$

$$\Delta\varepsilon(t) = -\hat{g}_l B\,, \qquad t = \hat{t}_l \in \hat{\mathcal{T}}. \tag{9}$$

If the impulses in the plant are known, then the observer impulses could be chosen so that $\hat{t}_k = t_k$ and $\hat{g}_k = g_k$. In this case the state estimation error for $t \geq 0$ is given by $\varepsilon(t) = e^{Dt}\varepsilon(0)$.

However, in the case investigated in the present paper, the plant impulses are unknown. To solve the problem with unknown impulses, it is assumed that, at any time t, the future output values $y(\cdot)$ within a sliding window $y(\theta), \theta \in [t, t+\tau)$ are made available to the observer. This is possible e.g. if the observer is run off-line or the state estimates are allowed to be delayed τ time units. A periodic mode in the plant (as in [9]) also opens up for an application of the present technique. Furthermore, it is assumed that $\tau < \Phi$, so that there is at most one plant impulse in the sliding window at any time t.

At time t, the observer will utilize a finite-memory convolution operator on the output $y(\theta)$, $\theta \in [t, t+\tau)$, to decide whether or not an observer impulse should be added in the current interval. If so, the same operator is used to evaluate the observer impulse time and weight.

A. IMPULSIVE OBSERVER ALGORITHM

A general outline of the algorithm is given next with the details provided in the subsequent sections.

1) Propagate the observer according to (4), until the condition for adding an observer impulse is met (see Section III-C). Let t_o be the time when the condition was met.
2) Determine the observer impulse time $\hat{t}_l \in [t_o, t_o + \tau)$ and weight $\hat{g}_l$. (See Section III-D.)
3) Propagate the observer according to (4)-(5), until $t = t_o + \tau$.
4) Go to Step 1.

B. THE FINITE-MEMORY CONVOLUTION OPERATOR

In this contribution, the finite-memory convolution operator

$$(Pf)(\lambda, \tau; t) = \int_{t-\tau}^{t} e^{\lambda(t-\theta)} f(\theta) d\theta \tag{10}$$

will be utilized. In virtue of being a pseudodifferential operator, $(Pf)(\lambda, \tau; t)$ is characterized by the symbol

$$p(\lambda, \tau, s) = \frac{1 - e^{(\lambda - s)\tau}}{s - \lambda}, \tag{11}$$

so that

$$\mathcal{L}\{(Pf)(\lambda, \tau; t)\} = p(\lambda, \tau, s)F(s),$$

where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform and $F(s) = \mathcal{L}\{f(t)\}$. This operator has previously been used for e.g. exact (deadbeat) state estimation, see e.g. [12], finite spectrum assignment control of time-delay systems [13], and also impulse detection [11].

Notice that the symbol in (11) can also be employed for defining a matrix function, i.e. for a square matrix M

$$p(\lambda, \tau, M) = \left(I - e^{(\lambda I - M)\tau}\right)(M - \lambda I)^{-1}.$$

Let $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$ be a set of real and distinct elements, and introduce the following notation

$$W(t) \triangleq \begin{bmatrix} Cp(\lambda_1, t, D) \\ \vdots \\ Cp(\lambda_m, t, D) \end{bmatrix}.$$

Also let $\mathcal{V} = W^T(\tau)W(\tau)$ and

$$V(t) = W(t)e^{Dt}.$$

In what follows, Λ is assumed to contain at least n distinct elements and be disjoint from the spectrum of the matrix D, i.e.

$$\Lambda \cap \sigma(D) = \emptyset, \tag{12}$$

where $\sigma(D)$ is the spectrum of D. This assumption is not restrictive since the elements of Λ can always be selected to satisfy the condition above.

Assumption (12) also implies that $\mathcal{V}$ is positive definite, so that a weighted vector norm

$$\|x\|_{\mathcal{V}}^2 = \|W(\tau)x\|^2 = x^T\mathcal{V}x$$

can be introduced together with the usual Euclidean one: $\|x\|^2 = x^T x$. It is straightforward to show (see e.g. [11]) that given the set Λ, it is possible to find a nonsingular state transformation matrix T, such that $\mathcal{V}$ for the transformed system and observer is equal to the identity matrix and resulting in $\|x\| = \|x\|_{\mathcal{V}}$.

