

# Optimized Distributed Control and Topology Design for Hierarchically Interconnected Systems

Martin Jilg<sup>1</sup> and Olaf Stursberg<sup>1</sup>

**Abstract**—A method to reduce the computational complexity for the simultaneous design of a communication topology and feedback control laws for large scale systems is proposed. In general, such combined procedures may be posed as Mixed-Integer Programs (MIPs), which suffer from high combinatorial complexity when the number of possible communication links grows large. Although some explicit solutions for MIP formulations exist, these are either based on very restrictive assumptions or yield an iterative LMI procedure, which is computationally expensive. The presented scheme tackles the problem by pre-analyzing the coupling structure of the plant and dividing it into hierarchically coupled, distinct groups (clusters). This enables one to decompose the global MIP into a set of smaller MIPs, which can then be solved independently. In contrast to existing approaches, a global LMI optimization has to be solved only once, not repeatedly.

## I. INTRODUCTION

The control of distributed, interconnected systems, which are often referred to as *large-scale systems*, is a very challenging problem and has attained considerable attention in the control community. First approaches to control large-scale dynamical systems were based on *decentralized* control [18]. Decentralized control aims to stabilize a large-scale system by a set of local controllers, where only local inputs and outputs are available to each controller. For this concept, the focus was on the question how to divide an interconnected system such that stability analysis and control design may be performed independently for single subsystems, i.e. without explicitly taking into account the overall system. This led to the development of graph-based decomposition methods [19].

Current approaches make explicit use of the possibility to exchange information over the communication networks mentioned before [3]. These *distributed* control laws take into account the states of neighboring subsystems, what typically results in a better performance of the closed-loop system. Moreover, a distributed controller is capable of stabilizing a system when no stabilizing decentralized controller exists, which is the case if the system contains unstable decentralized fixed modes (DFMs) [2].

There are several known approaches that consider the problem of designing an optimal distributed controller, i.e. a distributed control law that minimizes a global cost function. For the special case of identical and dynamically decoupled subsystems, the global LQR problem can be reduced to a single subproblem [4]. In [14], dual decomposition is used to

obtain a distributed algorithm that optimizes local controllers in an iterative scheme. In these approaches, it is assumed that the communication topology of the controller, which actually is an additional degree of freedom in distributed control design, is given a priori.

There exist novel approaches that explicitly incorporate the design of the communication topology in the overall control design. Since the decision whether to involve a specific communication link in the control design or not is binary, it is natural that these approaches result in MIP formulations. The approach in [13] considers the special case of topology design for optimal average-consensus, and the global control design procedure is formulated as a MISDP. In [8], the authors consider discrete time LTI systems that are coupled through states, inputs, and a quadratic performance criterion. They propose an MISDP approach for the simultaneous optimization of a distributed controller and the underlying communication topology, which is based on a convex LMI reformulation of the originally nonconvex problem. Furthermore, communication cost and constraints on the network topology are taken into account.

In general, MIP formulations suffer from the exponentially growing number of possible combinations for large numbers of binary decision variables. Therefore, explicit solutions have been proposed to avoid this issue. The approach in [9] proposes a performance-oriented control design for continuous time LTI systems, where a stabilizing decentralized controller is assumed to exist. Additional communication links are added in a subsequent step, where the convergence rate of the system is used as performance measure. However, the proposed explicit design procedure requires very restrictive assumptions, like scalar and homogeneous subsystems, a symmetric system matrix  $A$ , bidirectional communication and equal cost of each link. The objective in [16] is to find a minimal number of communication links that guarantees a pre-defined  $\mathcal{H}_\infty$ -performance for the closed-loop, where the system is considered as a set of dynamically coupled LTI subsystems with local inputs. The authors propose a convex relaxation of the cost function, which renders the problem non-combinatorial. However, the optimization problem involves a nonlinear matrix inequality, which is tackled by an iterative LMI scheme. Thus, the proposed approach still requires a high computational effort when the system dimension gets large.

A tradeoff between sparsity and  $\mathcal{H}_2$ -performance of a state feedback controller is sought in [10]. The proposed algorithm iterates over a set of Lyapunov and Sylvester equations. However, constraints and fixed costs on single

<sup>1</sup>M. Jilg and O. Stursberg are with the Institute of Control and System Theory, University of Kassel, Germany. E-Mail: {martin.jilg, stursberg}@uni-kassel.de

communication links are not considered, and the global system description is used for the optimization.

This paper proposes a new approach to reduce the computational complexity for the design of a topology-optimized controller for an interconnected system. The system is considered as a collection of discrete time LTI subsystems, coupled via states and a quadratic performance index, as defined in Sec. II. Some preliminary results are briefly reviewed in Sec. III. In Sec. IV, a decomposition algorithm is proposed that analyzes the interconnection structure of the plant and groups the subsystems into hierarchically interconnected clusters. By solving a global LMI optimization, preliminary controllers are designed for the clusters. These preliminary controllers are used afterwards to obtain a cost function for each cluster. Hence, the global problem is divided into independent, local problems. Subsequently, controllers with optimized communication topology are computed for the clusters, based on an extension of [8]. The simulation results in Sec. V show that the obtained performance is quite similar compared to a global MISDP approach, while the computation time is significantly reduced. Finally, the presented work is summarized and concluded in Sec. VI.

