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A barrier function based continuous-time algorithm for linear model predictive control

Christian Feller and Christian Ebenbauer *

Abstract— **In this paper, we present a novel linear model predictive control (MPC) scheme that relies on a continuous-time, barrier function based algorithm which asymptotically tracks the solution of a time-varying open-loop optimal control problem. In particular, the control input is obtained as the sampled output of a continuous-time dynamical system and no iterative optimization algorithm is needed in the on-line implementation. In addition, we present a new approach towards stabilizing MPC based on gradient recentered logarithmic barrier functions that allows enlargement of the employed terminal set.**

I. Introduction

Model predictive control (MPC) is a control strategy that computes the control action in a receding horizon fashion by solving on-line repeatedly a finite horizon open-loop optimal control problem in which both system dynamics and constraints are taken into account. We refer the reader to the survey paper [1] and references therein for a detailed account of the theory and history of MPC. Recently, arising from the generally growing demand for advanced control concepts and promoted by both more efficient optimization algorithms and faster processors, MPC is more and more applied to fast and/or embedded systems. Hence, there is an active research in the area of algorithmic MPC implementations that allow rapid computation of the optimal control input while, if possible, still guaranteeing stability of the closed loop, e.g., [2], [3], [4], [5], [6], [7] to only name a few.

In this paper, we present an MPC scheme for constrained linear systems that relies on a continuous-time, barrier function based algorithm. The main idea of this scheme is as follows. In MPC, the underlying open-loop optimal control problem has to be solved repeatedly with a constantly changing initial state for the prediction. Thus, MPC can be considered as a parameter-dependent or time-varying optimization problem. The idea of this work is to make use of the fact that the dynamics of the initial prediction state is governed by the underlying continuous-time dynamics of the plant. In particular, a continuous-time Newton-based algorithm is proposed that converges to as well as tracks the optimal solution of a time-varying optimization problem which is a barrier function based, and therefore *smoothed*, approximation of the original open-loop MPC problem for the discretized plant dynamics. The resulting control input is given as the sampled output of a dynamical system and guarantees strict satisfaction of all imposed input, state, and terminal constraints. On the one hand, if initialized with an optimal solution, the algorithm tracks the optimal solution exactly, and, on the other hand, a direct (Newton-based) method is used for optimization in the case of a feasible but suboptimal initialization of the algorithm. Thus, one could refer to the proposed method as a direct continuation method [8]. Note that, while the proposed continuous-time tracking algorithm explicitly relies on the underlying continuous-time plant dynamics, both the design and the stability analysis of the corresponding barrier function based MPC scheme are based on a discretized model of the plant. A schematic illustration of the proposed MPC scheme is depicted in Fig. 1 on the next page. More details will be discussed in the course of this paper.

Closely related to the discussed method are the works by [4] and [6], which consider real-time implementation of MPC in the context of nonlinear systems. In [4], the author applies an *indirect* continuation method in order to track the solution of the two-point boundary-value problem that is related to the finite-horizon Euler-Lagrange equations of the discretized open-loop optimal control problem. In addition, both a fast integration algorithm and an error analysis for the numerical implementation is presented. However, the proposed method considers only equality constraints and stability of the closed-loop is not discussed. In [6], the authors present a real-time framework for MPC of continuous-time nonlinear systems that uses a barrier function and gradient based solution approach to obtain a sampled-data like feedback with a variable and arbitrarily coarse parameterization of the input trajectory. The proposed algorithm guarantees asymptotic convergence of the closed-loop system to a set around the origin while ensuring pointwise satisfaction of all input and state constraints. However, optimality is not considered explicitly and no performance guarantees can be given. In contrast to both of these works, we restrict ourselves to the class of linear systems and prove, for the case of optimal initialization, both feasibility and optimality of the resulting control law as well as asymptotic stability of the closed-loop system with respect to a discretized plant model.