In order to choose the observer impulses, operator (10) will be applied to the output error $\bar{y}(t)$ that would be present if there were no observer impulse after some time t_o. That is

$$\bar{y}(t) = C\bar{\varepsilon}(t),$$

where $\bar{\varepsilon}(t)$ is the solution to (7)-(8) with the initial value $\bar{\varepsilon}(t_o) = \varepsilon(t_o)$. Define

$$R(t_o) \triangleq \begin{bmatrix} (P\bar{y})(\lambda_1, \tau; t_o + \tau) \\ \vdots \\ (P\bar{y})(\lambda_m, \tau; t_o + \tau) \end{bmatrix}. \tag{13}$$

Notice that $\bar{y}(t)$, and thus $R(t_o)$, can be computed if $\hat{x}(t_o)$ is known together with the output $y(t)$ for $t \in [t_o, t_o + \tau)$.

Proposition 3.1: If $\mathcal{T} \cap [t_o, t_o + \tau) = \{t_k\}$, then

$$R(t_o) = V(\tau)\varepsilon(t_o) + g_k V(t_o + \tau - t_k)B,$$

and if $\mathcal{T} \cap [t_o, t_o + \tau) = \emptyset$, then

$$R(t_o) = V(\tau)\varepsilon(t_o).$$

Proof: Cf. Appendix I in [11]. ■

Proposition 3.1 shows in what way $R(t)$ is affected when a plant impulse enters the sliding window $[t, t + \tau)$. It is also of interest to see how $R(t)$ is related to the state estimation error $\varepsilon(t)$.

Proposition 3.2: Assume that $\mathcal{T} \cap [t_o, t_o + \tau) = \{t_k\}$ and $\hat{\mathcal{T}} \cap [t_o, t_o + \tau) = \{\hat{t}_l\}$. Then

$$\begin{aligned} W(\tau)\varepsilon(t_o + \tau) &= V(\tau)\varepsilon(t_o) \\ &+ g_k \left(V(t_o + \tau - t_k)B + E(t_k - t_o)\right) \\ &- \hat{g}_l \left(V(t_o + \tau - \hat{t}_l)B + E(\hat{t}_l - t_o)\right), \end{aligned} \tag{14}$$

where

$$E(t) = (W(\tau) - W(\tau - t))e^{D(\tau - t)}B.$$

Furthermore, for any $\epsilon, r > 0$, it is possible to choose the set Λ such that each row of $V(t_o + \tau - \hat{t})B$ and $E(\hat{t} - t_o)$ satisfy

$$|E_i(\hat{t} - t_o)| \le r|V_i(t_o + \tau - \hat{t})B|$$

when $\hat{t} - t_o \le \tau - \epsilon$.

Proof: See Appendix I. ■

Proposition 3.2 justifies the following important approximation. If Λ is chosen such that the proposition holds for a small enough r, and there are no impulses in $[t_o + \tau - \epsilon, t_o + \tau)$, then the terms $g_k E(t_k - t_o)$ and $\hat{g}_l E(\hat{t}_l - t_o)$ in (14) are negligible. Thus it follows from Proposition 3.1 that

$$W(\tau)\varepsilon(t_o + \tau) \approx R(t_o) - \hat{g}_l V(t_o + \tau - \hat{t}_l). \tag{15}$$

C. Step 1: Deciding on Observer Impulse

This section deals with the first step in the impulsive observer algorithm of Section III-A. In this step, the observer propagates the state estimation $\hat{x}(t)$ according to (4) and computes $R(t)$, defined in (13), for each t. This continues until some time t_o, when it is decided that there should be an observer impulse in the interval $[t_o, t_o + \tau)$.

In order to see whether or not an observer impulse should be added, $\|R(t)\|$ will be considered. Notice that if $\varepsilon(t) = 0$ and $\mathcal{T} \cap [t, t + \tau) = \emptyset$, then it follows from Proposition 3.1 that $R(t) = 0$. However, as soon as an impulse at t_k enters the sliding window $[t, t + \tau)$, the norm of $R(t)$ will start to increase according to the relationship

$$\|R(t)\| = |g_k| \, \|V(t + \tau - t_k)B\|, \quad 0 < t_k - t < \tau. \tag{16}$$

Thus, if $\|R(t)\| > 0$, then $\varepsilon(t) \ne 0$ and/or $\mathcal{T} \cap [t, t + \tau) \ne \emptyset$. In both cases, there is a reason to add an observer impulse. If $\varepsilon(t_o) \ne 0$, then the observer impulse could be used to reduce the state estimation error, and if there is a plant impulse within $[t, t + \tau)$, then the observer should counter it with an observer impulse.