## II. PROBLEM DEFINITION

Assume that the global plant  $\mathcal{S}$  is given as an LTI system in discrete time of the form

$$\mathcal{S}: \quad x_{k+1}^{(g)} = Ax_k^{(g)} + Bu_k^{(g)}, \quad (1)$$

with global state vector  $x_k^{(g)} := x^{(g)}(kT) \in \mathbb{R}^{n_g}$ ,  $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ , discretization time  $T \in \mathbb{R}$ ,  $T > 0$ , and global input vector  $u_k^{(g)} \in \mathbb{R}^{m_g}$ . The matrices  $A \in \mathbb{R}^{n_g \times n_g}$  and  $B \in \mathbb{R}^{n_g \times m_g}$  are partitioned as follows:

$$A = \begin{bmatrix} A_{1,1} & \dots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \dots & A_{N,N} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_N \end{bmatrix}. \quad (2)$$

Hence, the global system  $\mathcal{S}$  is a collection of  $N$  interconnected subsystems

$$\mathcal{S}_i: \quad x_{k+1}^{(i)} = A_{i,i}x_k^{(i)} + B_i u_k^{(i)} + \sum_{j=1, j \neq i}^N A_{i,j}x_k^{(j)} \quad (3)$$

with local state vector  $x_k^{(i)} \in \mathbb{R}^{n_i}$ , local input vector  $u_k^{(i)} \in \mathbb{R}^{m_i}$ ,  $A_{i,j} \in \mathbb{R}^{n_i \times n_j}$  and  $B_i \in \mathbb{R}^{n_i \times m_i}$ . Note that the block diagonal structure of  $B$  implies that the subsystems  $\mathcal{S}_i$  are not coupled via inputs<sup>1</sup>. In addition, we consider the following assumptions to hold.

**Assumption 1** *The pair  $(A, B)$  is stabilizable.*

**Assumption 2**  *$A_{i,i} \neq 0^{n_i \times n_i} \quad \forall i$  and  $B_i \neq 0^{n_i \times m_i} \quad \forall i$*

**Assumption 3** *Each subsystem  $\mathcal{S}_i$  is able to measure its local state vector  $x_k^{(i)}$  at time  $t = kT$ .*

**Assumption 4** *The communication network is not subject to failure nor to time delay, such that any  $\mathcal{S}_i$  has timely knowledge of any connected state of another subsystem  $\mathcal{S}_{j \neq i}$ .*

<sup>1</sup>The proposed procedure may also deal with input couplings as long as they do not couple any two subsystems that are not in the same cluster (see sec. IV).

The control goal is to determine a distributed linear state-feedback controller of the form

$$u_k^{(g)} = Kx_k^{(g)} = \begin{bmatrix} K_{1,1} & \dots & K_{1,N} \\ \vdots & \ddots & \vdots \\ K_{N,1} & \dots & K_{N,N} \end{bmatrix} x_k^{(g)}, \quad (4)$$

such that the infinite horizon quadratic cost function

$$V^*(x_k^{(g)}) = \min_{u^{(g)}} \sum_{l=0}^{\infty} \left( x_{k+l}^{(g)} \right)^T Q x_{k+l}^{(g)} + \left( u_{k+l}^{(g)} \right)^T R u_{k+l}^{(g)} \quad (5)$$

for the closed-loop system resulting from (1) with (4) is minimized. In (5),  $Q = Q^T > 0$ ,  $Q \in \mathbb{R}^{n_g \times n_g}$  and  $R = R^T > 0$ ,  $R \in \mathbb{R}^{m_g \times m_g}$  are weighting matrices for the states and the inputs, respectively.

The controller (4) requires the exchange of state information between the subsystems. More precisely, a communication link from subsystem  $\mathcal{S}_j$  to subsystem  $\mathcal{S}_i$  is required for each  $K_{i,j} \neq 0^{m_i \times n_j}$ . The communication topology that is associated with a controller  $K$  is captured in a directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  with a finite set of nodes  $\mathcal{V} = \{1, \dots, N\}$  and directed edges  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ . Hence, a communication link from subsystem  $\mathcal{S}_i$  to subsystem  $\mathcal{S}_j$  is represented by the directed edge  $(i, j) \in \mathcal{E}$ . Note that assumption 3 implies that  $(i, i) \in \mathcal{E}$ , i.e. each subsystem can access its local states. In the following, let the directed graph  $\mathcal{G}$  be represented by its associated adjacency matrix  $D = [\delta_{i,j}]$ ,  $D \in \{0, 1\}^{N \times N}$ . The entries  $\delta_{i,j}$  of  $D$  are determined by the rule [7]:

$$\delta_{i,j} = \begin{cases} 1 & \text{if } (j, i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

An entry  $\delta_{i,j}$  indicates whether or not information is communicated from  $\mathcal{S}_j$  to  $\mathcal{S}_i$ . This results in the following logical implication on the structure of the controller (4):

$$(\delta_{i,j} = 0) \Rightarrow (K_{i,j} = 0). \quad (7)$$

Moreover, we introduce the communication cost function  $J_{com}$  to capture the total cost of a topology  $D$ :

$$J_{com}(D) = \sum_{i=1}^N \sum_{j=1, j \neq i}^N c_{i,j} \delta_{i,j}. \quad (8)$$

The weights  $c_{i,j} \geq 0$  of the weight matrix  $C = [c_{i,j}] \in \mathbb{R}^{N \times N}$ , with  $c_{i,i} = 0 \quad \forall i$ , encode the cost for a communication link between the  $j$ -th and the  $i$ -th subsystem.  $C$  may be based on hardware cost or the spatial distance of the corresponding subsystems. Finally,  $\mathcal{D} \subseteq \{0, 1\}^{N \times N}$  defines a set of admissible network topologies. Substituting (4) into (5), the considered problem can now be stated as:

$$\min_{K, D} \sum_{l=0}^{\infty} x_{k+l}^{(g)T} (Q + K^T R K) x_{k+l}^{(g)} + J_{com}(D) \quad (9)$$

$$\text{s.t. (1), (7), } D \in \mathcal{D}, \|\lambda_p(A + BK)\|_2 < 1 \quad \forall p \in \{1 \dots n_g\},$$

where  $\|m\|_2$  denotes the 2-norm of a complex number  $m \in \mathbb{C}$  and  $\lambda_p(M)$  denotes the  $p$ -th eigenvalue of a matrix  $M$ .

## III. PRELIMINARY RESULTS

### A. Hierarchical LBT-Decomposition

In [19], the author proposes various kinds of decomposition methods to partition a large system into subsystems. Decomposition algorithms provide additional insight into the

system structure and, in most cases, only require relatively low computational effort. Subsequently, we focus on the *Hierarchical LBT decomposition* of a dynamical system due to its special properties.