Due to its continuous-time nature, the proposed algorithm may not be suitable for direct implementation on a digital computer, but it provides a system-theoretic MPC approach which takes the dynamics of the optimization algorithm into account and guarantees stability of the closed loop for, in principle, arbitrarily fast system dynamics. The main but only limitation concerning the implementation is whether one

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* Institute for Systems Theory and Automatic Control, University of Stuttgart, Pfaffenwaldring 9, 70550 Stuttgart, {christian.feller,ce}@ist.uni-stuttgart.de

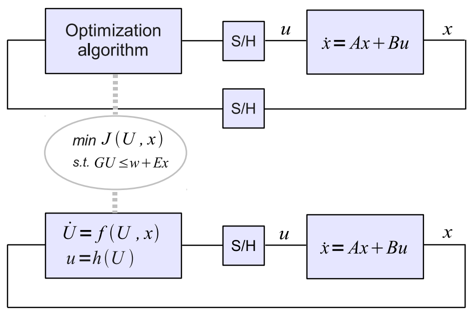

Fig. 1. Standard MPC and the proposed control algorithm based on a continuous-time dynamical system. The algorithm dynamics $f(U, x)$ and the output function $h(U)$ will be specified and discussed later.

is able to numerically integrate the continuous-time control algorithm properly. Thus, the proposed scheme is a first step towards an alternative, system-theoretic approach for fast and embedded MPC solutions. In addition, we also present a new approach towards stabilizing barrier function based MPC that allows to maximize the size of the terminal set and is not restricted to the specific MPC approach discussed in this paper. As we will discuss in Section III, some closely related results on unconstrained MPC implementations based on barrier functions can be found in [9].

Thoughout the paper we will make use of the following notation: $\|x\|_M := \sqrt{x^T M x}$ for any symmetric positive semi-definite matrix M; for any arbitrary set S, S° will denote the open interior and ∂S the limiting boundary.

II. Preliminaries

In the following, we summarize some basic background on both theoretical and practical aspects of linear MPC. Consider the discrete-time linear system

$$x(k+1) = A_D x(k) + B_D u(k), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ refer to the vectors of system states and inputs, respectively, and $A_D \in \mathbb{R}^{n \times n}$, $B_D \in \mathbb{R}^{n \times m}$ are real matrices with (A_D, B_D) stabilizable that are be obtained by discretizing the continuous-time plant dynamics $\dot{x}(t) = Ax(t) + Bu(t)$ with a sampling time $T_s > 0$. In general, the linear MPC open-loop optimal control problem for a finite prediction horizon N is then formulated as

$$\begin{aligned} J^*(x) = \min_{\boldsymbol{u}} & \sum_{k=0}^{N-1} \ell(x_k, u_k) + F(x_N) \\ & \text{subject to} \\ & \quad u_k \in \mathcal{U},\ k = 0, \dots, N-1, \\ & \quad x_k \in \mathcal{X},\ k = 1, \dots, N, \\ & \quad x_N \in \mathcal{X}_f \subset \mathcal{X},\ x_0 = x(t_k) = x, \\ & \quad x_{k+1} = A_D x_k + B_D u_k,\ k = 0, \dots, N-1, \end{aligned} \tag{2}$$

where the stage cost $\ell(x, u)$ and the terminal cost $F(x)$ are given by $\ell(x, u) = \|x\|_Q^2 + \|u\|_R^2$ and $F(x) = \|x\|_P^2$ for appropriately chosen weight matrices $Q = Q^T \succeq 0$, $R = R^T \succ 0$, and $P = P^T \succ 0$, and $\mathcal{X}_f$ refers to a closed and convex terminal constraint set. Moreover, $\boldsymbol{u} := \{u_0, u_1, \dots, u_{N-1}\}$ denotes the sequence of control inputs. A standard approach to ensure stability in the context of MPC is to employ the associated value function $J^*(x)$ as a Lyapunov function of the closed-loop system. As discussed in [1], this often requires to define a stabilizing local controller $u = Kx$ and choose $\mathcal{X}_f$ as an invariant subset of the feasible set $\mathcal{X}$ in which $u = Kx$ satisfies the input constraints, while the terminal cost $F(x)$ is chosen in such a way that it serves as a local control Lyapunov function (CLF) inside the terminal region. In particular, for the discrete-time case, the following conditions are sufficient for asymptotic stability of the closed-loop [1].