Thus, a condition for choosing t_o is that $\|R(t_o)\| > 0$. For robustness sake, zero in the right-hand side of the inequality can be replaced by

$$\|R(t_o)\| > \eta,$$

for some threshold $\eta > 0$.

It is also desirable, as motivated in Section III-D, that (15) holds at time t_o. Due to Proposition 3.2, it is thus preferable to choose t_o so that there is no plant impulse in $[t_o + \tau - \epsilon, t_o + \tau)$, and the designer should take this into account when picking η.

Since the way in which $\|R(t)\|$ is affected by an impulse is known beforehand, this is usually not a problem, at least when some bounds on $|g_k|$ are known. For instance, the interval of admissible impulse weights is always known in pulse-modulated control. Notice that if $|g_k| \le g_M$ for all k, then η could be chosen so that

$$\eta > g_M \left\|V(\tau - \hat{t})B\right\|, \quad \text{for } \tau - \epsilon < \hat{t} < \tau,$$

cf. (16). However, if η is chosen too large, small impulses might be missed.

It is often possible to devise a more advanced condition for choosing t_o, by studying the graph of

$$\|V(\tau - t_k)B\|, \quad 0 < t_k < \tau.$$

In this way it might be possible to choose t_o in a way that does not depend on the impulse weights. An example of this is provided in Section IV.

D. Step 2: Evaluating the Observer Impulse

This section deals with the second step in the impulsive observer algorithm of Section III-A.

Suppose that it has been decided at time t_o that there should be an observer impulse in $[t_o, t_o + \tau)$, see Section III-C. Let $\varepsilon_o = \varepsilon(t_o)$. Assume also that there is a plant impulse with weight g_k at time $t_k \in [t_o, t_o + \tau)$. If this is not the case, then $g_k = 0$ throughout this section.

When $\varepsilon_o = 0$, it follows from Proposition 3.1 that t_k and g_k can be found by solving

$$R(t_o) = \hat{g}_l V(t_o + \tau - \hat{t}_l)B. \tag{17}$$

However, for $\varepsilon_o \neq 0$, the above equation may not have a solution. Therefore, the following optimization problem is solved instead

$$\begin{aligned} \min_{\hat{t}_l, \hat{g}_l} \quad & \left\| R(t_o) - \hat{g}_l V(t_o + \tau - \hat{t}_l)B \right\|^2, \\ \text{s.t} \quad & t_o < \hat{t}_l < t_o + \tau. \end{aligned} \tag{18}$$

A suitable technique for solving this problem is discussed in Section III-E.

For $\varepsilon_o = 0$, (17) is always satisfied by choosing $\hat{t}_l = t_k$ and $\hat{g}_l = g_k$. Numerical experiments indicate that this solution is also unique, but this result is not formally proved here.

To analyze what happens when $\varepsilon_o \neq 0$, assume that an observer impulse is fired at $\hat{t}_l$. If Λ has been chosen so that Proposition 3.2 holds for a small enough r, and there is no impulse in $[t_o + \tau - \epsilon, t_o + \tau)$ (see Section III-C), then it can be concluded from (15) that

$$\|\varepsilon(t_o + \tau)\|_{\mathcal{V}}^2 \approx \left\| R(t_o) - \hat{g}_l V(t_o + \tau - \hat{t}_l)B \right\|^2. \tag{19}$$

The right-hand side of (19) is exactly the quantity that is minimized in (18). Also note that thanks to the minimization,

$$\left\| R(t_o) - \hat{g}_l V(t_o + \tau - \hat{t}_l)B \right\| \leq \left\| e^{D\tau} \varepsilon_o \right\|_{\mathcal{V}}.$$

Hence, if $\varepsilon_o \neq 0$, the observer might add an impulse that does not correspond to an impulse in the plant. However, in this case the "false" impulse is chosen so that

$$\|\varepsilon(t_o + \tau)\|_{\mathcal{V}} \lesssim \left\| e^{D\tau} \varepsilon_o \right\|_{\mathcal{V}}$$

where the right-hand side is the state estimation error that is acquired when the observer impulse coincides with the plant impulse.