Consider the system defined in (1). The Hierarchical LBT decomposition permutes the block rows and columns of  $A$  such that it is brought into a lower block triangular (LBT) form. This decomposition corresponds to identifying the strongly connected components of the interconnection graph underlying the system. Each block on the diagonal of the LBT matrix  $A$  represents a set of strongly coupled subsystems. The obtained system structure can be interpreted as a hierarchy of subsystems, where subsystems on any layer of the hierarchy are not influenced by subsystems belonging to the lower layers. This is illustrated in figure 1 for a system consisting of three subsystems.

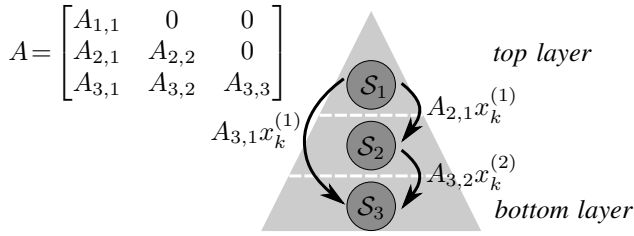


Fig. 1. Subsystem hierarchy imposed by a lower block-triangular structure of the system matrix  $A$ .

According to [19], a hierarchical LBT decomposed system has the following properties:

- The controllability of each subsystem implies the controllability of the overall system
- The stability of each subsystem implies the stability of the overall system

The decomposition method that will be presented in Sec. III is based on the upper block triangular (UBT) form of a matrix. The following lemma states that both forms are similar to each other, which means that the properties of a system in LBT form also apply to a system in UBT form.

**Lemma 1 Similarity of UBT and LBT form.** *Each system of the form (1) with  $A$  upper block triangular and  $B$  block diagonal can be transformed into a system  $(\hat{A}, \hat{B})$  with  $\hat{A}$  lower block triangular and  $\hat{B}$  block diagonal and vice versa. Let the total number and the dimensions of diagonal blocks  $A_{i,i}$  of  $A$  be denoted by  $N$  and  $n_i$ , respectively,  $i \in \{1 \dots N\}$ . Let  $B = \text{diag}(B_i)$  with  $B_i \in \mathbb{R}^{n_i \times m_i}$ . Then, there exist permutation matrices  $T$  and  $T_B$  such that:  $\hat{A} = T^{-1}AT$  and  $\hat{B} = T^{-1}BT_B$ , where  $T^{-1} := \text{adiag}(I_{n_1} \dots I_{n_N}) = T^T \in \mathbb{R}^{n_g \times n_g}$ ,  $n_g = \sum n_i$ , is a block anti-diagonal matrix with unit matrices  $I_{n_i}$  of dimension  $n_i$  on the anti-diagonal and  $T_B := \text{adiag}(I_{m_N} \dots I_{m_1})$ .  $\square$*

The proof of lemma 1 is based on the inherent properties of  $T$  and  $T_B$  and is omitted due to space limitations.

### B. Optimized Distributed Control and Topology Design

This subsection briefly summarizes the design procedure of a distributed state-feedback controller with optimized network topology, which is proposed in [8].

Consider the system dynamics (1) with controller (4). The goal is to find a distributed control law  $K$  with underlying communication topology  $D \in \mathcal{D}$  that is optimal with respect to the sum of the quadratic infinite horizon cost (5) and communication cost (8). Utilizing the fact the the optimal infinite horizon cost function is a quadratic function [1]:

$$V^*(x_k^{(g)}) = (x_k^{(g)})^T P^* x_k^{(g)}, \quad (10)$$

with  $P^* = P^{*T} > 0$ , leads to the following proposition:

**Proposition 1** [8] *The pair  $P$  and  $K$  is optimal w.r.t. (5), and only if it minimizes the following optimization problem:*

$$\min_{P,K} \text{trace}(P) \quad \text{s.t.} \quad P = P^T > 0, \quad (11)$$

$$(P - (A + BK)^T P (A + BK) - Q - K^T R K) \geq 0 \quad (12)$$

$\square$

Applying this proposition, optimization (9) can be rewritten:

$$\min_{P,K,D} \text{trace}(P) + J_{\text{com}}(D) \quad (13)$$

$$\text{s.t. (7), (12), } D \in \mathcal{D}, P = P^T > 0$$

Since the optimality constraint (12) is a BMI in  $P$  and  $K$ , the resulting optimization problem is nonconvex. In [8], Groß and Stursberg propose a convex LMI reformulation of this BMI constraint, and implement the logical implication constraint (7) by means of a Big-M method [20]. Their result is summarized in the following theorem.

**Theorem 1** [8] *Suppose the matrices  $\hat{P}, Y, G, L, D$  are an optimal and feasible solution of the optimization problem:*

$$\min_{\hat{P}, Y, G, L, D} \text{trace}(\hat{P}) + J_{\text{com}}(D) \quad \text{s.t.} \quad (14)$$

$$\begin{bmatrix} G + G^T - Y & (AG + BL)^T & G^T & L^T \\ AG + BL & Y & 0 & 0 \\ G & 0 & Q^{-1} & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} > 0 \quad (15)$$

$$\begin{bmatrix} \hat{P} & I \\ I & Y \end{bmatrix} > 0 \quad (16)$$

$$Y = Y^T > 0, \hat{P} = \hat{P}^T > 0, G > 0, D \in \mathcal{D}$$

$$-M\delta_{i,j} \preceq L_{i,j} \preceq M\delta_{i,j} \quad \forall i, j \in \mathcal{N}$$

$$-M\delta_{i,j} \preceq G_{i,j} \preceq M\delta_{i,j} \quad \forall i, j \in \mathcal{N}$$

$$-M(\delta_{i,j} - \delta_{i,z} + 1) \preceq L_{i,j} \preceq M(\delta_{i,j} - \delta_{i,z} + 1) \quad \forall i, j, z \in \mathcal{N}$$

where  $I$  and  $0$  denote unit and zero matrices of appropriate dimensions, respectively,  $M$  is a matrix of sufficiently large numbers and appropriate dimensions,  $\preceq$  denotes an elementwise inequality, and  $\mathcal{N} := \{1 \dots N\}$ . Then, it holds that:

A) The non-convex constraint (12) is satisfied with  $P = Y^{-1}$  and  $K = LG^{-1}$ .