Definition 1 (Stability conditions)

A1: $\mathcal{X}_f \subset \mathcal{X}$, $\mathcal{X}_f$ *closed,* $0 \in \mathcal{X}_f$
A2: $Kx \in \mathcal{U}\ \forall x \in \mathcal{X}_f$
A3: $(A_D + B_D K)x \in \mathcal{X}_f\ \forall x \in \mathcal{X}_f$
A4: $F((A_D + B_D K)x) - F(x) \leq -\ell(x, Kx)\ \forall x \in \mathcal{X}_f$,
where $K \in \mathbb{R}^{n \times m}$ *is an arbitrary stabilizing local control gain, i.e.,* $(A_D + B_D K)$ *is Schur.*

For problem (2), these conditions are, for example, satisfied when K and P are obtained from the algebraic Riccati equation of the unconstrained LQR problem and $\mathcal{X}_f$ is chosen to be the maximal positive invariant set for the dynamics $x_{k+1} = (A_D + B_D K)x_k$.

By stacking the input sequence in the extended input vector $U := [u_0^T, \cdots, u_{N-1}^T]^T \in \mathbb{R}^{Nm}$ and eliminating the predicted system states $x_k, k = 1, \dots, N$, via $x_k(U, x) = A_D^k x + \sum_{j=0}^{k-1} A_D^j B_D u_{k-j-1},\ k = 1, \dots, N$, problem (2) can be rewritten as a strongly convex quadratic program (QP) which depends on the current system state x:

$$J^*(x) = \min_U \frac{1}{2} U^T H U + x^T F U + x^T Y x \tag{3a}$$

$$\text{s. t.}\quad GU \leq w + Ex\ . \tag{3b}$$

Here, the problem matrices $0 \prec H = H^T \in \mathbb{R}^{n_U \times n_U}$, $F \in \mathbb{R}^{n \times n_U}$, $Y \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{q \times n_U}$, $w \in \mathbb{R}^q$, and $E \in \mathbb{R}^{q \times n}$ with $n_U = Nm$ can be constructed from (2) by means of simple matrix operations [2]. Note that when solving problem (3) for the optimal control input vector U, the term $x^T Y x$ can of course be neglected. For unstable systems and long prediction horizons, the above QP may be rather ill-conditioned, which can, however, be resolved by a suitable prestabilization, see [10].

Definition 2 *Let the feasible sets* $\mathcal{U}_N(x)$ *and* $\mathcal{X}_N$ *be defined as* $\mathcal{U}_N(x) := \{U \in \mathbb{R}^{n_U} : u_k \in \mathcal{U}, k = 0, \dots, N-1, x_k(U, x) \in \mathcal{X}, k = 1, \dots, N-1\ , x_N(U, x) \in \mathcal{X}_f\}$ *and* $\mathcal{X}_N := \{x \in \mathcal{X} : \mathcal{U}_N(x) \neq \emptyset\}$.

Definition 3 *For* $U \in \mathbb{R}^{n_U}$*, let the projection* $\mathscr{P}(U) : \mathbb{R}^{n_U} \to \mathbb{R}^m$ *be defined as* $\mathscr{P}(U) = \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix} U$.

The standard implementation of linear MPC is then:

1) Measure the current state $x = x(k)$.
2) Solve the QP (3) to obtain the optimizer $U^*(x)$.

3) Apply $u(k) = u_0^*(x) = \mathscr{P}(U^*(x))$ to the plant.
4) Return to Step 1.

Note that since problem (3) depends on the current state x and needs to be solved at every sampling instant, it is necessary to measure or estimate the system state $x(t)$ during the on-line implementation.

III. Stabilizing barrier function based MPC

The general idea of barrier function based MPC relies on the standard approach that is used in interior point based convex optimization algorithms, in which inequality constraints are incorporated into the cost function by using suitable barrier functions with a corresponding weighting factor. As a consequence, the resulting optimization problem is unconstrained (or equality constrained) and can be solved by means of efficient optimization techniques like the Newton method. In general, the exact solution to the original problem is recovered when the weighting factor of the barrier functions goes to zero [11].

Our motivation to use a barrier function based method stems from the wish that the algorithm that we will discuss in Section IV should be able to track the solution of the MPC open-loop optimal control problem in form of a time-varying unconstrained optimization problem. This implicitly requires that one needs to be able to asymptotically track the underlying Karush-Kuhn-Tucker optimality conditions. However, in the standard setup, these conditions are nonsmooth which makes an asymptotic tracking very difficult. Barrier function based problem formulations have the nice property that they rely on a *smoothed* version of the KKT conditions, which will allow us in Section IV to derive a continuous-time gradient based algorithm with asymptotic tracking behavior.