In practice, the observer usually adds "false" impulses when the state estimation error is large, resulting in a faster convergence (see also Section IV). As the state estimation error decays over time, the observer impulses will get closer to the true plant impulses, as seen in Section IV.

E. SOLVING THE OPTIMIZATION PROBLEM

Solving optimization problem (18) is an important part of the observer described above. Notice that the cost function depends linearly on $\hat{g}_l$. It follows that, if the pair $\hat{t}_l, \hat{g}_l$ minimizes the cost function in (18) then

$$\hat{g}_l = (B^T V^T V B)^{-1} B^T V^T R(t_o),$$

where $V = V(t_o + \tau - \hat{t}_l)$. Inserting this into (18) gives rise to a nonlinear optimization problem in one variable. Thanks to the constraint $\hat{t}_l \in [t_o, t_o + \tau)$, this problem can be solved with an arbitrary accuracy by gridding the interval, evaluating the cost function at each point, and selecting the solution that corresponds to the least cost function value.

F. THE OBSERVER PARAMETERS

Besides the condition discussed in Section III-C (e.g. choosing the threshold η), the designer has to choose the observer gain matrix K, the length of the sliding window τ, and the set Λ.

The proof of Proposition 3.2 shows that letting the elements of Λ tend to negative infinity ensures that the approximation in (15) is valid. However, other aspects have also to be taken into account. In [13] and [11], the low-pass characteristics and disturbance attenuation of the operator (10) are studied. It is demonstrated that the operator in general is less sensitive to measurement noise when $|\lambda|$ is small. Thus there is a trade-off between making r in Proposition 3.2 small, and decreasing the sensitivity to noise. Fortunately, this trade-off can usually be handled in practice, as shown in Section IV. It is also seen that the sensitivity to high frequency noise is reduced when τ is increased. However, increasing τ leads to a longer lag in the state estimate. Furthermore, τ must be chosen less than Φ, which limitation is set by the plant characteristics.

Following [14], it is also of interest to study the sensitivity of (10) to structured uncertainty in the system matrix of the plant, and how it is influenced by the choice of Λ and τ.

Finally, the observer gain K has to be selected. This will mainly affect the state estimates behaviour in between impulses, and could be chosen e.g. as in a (steady-state) Kalman filter.

IV. NUMERICAL EXAMPLE

In this section, the proposed technique for state estimation is validated on a numerical example, both with and without measurement noise. The observer is applied to sampled data, with fast sampling (0.01 time units between each sample), to imitate continuous execution.

Assume the following values in (1)-(3)

$$A = \begin{bmatrix} -0.08 & 0 & 0 \\ 2 & -0.15 & 0 \\ 0 & 0.5 & -0.2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \qquad x_o = \begin{bmatrix} 4 \\ 20 \\ 60 \end{bmatrix}$$

with four impacting impulses

$$\begin{array}{llll} t_1 = 10 & t_2 = 40, & t_3 = 80 & t_4 = 140 \\ g_1 = 4 & g_2 = 5 & g_3 = 7 & g_4 = 6. \end{array}$$

To see the effect of measurement noise, white noise of zero mean and variance 10 was added to the output. The output $y(t)$ of the system is shown in Fig. 1. For the observer, the gain was chosen as

$$K = \begin{bmatrix} 0.0002 \\ 0.0048 \\ 0.0200 \end{bmatrix},$$

so that the eigenvalues of D are placed at $-0.1, -0.15$ and -0.2. The length of the sliding window was set to $\tau = 20$.

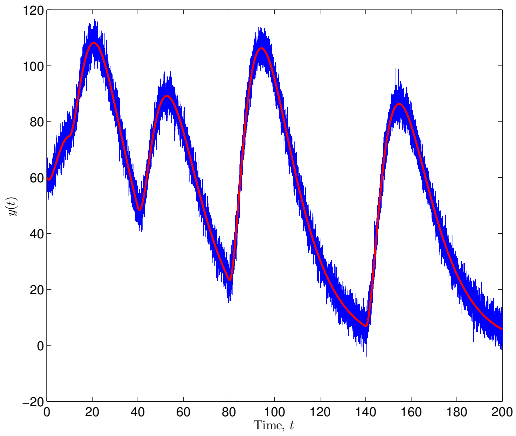

Fig. 1. The output signal of the plant, with noise (blue line) and without (red line).