B) The controller  $K = LG^{-1}$  fulfills the structural constraint (7) imposed by the communication graph  $\mathcal{G}$ .  $\square$

## IV. PROPOSED METHOD

The proposed control design procedure is organized in five steps. At first, a decomposition algorithm performs an analysis of the interconnection structure of the plant. This algorithm ends up with clusters of subsystems, which exhibit the properties of the mentioned hierarchical LBT

decomposition. In the subsequent step, a preliminary controller is designed for the clustered system. Therefore, the communication topology is fixed to full communication within each cluster and no communication between any two distinct clusters. Based on the preliminary controller, local cost functions are derived for the clusters. This enables one to solve an extended version of (14) for every single cluster, which leads to the desired topology optimized controllers. Summing up, the mentioned procedure provides a way to divide the global MISDP (9) into separate, independent MISDPs of considerably lower dimension and smaller combinatorial complexity. In the final step, the results of these independent local problems are merged to a global controller and the global performance is evaluated.

### A. Subsystem Clustering

The proposed clustering algorithm divides the global system  $\mathcal{S}$  into  $N_c$  distinct groups of subsystems. We call such a group of subsystems a *cluster*. With each cluster, an index set  $\mathcal{C}_i \subseteq \{1 \dots N\}$ ,  $i \in \{1 \dots N_c\}$  is associated, which contains the indices  $j$  of all subsystems  $\mathcal{S}_j$  that belong to this cluster.