In order to apply the barrier function approach to linear MPC, we consider in the spirit of [9] the following modified open-loop optimal control problem

$$\tilde{J}^*(x) = \min_{\boldsymbol{u}} \left\{ \tilde{\ell}_0(x_0, u_0) + \sum_{k=1}^{N-1} \tilde{\ell}(x_k, u_k) + \tilde{F}(x_N) \right\}$$
$$\text{subject to} \qquad (4)$$
$$x_{k+1} = A_D x_k + B_D u_k,\ x_0 = x(t_k) = x,$$

$$\begin{aligned} \text{where} \quad \tilde{\ell}_0(x,u) &:= \ell(x,u) + \varepsilon B_u(u) \\ \tilde{\ell}(x,u) &:= \ell(x,u) + \varepsilon B_u(u) + \varepsilon B_x(x) \\ \tilde{F}(x) &:= F(x) + \varepsilon B_f(x). \end{aligned}$$

Here, $B_u(.)$, $B_x(.)$ and $B_f(.)$ are suitable convex barrier functions with domains $\mathcal{U}^\circ$, $\mathcal{X}^\circ$, and $\mathcal{X}_f^\circ$ with $B_u(u) \to \infty$ for $u \to \partial\mathcal{U}$, $B_x(x) \to \infty$ for $x \to \partial\mathcal{X}$, and $B_f(x) \to \infty$ for $x \to \partial\mathcal{X}_f$. The positive scalar $\varepsilon > 0$ is the barrier function weighting parameter which determines the influence of the barrier function values on the cost objective. As outlined above, the solution of (4) will for a given x converge to the solution of (2) as $\varepsilon \to 0$. However, as pointed out in [11], small values of ε will usually result in ill-conditioned problems and do, therefore, not work well in practice. Hence, it is desirable to choose P, $\mathcal{X}_f$, and the barrier functions $B_u(u)$, $B_x(x)$, and $B_f(x)$ in such a way that a linear MPC scheme based on (4) asymptotically stabilizes the origin for any $\varepsilon > 0$. If we want to employ standard MPC stability concepts which are based on using the value function $\tilde{J}^*(x)$ as a Lyapunov function for the closed-loop system, this obviously results in the following conditions: (i) $\tilde{J}(\boldsymbol{u}, x)$ has to be a positive definite function with minimum $\tilde{J}(\mathbf{0}, 0) = 0$; and (ii) the conditions A1–A4 in Definition 1 have to be satisfied, in particular, $\tilde{F}(x) = x^T P x + \varepsilon B_f(x)$ has to be a local CLF in $\mathcal{X}_f$.

Following [9], we introduce the concept of gradient recentered barrier functions, which can be used to ensure that condition (i) is satisfied.

Definition 4 (Gradient recentered barrier function) *Let $B : D \to \mathbb{R}$ be a convex barrier function on an open convex set D with $0 \in D$. Then, the function $\bar{B} : D \to \mathbb{R}$ defined as*

$$\bar{B}(z) = B(z) - B(0) - [\nabla B(0)]^T z \qquad (5)$$

is called the gradient recentered barrier function for B around the origin.

In the following, we assume $B_x(x)$, $B_u(u)$, and $B_f(x)$ to be gradient recentered barrier functions for the domains $\mathcal{U}^\circ$, $\mathcal{X}^\circ$, and $\mathcal{X}_f^\circ$, which ensures that $\tilde{J}(\boldsymbol{u}, x)$ is a positive definite function with minimum $\tilde{J}(\mathbf{0}, 0) = 0$, i.e., that condition (i) is satisfied. As in the standard linear MPC case, the conditions A1–A3 can be easily satisfied by choosing a stabilizing local controller and a corresponding invariant terminal set. However, the main difficulty is to satisfy the CLF condition on $\tilde{F}(x)$ since both the stage cost barrier functions $B_x(x)$, $B_u(u)$ and the barrier function $B_f(x)$ for the terminal set constraint are involved in A4 as discussed in the following. Let an arbitrary stabilizing local control law be given by

$$u = Kx, \qquad (6)$$

where $K \in \mathbb{R}^{m \times n}$. Let A_K be defined as $A_K := A_D + B_D K$ and let

$$\mathcal{X}_K := \{x \in \mathcal{X} : Kx \in \mathcal{U}\}\ . \qquad (7)$$

Furthermore, define the function

$$B_K(x) = B_x(x) + B_u(Kx) \qquad (8)$$

as the gradient recentered barrier function for $\mathcal{X}_K^\circ$. Then, the CLF condition A4 for problem (4) is given by

$$\begin{aligned} &\|A_K x\|_P^2 - \|x\|_P^2 + \|x\|_Q^2 + \|Kx\|_R^2 \qquad (9) \\ &\quad + \varepsilon B_f(A_K x) - \varepsilon B_f(x) + \varepsilon B_K(x) \le 0 \ \forall x \in \mathcal{X}_f^\circ\ . \end{aligned}$$