Finally, Λ was chosen as

$$\Lambda = \{-2.65, -2.70, -2.75, -2.80, -2.85, -2.90\}.$$

For this set of parameters, it holds that

$$|E_i(\hat{t} - t_o)| < 10^{-7} |V_i(t_o + \tau - \hat{t})B|,$$

when $\hat{t} - t_o \leq 15$, cf. Proposition 3.2.

Thus, if the optimization problem in (18) is only solved when $\mathcal{T} \cap [t_o + 15, t_o + 20) = \emptyset$, then (19) is a good approximation.

Next a condition for adding an observer impulse should be chosen. In light of the discussion in Section III-C, first consider $\|V(t_o + \tau - t_k)B\|$, which quantity is plotted in Fig. 2. The maximum in Fig. 2 occurs when $t_k - t_o \approx 5.8$, i.e. well below 15. This suggests choosing the optimization

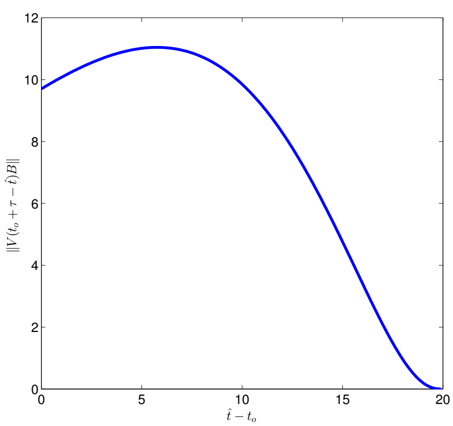

Fig. 2. $\|V(t_o + \tau - t_k)B\|$ for $0 < t_k - t_o < \tau$.

instants t_o in the observer such that the quantity $\|R(t)\|$ is at, or near, a maximum. Also, with this condition, the chosen optimization instants will not depend on the impulse weights. Hence, in this example, Step 1 of the algorithm in Section III-A will continue until $\|R(t)\|$ reaches a maximum.

First the noise-free output was tested. In Fig. 3, the state estimates produced by the observer initialized with

$$\hat{x}(0) = \begin{bmatrix} 7 \\ 30 \\ 70 \end{bmatrix}$$

are shown together with true values of the plant states. The state estimation error $x(t) - \hat{x}(t)$, together with the quantity $e^{Dt}(x(0) - \hat{x}(0))$ are provided in Fig. 4. The latter characterizes the state estimation error that the observer would have with all observer impulses being identical to the plant impulses. It can be seen that the proposed observer

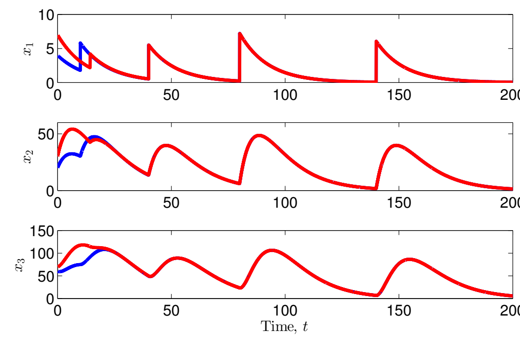

Fig. 3. True states $x(t)$ (blue line), and estimated states $\hat{x}(t)$ (red line).

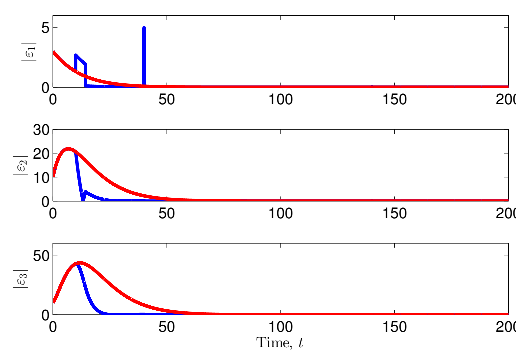

Fig. 4. State estimation error $|x(t) - \hat{x}(t)|$ (blue line), and the state estimation error for an observer with exact knowledge of the plant impulses (red line).

is completely off on the first impulse, but yet converges to the true state vector faster than it would if the first impulse were identical to the true plant impulse.