1.) Map the coupling topology of the plant into a matrix  $\Gamma^0 = [\gamma_{i,j}^0]$ ,  $\Gamma^0 \in \mathbb{N}_0^{N \times N}$ :

```

for  $i = 1 \dots N$  do
  for  $j = 1 \dots N$  do
     $\gamma_{i,j}^0 :=$  number of entries  $\neq 0$  in matrix  $A_{i,j}$ .
  end for
end for

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2.) Apply the *Dulmage-Mendelsohn-Decomposition* (DM-Decomposition) [12] to matrix  $\Gamma^0$ , which permutes the rows and columns of  $\Gamma^0$  such that the permuted matrix  $\hat{\Gamma} = [\hat{\gamma}_{i,j}]$  is UBT. Mathematically, the permutation is either described by a left and a right permutation matrix,  $U^0, V^0 \in \{0, 1\}^{N \times N}$ , respectively, with  $\hat{\Gamma} = U^0 \Gamma^0 V^0$ , or by a pair of index vectors  $p^0, q^0 \in \mathbb{N}^N$ , with  $\hat{\gamma}_{i,j} = \gamma_{p_i^0, q_j^0}^0$ , that contain the permuted indices of the rows and columns of  $\Gamma^0$ , respectively. Note that assumption 2 ensures that  $p^0 = q^0$  [12].

3.) Denote the number of diagonal blocks of  $\hat{\Gamma}$  by  $\hat{N}_c$ . Create an index set  $\hat{\mathcal{C}}_i$ ,  $i \in \{1 \dots \hat{N}_c\}$ , for every diagonal block of  $\hat{\Gamma}$  and store the indices of all subsystems that belong to the  $i$ -th block in  $\hat{\mathcal{C}}_i$ .

4.) Choose a minimal cluster size<sup>2</sup>  $1 < c_{min} < N$ ,  $c_{min} \in \mathbb{N}$ , to classify each cluster  $\hat{\mathcal{C}}_i$  either as a *fixed* cluster  $\mathcal{C}_j$  or as a *free* cluster  $\tilde{\mathcal{C}}_k$ , respectively. The index sets  $\mathcal{I}_j$  and the vector  $\tilde{\eta}_k$  are needed to maintain the hierarchical structure in the final clustering step:

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Define  $j := 1, k := 1$ 
for  $i = 1 \dots \hat{N}_c$  do
  if  $|\hat{\mathcal{C}}_i| \geq c_{min}$  then  $\mathcal{C}_j := \hat{\mathcal{C}}_i$ ,  $\mathcal{I}_j := \{i\}$ ,  $j := j + 1$ 
  else  $\tilde{\mathcal{C}}_k := \hat{\mathcal{C}}_i$ ,  $\tilde{\eta}_k := i$ ,  $k := k + 1$ 
end for
Define  $N_c := j - 1$ ,  $\tilde{N}_c := k - 1$ 

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<sup>2</sup> $c_{min}$  is typically chosen small w.r.t  $N$  to allow enough freedom in the clustering process.

5.) Add every free cluster to a fixed cluster. This decision is based on the number of interconnections between a free cluster  $\tilde{\mathcal{C}}_k$  and a fixed cluster  $\mathcal{C}_j$ . A free cluster  $\tilde{\mathcal{C}}_k$  will be added to the fixed cluster  $\mathcal{C}_j$  with the highest number of interconnections to  $\tilde{\mathcal{C}}_k$ . The size of the resulting *final* clusters  $\mathcal{C}_j$ ,  $i \in \{1 \dots N_c\}$ , is upper bounded by the design parameter  $c_{max} \geq \max\{c_{min} + 1, \max_j |\mathcal{C}_j|\}$ ,  $c_{max} \in \mathbb{N}$ :

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for  $k = 1 \dots \tilde{N}_c$  do
  Define  $\tilde{\mathcal{I}}_k := \{\tilde{\eta}_k - 1, \tilde{\eta}_k + 1\}$ 
  for  $j = 1 \dots N_c$  do
     $\phi_{k,j} := \sum_{p \in \tilde{\mathcal{C}}_k} \sum_{q \in \mathcal{C}_j} (\gamma_{p,q}^0 + \gamma_{q,p}^0)$ 
  end for
   $i := \arg \max_i \phi_{k,i}$  s.t.  $|\mathcal{C}_i| \leq c_{max} - |\tilde{\mathcal{C}}_k|$ ,  $\tilde{\mathcal{I}}_k \cap \mathcal{I}_i \neq \emptyset$ 
  if  $i \neq \emptyset$  then  $\mathcal{C}_i := \mathcal{C}_i \cup \tilde{\mathcal{C}}_k$ ,  $\mathcal{I}_i := \mathcal{I}_i \cup \tilde{\eta}_k$ 
  else  $N_c := N_c + 1$ ,  $\mathcal{C}_{N_c} := \tilde{\mathcal{C}}_k$ ,  $\mathcal{I}_{N_c} := \tilde{\eta}_k$ 
end for

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This step of the clustering algorithm ends up with  $N_c$  hierarchically coupled final clusters  $\mathcal{C}_i$ . Note that the clustering procedure will end up in a single cluster ( $N_c = 1$ ) if the system (1) does not omit a hierarchical coupling structure.

Each final cluster  $\mathcal{C}_i$  defines a collection of subsystems  $\mathcal{S}_i^c = \{\mathcal{S}_j : j \in \mathcal{C}_i\}$  modeled by:

$$\mathcal{S}_i^c : \quad x_{k+1}^{(C_i)} = A_{i,i}^c x_k^{(C_i)} + B_i^c u_k^{(C_i)} + \sum_{j=1, j \neq i}^{N_c} A_{i,j}^c x_k^{(C_j)} \quad (17)$$

with state  $x_{k+1}^{(C_i)} \in \mathbb{R}^{n_i^c}$ , input vector  $u_k^{(C_i)} \in \mathbb{R}^{m_i^c}$ ,  $A_{i,j}^c = [A_{p,q}]_{p \in \mathcal{C}_i, q \in \mathcal{C}_j}$  and  $B_i^c = \text{diag}(B_p)$ ,  $p \in \mathcal{C}_i$ . Before proceeding, the stabilizability of each  $\mathcal{S}_i$  has to be verified, e.g. by means of computing the controllability staircase form of each pair  $(A_i^c, B_i^c)$  and determining the eigenvalues of the noncontrollable system part, cf. [15]. In case any system  $\mathcal{S}_i$  is not stabilizable, steps 4 and 5 of the clustering procedure have to be repeated either with modified  $c_{min}, c_{max}$  or by directly incorporating the stabilizability check<sup>3</sup>.

For a given set of final clusters  $\mathcal{C}_i$ , the permutation matrices  $\hat{U}, \hat{V} \in \{0, 1\}^{n_g \times n_g}$ ,  $\hat{W} \in \{0, 1\}^{m_g \times m_g}$  are constructed such that  $\hat{U} A \hat{V} = [A_{i,j}^c]$  and  $\hat{U} B \hat{W} = \text{diag}(B_i^c)$ , and the permuted subsystem indices are captured in the index vector  $\hat{p} \in \mathbb{N}^{n_g}$ . For the subsequent sections, simplify the notation to  $A := [A_{i,j}^c]$ ,  $B := \text{diag}(B_i^c)$ ,  $Q := \hat{U} Q \hat{V}$  and  $R := \hat{W} R \hat{W}^T$ , and define  $\mathcal{C}_i^c := [c_{p,q}]_{p,q \in \mathcal{C}_i}$ . Let  $D_i^c$  be the topology matrix  $D$  reduced to the rows and columns in  $\mathcal{C}_i$ , and let  $\mathcal{D}_i^c$  denote the set of such reduced matrices for all  $D \in \mathcal{D}^4$ .

### B. Preliminary Controller

Once an appropriate subsystem clustering is obtained, a preliminary control law  $u_k = \tilde{K} x_k$  has to be designed for the clustered overall system. That is, for every cluster  $\mathcal{C}_i$ , we search a preliminary controller  $\tilde{K}_{i,i}^c$  with full communication