In the following, we will show how satisfaction of condition (9), and hence asymptotic stability of the closed-loop, can be achieved by ensuring $B_f(A_K x) - B_f(x) \le 0\ \forall x \in \mathcal{X}_f^\circ$ and bounding $B_K(x)$ by a quadratic function that can then be compensated for by a suitable choice of the terminal cost matrix P. We shortly summarize the results presented in [9] before proposing a new approach that allows to maximize the size of the resulting terminal region $\mathcal{X}_f$.

A. Approach from [9]

In [9], the authors present a stabilizing MPC approach that is based on exploiting some properties of self-concordant functions and the corresponding so-called Dikin ellipsoid. More specific, for a given stabilizing local controller $u = Kx$, the terminal set $\mathcal{X}_f(\alpha) := \{x \in \mathbb{R}^n : \|x\|_P^2 \le \alpha^2\}$ is chosen as an ellipsoidal subset of the open unit Dikin ellipsoid, in which the barrier function B_K can be bounded by a quadratic function. By choosing P as the solution of a suitable Lyapunov equation, the terminal cost allows to compensate for B_K and guarantees both invariance of $\mathcal{X}_f$ and satisfaction of the CLF condition $B_f(A_K x) - B_f(x) \le 0 \ \forall x \in \mathcal{X}_f^\circ$. Hence, as proven in [9], the outlined choice of the control parameters satisfies the conditions A1–A4 and leads to asymptotic stability of the closed-loop system. However, one disadvantage of this approach is the fact that the size of the resulting terminal set $\mathcal{X}_f$ is limited by the size of the Dikin ellipsoid, which may be considerably smaller than the feasible set $\mathcal{X}_K$. For more details we refer the reader to [9] and references therein.

B. A new approach with enlarged terminal set

In the following, we will present a new approach towards stabilizing MPC based on gradient recentered logarithmic barrier functions. In contrast to the approach from [9], our MPC design does not rely on the Dikin ellipsoid and allows, in principle, to choose $\mathcal{X}_f$ as the maximal volume invariant ellipsoid contained within $\mathcal{X}_K$. In fact, the following results allow a trade-off between the size of the resulting terminal region and the impact of the terminal cost that is needed to guarantee stability. Note that the following results are, with some minor modifications, also applicable to nonlinear systems as long as the constraints are given in form of linear inequalities.

Assumption 1 *The state and input constraints are given in form of compact polytopic sets that contain the origin in their interior, i.e.,* $\mathcal{U} = \{u \in \mathbb{R}^m : C_u u \le d_u\}$ *and* $\mathcal{X} = \{x \in \mathbb{R}^n : C_x x \le d_x\}$ *with* $C_x \in \mathbb{R}^{q_x \times n}$, $C_u \in \mathbb{R}^{q_u \times m}$ *and* $d_u > 0,\ d_x > 0$. *Moreover, we assume that the feasible sets* $\mathcal{X}_N$ *and* $\mathcal{U}_N(x)$ *have nonempty interior, i.e.,* $\mathcal{X}_N^\circ \neq \emptyset$ *and* $\mathcal{U}_N^\circ(x) \neq \emptyset \ \forall\, x \in \mathcal{X}_N^\circ$.

Consider the following gradient recentered logarithmic barrier functions for the polytopic input and state constraints:

$$B_{x,i}(x) = -\ln(-c_{x,i}^T x + d_{x,i}) + \ln(d_{x,i}) - \frac{c_{x,i}^T x}{d_{x,i}} \quad \text{(10a)}$$

$$B_{u,j}(u) = -\ln(-c_{u,j}^T u + d_{u,j}) + \ln(d_{u,j}) - \frac{c_{u,j}^T u}{d_{u,j}} \quad \text{(10b)}$$

with $i = 1, \dots, q_x$ and $j = 1, \dots, q_u$. Note that when applying the stabilizing local control law $u = Kx$, the barrier functions for the input constraints (10b) can be represented as barrier functions of the form (10a) with $c_x^T = c_u^T K$.