The observer was also tested with white measurement noise of variance 10 added to the output, see Fig. 1. The resulting state estimation error is presented in Fig. 5. Obviously, the estimated impulses are quite far from the true ones because of the measurement noise. In Section III-F it is seen that one way to reduce the sensitivity to noise is to move the elements of Λ closer to the origin. Therefore, the observer was also tested with

$$\Lambda = \{-0.65, -0.70, -0.75, -0.80, -0.85, -0.90\}$$

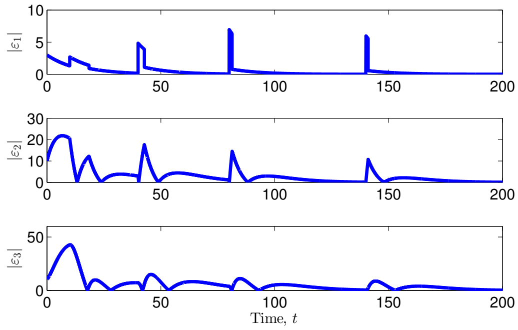

Fig. 5. State estimation error $|x(t) - \hat{x}(t)|$, with original Λ, when measurement noise is added to the output.

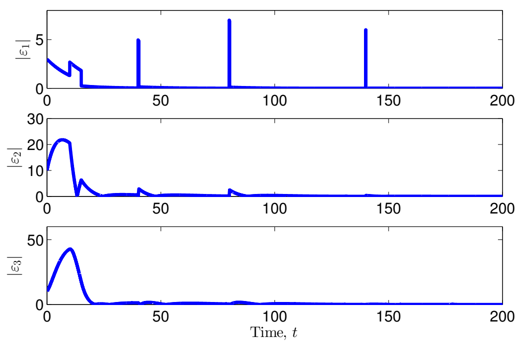

Fig. 6. State estimation error $|x(t) - \hat{x}(t)|$, with modified Λ, when measurement noise is added to the output.

and the rest of the parameters left unchanged. The result for these parameters, with the noisy measurements of the output, is shown in Fig. 6. Clearly, the estimated impulses are much closer to the actual impulses in this case.

V. Conclusion

An observer for state estimation in linear time-invariant systems with unknown input impulses is suggested. The core of the method is to use a standard linear observer coupled with a state estimation algorithm employing finite-memory convolution operators. A numerical example demonstrates the feasibility of the proposed observer.

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Appendix I
Proof of Proposition 3.2

The relation in (14) follows from the fact that

$$\varepsilon(t_o + \tau) = e^{D\tau}\varepsilon(t_o) + g_k e^{D(t_o+\tau-t_k)}B - \hat{g}_l e^{D(t_o+\tau-\hat{t}_l)}B.$$

To prove the last part of the proposition assume, without loss of generality, that $t_o = 0$.

Let $C_i = C(\lambda_i I - D)^{-1}$. The rows of $E(\hat{t})$ are given by

$$E_i(\hat{t}) = C_i\left(e^{(\lambda_i I-D)\tau} - e^{(\lambda_i I-D)(\tau-\hat{t})}\right)e^{D(\tau-\hat{t})}B = e^{\lambda_i(\tau-\hat{t})}C_i\left(e^{(\lambda_i I-D)\hat{t}} - I\right)B.$$

Notice that, for fixed $\hat{t} > 0$,

$$\left|C_i(e^{(\lambda_i I-D)\hat{t}} - I)B\right| \to 0, \text{ as } \lambda_i \to -\infty.$$

Similarily, the rows of $V(\tau - \hat{t})B$ are found to be

$$V_i(\hat{t})B = C_i\left(e^{(\lambda_i I-D)(\tau-\hat{t})} - I\right)e^{D(\tau-\hat{t})}B = -e^{\lambda_i(\tau-\hat{t})}C_i\left(e^{(D-\lambda_i I)(\tau-\hat{t})} - I\right)B,$$

Note that, for fixed $0 < \hat{t} < \tau$,

$$\left|C_i(e^{(D-\lambda_i I)(\tau-\hat{t})} - I)B\right| \to \infty, \text{ as } \lambda_i \to -\infty.$$

Hence, it can be concluded that, for $0 < \hat{t} < \tau$,

$$\frac{|E_i(\hat{t})|}{|V_i(\hat{t})B|} = \frac{|C_i(e^{(\lambda_i I-D)\hat{t}} - I)B|}{|C_i(e^{(D-\lambda_i I)(\tau-\hat{t})} - I)B|} \to 0, \text{ as } \lambda_i \to -\infty$$

and thus the result of the proposition follows.