<sup>3</sup>Besides the mentioned conditions for adding a free cluster to a fixed cluster, additional or alternative conditions could be implemented. For instance, for distributed plants, a condition based on the spatial arrangement of subsystems may be reasonable.

<sup>4</sup>At this point, it is assumed that the decomposition of  $\mathcal{D}$  induces no couplings between any  $\mathcal{D}_i^c$  and  $\mathcal{D}_j^c$  with  $i \neq j$ .

graph  $\hat{\mathcal{G}}_i^c$ , while there is no communication between any two clusters  $\mathcal{C}_i$  and  $\mathcal{C}_j$  with  $i \neq j$ . This results in the communication structure  $\hat{D} = \text{diag}(\mathbf{1}_{|\mathcal{C}_1|} \dots \mathbf{1}_{|\mathcal{C}_{N_c}|})$ , where  $\mathbf{1}_n := \mathbf{1}^{n \times n}$ . The preliminary controller  $\hat{K}$  can be found by solving (14) for  $A = [A_{i,j}^c]$ ,  $B = \text{diag}(B_i^c)$ ,  $J_{com} = 0$  and  $D = I_{N_c}$ . Since the communication topology  $D$  is fixed, the MISDP is reduced to a single SDP problem. Finally, we define  $\hat{J}^* := \text{trace}(\hat{P})$  with  $\hat{P} = (A + B\hat{K})^T \hat{P} (A + B\hat{K}) + Q + \hat{K}^T R \hat{K}$ . Solving a separate discrete Lyapunov equation is necessary to obtain the global cost, since optimization problem (14) minimizes an upper bound on  $\hat{P}$  (cf. (16)).

### C. Local Inverse Optimal Control Problems

Eventually, the goal is to design an optimized controller  $K_{i,i}^c$  with optimized topology for each cluster  $\mathcal{C}_i$  by solving a set of independent subproblems. While the obtained clusters may be treated as decoupled under the aspect of global system stability (cf. Sec. IV - A), they are still coupled through the global cost function (5). Hence, there is still a lack of local performance criteria for the intended subproblems.

To overcome this, local performance criteria are computed such that the cost emerging from the interactions between the clusters are approximated, assuming that the remaining clusters are equipped with their respective preliminary controllers. The local performance criteria are obtained from the preliminary controllers  $\hat{K}_{i,i}^c$  by solving an inverse optimal control problem, i.e. by determining local cost functions  $V_i^c$  given  $\hat{K}_{i,i}^c$  as the solution of the associated local optimal control problems.

For each cluster, define the local cost function:

$$V_i^c(x_k^{(C_i)}) := \min_{u_k^{(C_i)}} \sum_{l=0}^{\infty} \left[ x_{k+l}^{(C_i)T} Q_i^c x_{k+l}^{(C_i)} + u_{k+l}^{(C_i)T} R_i^c u_{k+l}^{(C_i)} + 2 x_{k+l}^{(C_i)T} S_i^c u_{k+l}^{(C_i)} \right] = \min_{u_k^{(C_i)}} \sum_{l=0}^{\infty} l(x_{k+l}^{(C_i)}, u_{k+l}^{(C_i)}). \quad (18)$$

Each local cost function is parameterized by a set of weighting matrices  $Q_i^c, R_i^c, S_i^c$  of appropriate dimensions.

**Remark 1** *The additional bilinear cost term with weighting matrix  $S_i^c$  is motivated by the observation that a decentralized control law that is globally optimal w.r.t. the cost function (5) may not be locally optimal w.r.t. a local cost function of the same form. The additional bilinear term allows searching for a local cost function in a more general class of quadratic functions.*

**Remark 2** *The proposed approach implies suboptimality of the obtained global solution, since the local cost functions are only valid if the controllers of the remaining clusters stay fixed. However, the computational complexity is significantly reduced, since  $N_c$  lower dimensional mixed-integer subproblems with  $|\mathcal{C}_i|^2 - |\mathcal{C}_i|$  binary variables have to be solved in contrast to one high dimensional mixed-integer problem with  $N^2 - N$  binary variables. Furthermore, it is shown in Sec. V that the obtained solutions omit satisfying performance.*

An inverse optimal control problem can be posed as a collection of LMIs [5]. Extended to the form of the cost

function (18), the LMI feasibility problem

$$P_i^c = (P_i^c)^T > 0, \quad R_i^c = (R_i^c)^T > 0, \quad Q_i^c = (Q_i^c)^T, \quad (19)$$

$$\begin{aligned} & \left( A_{i,i}^c + B_i^c \hat{K}_{i,i}^c \right)^T P_i^c \left( A_{i,i}^c + B_i^c \hat{K}_{i,i}^c \right) - P_i^c + Q_i^c \\ & + \left( \hat{K}_{i,i}^c \right)^T R_i^c \hat{K}_{i,i}^c + S_i^c \hat{K}_{i,i}^c + \left( S_i^c \hat{K}_{i,i}^c \right)^T = 0, \end{aligned} \quad (20)$$

$$(B_i^c)^T P_i^c B_i^c \hat{K}_{i,i}^c + R_i^c \hat{K}_{i,i}^c + (B_i^c)^T P_i^c A_{i,i}^c + (S_i^c)^T = 0, \quad (21)$$

$$\begin{bmatrix} Q_i^c & S_i^c \\ (S_i^c)^T & R_i^c \end{bmatrix} > 0, \quad (22)$$

with variables  $P_i^c, Q_i^c, R_i^c, S_i^c$  is obtained, which has to be solved for every  $i \in \{1 \dots N_c\}$ . In the inverse problem, equations (20) and (21) claim optimality of  $P_i^c, \hat{K}_{i,i}^c$  w.r.t.  $Q_i^c, R_i^c, S_i^c$  [6]. Applying the Schur complement [5] on LMI (22) yields the inequality  $Q_i^c - S_i^c R_i^c^{-1} S_i^c{}^T > 0$ , which is, like the constraints (19), a standard assumption when the cost function is given in the form of (18), cf. [6].

An analysis of the problem (19) - (22) reveals that it is under-determined. The matrices  $P_i^c, Q_i^c, R_i^c, S_i^c$  consist of a total number of  $\pi = 2(n_i^c)^2 + (m_i^c)^2 + n_i^c m_i^c$  free variables. Taking the symmetry conditions (19) as well as the equality constraints (20) and (21) into account, the number of fixed variables is  $\pi_f = n_i^c(n_i^c - 1) + \frac{1}{2}n_i^c(n_i^c + 1) + \frac{1}{2}m_i^c(m_i^c + 1) + m_i^c n_i^c$ , resulting in  $\pi - \pi_f = \frac{1}{2}n_i^c(n_i^c + 1) + \frac{1}{2}m_i^c(m_i^c + 1)$  free variables. Consequently, multiple solutions exist to the problem. This is not surprising, as the utilized formulation of the inverse optimal control problem only fixes the coordinates of the minimum, but neither the optimal cost value nor the curvature of the cost function. Fixing the optimal cost value is essential to preserve the ratio between the ‘‘performance cost’’ (caused by states and inputs) and the communication cost  $J_{com}$  which have both been defined in the global context. The ratio is preserved by scaling each local cost function as  $Q_i^c := \alpha_i Q_i^c, R_i^c := \alpha_i R_i^c, S_i^c := \alpha_i S_i^c$ , with  $\alpha_i := |\mathcal{C}_i| \cdot \text{trace}(\hat{P}) / (N \cdot \text{trace}(P_i^c))$  and setting  $C_i^c := |\mathcal{C}_i| C_i^c / N$ .