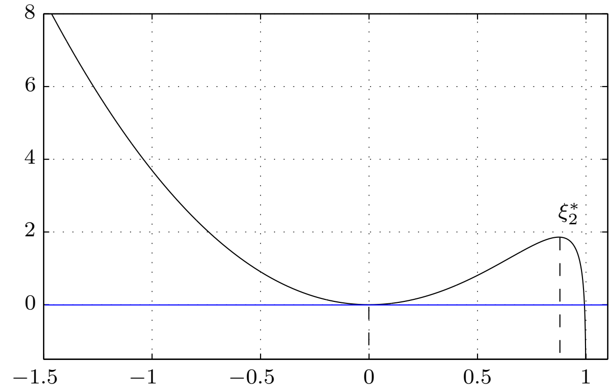

Fig. 2. The function $f(\xi)$ for $d_i = 1$ and $\gamma = 4$.

Lemma 1 *Let* $B_i(x) = -\ln(-c_i^T x + d_i) + \ln(d_i) - \frac{1}{d_i} c_i^T x$ *be a gradient recentered barrier function for the constraint* $c_i^T x \le d_i$ *with* $c_i^T \neq 0$, $d_i > 0$. *Then, for any scalar parameter* $\gamma > \frac{1}{2d_i^2}$ *it holds that*

$$B_i(x) \le \gamma\, x^T M_i x \quad \forall x \in \mathcal{H}_i\ , \quad (11)$$

where $\mathcal{H}_i$ *denotes the halfspace*

$$\mathcal{H}_i := \{x \in \mathbb{R}^n : c_i^T x \le d_i - \frac{1}{2\gamma d_i}\}. \quad (12)$$

and $M_i \in \mathbb{R}^{n\times n}$ *is defined as* $M_i := c_i c_i^T$.

Proof: Inserting the definitions of $B_i(x)$ and M_i into (11), we obtain the condition

$$-\ln(-c_i^T x + d_i) + \ln(d_i) - \frac{1}{d_i} c_i^T x \le \gamma\, x^T c_i c_i^T x\ , \quad (13)$$

which is equivalent to

$$\ln\left(1 - \frac{1}{d_i} c_i^T x\right) + \frac{1}{d_i} c_i^T x + \gamma x^T c_i c_i^T x \ge 0\ . \quad (14)$$

Consider now the function $f : (-\infty, d_i) \to \mathbb{R}$ defined as

$$f(\xi) = \ln\left(1 - \frac{1}{d_i}\xi\right) + \frac{1}{d_i}\xi + \gamma\xi^2 \quad (15)$$

where ξ replaces $c_i^T x$. The function $f(\xi)$ satisfies $\lim_{\xi\to-\infty} f(\xi) = \infty$ and $\lim_{\xi\to d_i} f(\xi) = -\infty$. By computing the zeros of the first derivative, we obtain the two local extrema

$$\xi_1^* = 0 \ \text{ and } \ \xi_2^* = d_i - \frac{1}{2\gamma d_i}. \quad (16)$$

We choose $\gamma > \frac{1}{2d_i^2}$, which ensures that $0 = \xi_1^* < \xi_2^* < d_1$. Furthermore, for this choice it is also guaranteed that

$$f''(\xi_1^*) = 2\gamma - \frac{1}{d_i^2} > 0\ , \quad f''(\xi_2^*) = 1 - 2\gamma d_i^2 < 0, \quad (17)$$

i.e., that $\xi_1^* = 0$ is a local minimum and $\xi_2^* = d_i - \frac{1}{2\gamma d_i}$ a local maximum. Hence, as illustrated in Fig. 2, we can conclude that $f(\xi) \ge 0\ \forall \xi \in (-\infty, \xi_2^*]$. Resubstituting $c_i^T x$ for ξ proves the claim. ∎

Remark 1 *Note that choosing $\gamma \leq \frac{1}{2d_i^2}$ would lead to the halfspace $\tilde{\mathcal{H}}_i := \{x \in \mathbb{R}^n : c_i^T x \leq 0\}$. We exclude this case since we want to use the halfspaces $\mathcal{H}_i$ in the sequel to construct a terminal set that contains the origin in its interior. Note also that the original constraint $c_i^T x \leq d_i$ can be approximated arbitrarily close by increasing γ. Similarly, a large γ is necessary if $d_i > 0$ is close to zero.*

Assumption 2 *The barrier functions $B_x(x)$ and $B_u(x)$ in problem (4) are given as $B_x(x) = \sum_{i=1}^{q_x} B_{x,i}(x)$ and $B_u(u) = \sum_{j=1}^{q_u} B_{u,j}(u)$, where $B_{x,i}(x)$ and $B_{u,j}(u)$ are gradient recentered logarithmic barrier functions of the form (10). Consequently, the combined barrier function $B_K(x)$ has the form $B_K(x) = \sum_{i=1}^{q_x} B_{x,i}(x) + \sum_{j=1}^{q_u} B_{u,j}(Kx)$.*

Lemma 2 *Let Assumption 2 be satisfied. Furthermore, let the matrices M_x and M_u be defined as $M_x := \sum_{i=1}^{q_x} c_{x,i} c_{x,i}{}^T$ and $M_u = \sum_{j=1}^{q_u} c_{u,i} c_{u,i}{}^T$, respectively, and let $\bar{\gamma}$ be a positive scalar with $\bar{\gamma} > \frac{1}{2d_{\min}^2}$, where $d_{\min} := \min_{i,j}\{\{d_{x,i}\},\{d_{u,j}\}\}$ with $i = 1,\ldots,q_x$, $j = 1,\ldots,q_u$. Then, it holds that*

$$B_K(x) \leq \bar{\gamma}\; x^T \left(M_x + K^T M_u K\right) x \qquad \forall x \in \mathcal{P}_K \;, \tag{18}$$

where $\mathcal{P}_K \subset \mathcal{X}_K$ is the convex polytope

$$\mathcal{P}_K := \left\{ x \in \mathbb{R}^n : \begin{array}{c} c_{x,i}^T x \leq d_{x,i} - \frac{1}{2\gamma d_{x,i}} \\ c_{u,j}^T K x \leq d_{u,j} - \frac{1}{2\gamma d_{u,j}} \end{array} \right\} \tag{19}$$

which can be interpreted as the intersection of all the halfspaces $\mathcal{H}_i$ in (12) with $i = 1,\ldots,q_x+q_u$.

Proof: The upper bound follows directly from applying Lemma 1 to each term of $B_K(x)$ as stated Assumption 2 and summing up the resulting quadratic forms. Moreover, the choice of $\bar{\gamma}$ guarantees that each bound is valid in the polytopic region $\mathcal{P}_K = \cup_i^{q_x+q_u} \mathcal{H}_i$. ■

Remark 2 *In the spirit of Remark 1, the feasible set $\mathcal{X}_K$ can be approximated arbitrarily close by increasing $\bar{\gamma}$. However, this will also lead to a very large bound for $B_K(x)$ and therefore, as we will see in the following, to an increased terminal cost. It is, of course, also possible to use individual γ_i instead of the single $\bar{\gamma}$, which may be relevant in cases where the d_i are distributed over a large scale.*

In the following, we use the bound in Lemma 2 to derive conditions on K, P, and $B_f(x)$ that ensure asymptotic stability of the closed loop of the discrete-time system (1). In contrast to the approach from [9], we do not use the terminal weighting matrix P in the terminal constraint but introduce a separate shape matrix P_f that allows to maximize the size of the ellipsoidal terminal set.
Let the matrices K and $P \succ 0$ be given as the solutions to the discrete algebraic Riccati equation

$$\begin{aligned} K &= -\left(\tilde{R} + B_D^T P B_D\right)^{-1} B_D^T P A_D \\ P &= (A_D + B_D K)^T P (A_D + B_D K) + K^T \tilde{R} K + \tilde{Q} \end{aligned} \tag{20}$$

with $\tilde{Q} = Q + \varepsilon\bar{\gamma} M_x$ and $\tilde{R} = R + \varepsilon\bar{\gamma} M_u$, where M_x and M_u are defined by Lemma 2. Furthermore, define the terminal region as the ellipsoid

$$\mathcal{X}_f = \left\{ x \in \mathbb{R}^n : \|x\|_{P_f} \leq 1 \right\} \tag{21}$$