The curvature of each local cost function should be adjusted such that the function approximates the change of global performance cost when the parameters of  $\hat{K}_{i,i}^c$  are changed, while  $\hat{K}_{j,j}^c$  is fixed for all  $j \neq i$ . Extending the inverse optimal control problem by a set of constraints in order to approximate the curvature of the global cost function is subject of current research. Possible approaches are the computation of sampling points of the global cost function used to formulate either hard- or soft-constraints on the inverse problem, or the formulation of a second order condition.

**Remark 3** *As stated in [1], there exist state feedback control laws that are not optimal w.r.t. to any (quadratic) cost function. Thus, if problem (19) - (22) has no solution for some cluster  $\mathcal{C}_i$ , a control law optimized w.r.t. the topology can not be computed. In this case, set  $K_{i,i}^c := \hat{K}_{i,i}^c$ .*

### D. Local Distributed Control and Topology Design

After obtaining the local cost functions (18), the decomposition of the global optimization problem into  $N_c$  local problems is completed.

Each local problem is now parametrized as follows: The system description is given by the pair  $A_i^c, B_i^c$ , the cost

function is parametrized by  $Q_i^c, R_i^c, S_i^c$ , communication cost and constraints are defined by  $C_i^c$  and  $D_i^c$ , respectively. Finally, the local optimization variables are denoted by  $P_i^c, Y_i^c, G_i^c, L_i^c, D_i^c$ . In the next step, a final distributed state feedback controller  $K_{i,i}^c$  with optimized performance and communication topology is designed for each of the local problems.

An analysis of the set of local variables reveals that its structure is similar to the one of the global problem (9), unless the quadratic cost function contains an additional bilinear term. Hence, the method described in theorem 1 is, in principle, suitable to solve the local problems.

The following theorem is an extension of theorem 1 which can deal with a bilinear cost term. In order to keep the notation simple, the result is presented by means of a global system representation:

**Proposition 2** *The pair  $P$  and  $K$  is optimal w.r.t. (18), if and only if it minimizes the following optimization problem:*

$$\min_{P,K} \text{trace}(P) \quad \text{s.t.} \quad P = P^T > 0, \quad (23)$$

$$\left( P - (A+BK)^T P (A+BK) - \tilde{Q} - \tilde{K}^T R \tilde{K} \right) \geq 0, \quad (24)$$

with  $\tilde{Q} := Q - SR^{-1}S^T$  and  $\tilde{K} := K + R^{-1}S^T$ .  $\square$

*Proof:* Consider the proof of proposition 1 in [8], which holds if it is shown that (24) is equivalent to

$$(x_k^{(g)})^T P x_k^{(g)} \geq l(x_k^{(g)}) + (x_{k+1}^{(g)})^T P x_{k+1}^{(g)}, \quad (25)$$

where  $l(x_k^{(g)})$  is the step cost of a quadratic performance index of form (18) with controller (4). Rewriting the step cost as proposed in [1] and substituting (4) leads to:

$$l(x_k^{(g)}, u_k^{(g)}) = (x_k^{(g)})^T Q x_k^{(g)} + (u_k^{(g)})^T R u_k^{(g)} + 2(x_k^{(g)})^T S u_k^{(g)} \\ = (x_k^{(g)})^T [\tilde{K}^T R \tilde{K} + \tilde{Q}] x_k^{(g)} = l(x_k^{(g)}) \quad (26)$$

with  $\tilde{Q} := Q - SR^{-1}S^T$  and  $\tilde{K} := K + R^{-1}S^T$ . Considering the system dynamics (1) and control law (4), as well as (26), it is now easy to verify that (24) is equivalent to (25).  $\blacksquare$

The following theorem states the optimization problem (14) in combination with a convex LMI reformulation of constraint (24).

**Theorem 2** *Suppose the matrices  $\hat{P}, Y, G \in \mathbb{R}^{n_g \times n_g}, L \in \mathbb{R}^{m_g \times n_g}$ , and  $D \in \{0, 1\}^{N \times N}$  are an optimal and feasible solution of the following optimization problem:*

$$\min_{\hat{P}, Y, G, L, D} \text{trace}(\hat{P}) + J_{com}(D) \quad \text{s.t.} \quad (27)$$

$$\begin{bmatrix} G+G^T-Y & (AG+BL)^T & (L+R^{-1}S^T G)^T & G^T \\ AG+BL & Y & 0 & 0 \\ L+R^{-1}S^T G & 0 & R^{-1} & 0 \\ G & 0 & 0 & \tilde{Q}^{-1} \end{bmatrix} > 0 \quad (28)$$

$$\begin{bmatrix} \hat{P} & I \\ I & Y \end{bmatrix} > 0$$

$$Y = Y^T > 0, \hat{P} = \hat{P}^T > 0, G > 0, D \in \mathcal{D}$$

$$-M\delta_{i,j} \preceq L_{i,j} \preceq M\delta_{i,j} \quad \forall i, j \in \mathcal{N}$$

$$-M\delta_{i,j} \preceq G_{i,j} \preceq M\delta_{i,j} \quad \forall i, j \in \mathcal{N}$$

$$-M(\delta_{i,j} - \delta_{i,z} + 1) \preceq L_{i,j} \preceq M(\delta_{i,j} - \delta_{i,z} + 1) \quad \forall i, j, z \in \mathcal{N}$$

where  $I$  and  $0$  denote unit and zero matrices of appropriate dimensions, respectively,  $\tilde{Q} := Q - SR^{-1}S^T$ ,  $M$  is a matrix

of sufficiently large numbers and appropriate dimension,  $\mathcal{N} := \{1 \dots N\}$ , and  $\preceq$  denotes an elementwise inequality. Then, it applies that:

A) The non-convex constraint (24) holds with  $P = Y^{-1}$  and  $K = LG^{-1}$ .

B) The controller  $K = LG^{-1}$  satisfies the structural constraint (7) imposed by the communication graph  $\mathcal{G}$ .  $\square$

*Proof:* Starting from proposition 2 under the assumption that  $R = R^T > 0$  and  $\tilde{Q} = \tilde{Q}^T > 0$ , the proof to obtain (28) is very similar to the proof of theorem 1 in [8] and is omitted due to space limitations.  $\blacksquare$

## E. Global Controller and Information Structure

Once a local controller  $K_{i,i}^c$  is determined for every cluster  $\mathcal{C}_i$ , the global control design is given by:

$$K = \text{diag}(K_{i,i}^c), \quad D = \text{diag}(D_i^c). \quad (29)$$

The performance index  $J(K, D)$  of the global controller is obtained from the solution of a Lyapunov equation:

$$(A+BK)^T P (A+BK) + Q + K^T R K = P \quad (30)$$

$$J(K, D) = \text{trace}(P) + J_{com}(D). \quad (31)$$

## V. NUMERICAL EXAMPLE

A randomly generated interconnected system consisting of  $N = 10$  subsystems of random dimension  $n_i \in \{2 \dots 5\}$  is used in order to emphasize the ability of the presented method to deal with a relatively large number of subsystems. The global system is of dimensions  $n_g = 35$ ,  $m_g = 26$ . It is fully reachable and has both stable and unstable eigenvalues. We intentionally omitted the numerical values of these large matrices due to space limitations. Instead, the appearing matrix structures are visualized by collections of colored squares in the following figures. Each square symbolizes a block  $[M_{i,j}]$  of a matrix  $M$ , where the upper left block coincides with  $M_{1,1}$ . A black colored square implies that  $M_{i,j} \neq 0$ , and a grey colored block implies  $M_{i,j} = 0$ .

Fig. 2 shows the structure of the global plant during the clustering process. The initial subsystem order (left graphic) does not allow any insight into the system structure. However, applying the DM-Decomposition reveals that the subsystems can be clustered into four hierarchically coupled groups of subsystems (middle).

These groups are modeled by the index sets:  $\hat{\mathcal{C}}_1 = \{1, 10\}, \hat{\mathcal{C}}_2 = \{4, 8\}, \hat{\mathcal{C}}_3 = \{7, 9\}, \hat{\mathcal{C}}_4 = \{5\}, \hat{\mathcal{C}}_5 = \{2, 3, 6\}$ . With  $c_{min} = 2$ , the fixed clusters  $\mathcal{C}_1 := \hat{\mathcal{C}}_1, \mathcal{C}_2 := \hat{\mathcal{C}}_2, \mathcal{C}_3 := \hat{\mathcal{C}}_3, \mathcal{C}_4 := \hat{\mathcal{C}}_5$  and the free cluster  $\tilde{\mathcal{C}}_1 := \hat{\mathcal{C}}_4$  are obtained.

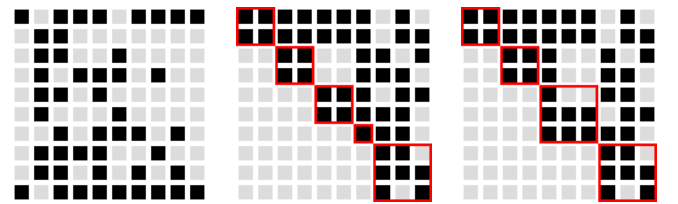
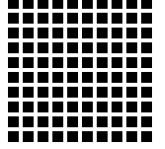
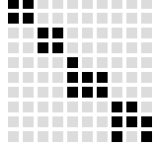
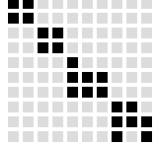


Fig. 2. Structure of the initial plant, after DM-Decomposition (middle), and after final clustering (right). Black boxes denote non-zero matrices  $A_{i,j}$ , the red squares show which subsystems belong to a cluster.

TABLE I  
COMPARATIVE STUDY WITH GLOBAL MISDP-APPROACH

$D$			
method	thm. 1, MISDP	decomp. method	thm. 1, $D$ fixed
trace( $P$ )	2537.3	3258.6	3659.8
$J_{com}(D)$	5000	1100	1100
total cost	7537.3	4358.6	4759.8
com. time	12473 min <sup>5</sup>	8 min	6 min

Setting  $c_{max} = 4$ ,  $\tilde{C}_1$  is added to  $C_3$ , i.e.  $C_3 := C_3 \cup \tilde{C}_1$ . This yields a set of four final clusters (right). The permuted order associated with the final clustering is captured in the index vector  $\hat{p} = [1 \ 10 \ 4 \ 8 \ 5 \ 7 \ 9 \ 2 \ 3 \ 6]$ .

The approach has been implemented in MATLAB, using YALMIP [11] and SeDuMi 1.3 [17], to perform a comparative study with the global approach in theorem 14. For the mentioned system, the global MISDP solution (cf. table I, left column) is compared with the approach on hand (middle). The communication cost is set to  $c_{i,j} = 50 \forall i, j$ . The global cost function is parameterized with  $R = I_{m_g}$ ,  $Q = \text{diag}(1 + 3j), j \in \{0 \dots n_g - 1\}$ . In order to validate the error induced by the local cost approximations, the global problem (cf. theorem 1) is additionally solved with  $D$  being fixed to the topology obtained from the presented approach (right).

The maximum number of branch-and-bound-iterations was set to 2000 in order to limit the computation time. For the global MISDP, the computation of the first 2000 iterations took almost 9 days on an AMD Phenom II X4 925 with 4GB of RAM, which emphasizes the need for a decomposition of the problem. A full communication graph was the best solution obtained so far. Not surprisingly, the global solution offers the best performance cost (trace( $P$ )), while the full usage of interconnections leads to a high total cost value. While the performance cost of the decomposition method is about 28% higher due to the relatively high number of unused communication links, the total cost is about 42% lower. Additionally, the computational time reduces to 8 min. Computing the optimized structured controller for the obtained topology  $D$  from a global viewpoint yields an unexpected result, namely a worse performance compared to the controller obtained from the decomposition approach. This is due to the fact that theorem 1 states a relaxation of the original non-convex optimization problem, rendering the obtained solutions suboptimal. Consequently, by solving the local problems, a global solution may be obtained that would violate the constraints of theorem 1 (due to the relaxation), but is a feasible solution of problem (9).

## VI. CONCLUSION AND FUTURE WORK

An extension and complexity reduction of an MISDP based approach for the simultaneous design of optimized

distributed state feedback controllers and the underlying communication topology is presented. The complexity reduction is accomplished by an analysis of the subsystem interconnection structure prior to the control design. This allows to divide the global control design problem into independent smaller ones. Although a global LMI optimization has to be solved once, the construction of a global MISDP is avoided. Ongoing research deals with the extension of the inverse optimal control problems to approximate the curvature of the global cost by the local cost function. Furthermore, the computation of either bounds or estimates on the suboptimality of the proposed scheme is an open problem. A numerical analysis is complicated by the lack of knowledge about the global optimum of the considered problem class and the high computation time of global approaches.

## VII. ACKNOWLEDGEMENTS

Partial financial support by the EU through the Network of Excellence HYCON2 is gratefully acknowledged.

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<sup>5</sup>Best feasible solution obtained after 2000 branch-and-bound iterations.