where $P_f \in \mathbb{R}^{n\times n}$, $P_f = P_f^T \succ 0$. We now want to choose the shape matrix P_f in such a way that $\mathcal{X}_f$ is invariant under the local control law $u = Kx$, i.e., that $\mathcal{P}_f$ satisfies the Lyapunov matrix inequality $A_K^T P_f A_K - P_f \preceq 0$, where again $A_K := A_D + B_D K$. In addition, we would like to maximize the size of the terminal set, i.e., the volume of the ellipsoid $\mathcal{X}_f$ under the restriction that it lies inside the polytope $\mathcal{P}_K$ where our quadratic bound for $B_K(x)$ holds, see (19). Without loss of generality, let the polytope $\mathcal{P}_K$ be given in normalized form, i.e., $\mathcal{P}_K = \{x \in \mathbb{R}^n : c_k^T x \leq 1,\ k = 1,\ldots,r\}$. Then, the condition $\mathcal{X}_f \subset \mathcal{P}_K$ can be expressed as $c_k^T P_f^{-1} c_k \leq 1$ $\forall k = 1,\ldots,r$. Since the volume of the ellipsoid $\mathcal{X}_f$ is proportional to $(\det P_f^{-1})^{\frac{1}{2}}$, we can find the optimal P_f by defining $X := P_f^{-1}$ and solving the following convex semi-definite programming problem

$$\begin{aligned} \min\; & -\log\det X \\ \text{s.t.}\; & c_k^T X c_k \leq 1 \;\forall k = 1,\ldots,r \\ & X \succ 0\;,\; \begin{bmatrix} X & A_K X \\ X A_K^T & X \end{bmatrix} \succeq 0\;, \end{aligned} \tag{22}$$

where we used the monotonic transformation $\log\det P_f^{-1}$ and the Schur complement in order to render the problem convex and include the mentioned Lyapunov inequality. Now, define the gradient recentered logarithmic barrier function for the terminal set $\mathcal{X}_f$ as

$$B_f(x) = -\ln(1 - \|x\|_{P_f}^2)\;, \tag{23}$$

where $P_f = X^{-1}$ is obtained from (22).

Theorem 1 *Let Assumption 1 and Assumption 2 be satisfied and let the stabilizing local feedback matrix K and the terminal weighting matrix $P \succ 0$ be given by (20). Furthermore, let the ellipsoidal terminal set $\mathcal{X}_f$ and the corresponding barrier function $B_f(x)$ be given by (21) and (23), respectively. Then, the origin of system (1) under the MPC feedback $u(x) = u_0^*(x)$ based on repeated solution of the barrier function based open-loop optimal control problem (4) is asymptotically stable for any $x_0 \in \mathcal{X}_N$.*

Proof: We prove asymptotic stability as summarized in [1] by using the value function $\tilde{J}^*(x)$ as a Lyapunov function for the closed-loop system. Clearly, the conditions A1, A2, and A3 stated in Definition 1 are satisfied by construction for the chosen K and $\mathcal{X}_f$. Let $x^+ = (A_D + B_D K)x = A_K x$. Then, we furthermore have $\forall x \in \mathcal{X}_f^\circ$

$$\begin{aligned} \tilde{F}(x^+) - \tilde{F}(x) + \tilde{\ell}(x, Kx) =& \|x^+\|_P^2 - \|x\|_P^2 \\ &+ \|x\|_Q^2 + \|Kx\|_R^2 + \varepsilon B_K(x) \\ &+ \varepsilon B_f(x^+) - \varepsilon B_f(x) \end{aligned}$$

Due to Lemma 2 and the appropriate choice of K, P and $\mathcal{X}_f$ in (20) and (21), we get $\|x^+\|_P^2 - \|x\|_P^2 + \|x\|_Q^2 + \|Kx\|_R^2 + \varepsilon B_K(x) \leq 0 \; \forall x \in \mathcal{X}_f^\circ$ and therefore $\tilde{F}(x^+) - \tilde{F}(x) + \tilde{\ell}(x, Kx) \leq \varepsilon B_f(x^+) - \varepsilon B_f(x)$. Since P_f satisfies by design the Lyapunov inequality $A_K^T P_f A_K - P_f \leq 0$, it holds that $B_f(x^+) - B_f(x) = \ln\left(\frac{1-\|x\|_{P_f}^2}{1-\|x\|_{A_K^T P_f A_K}^2}\right) \leq \ln\left(\frac{1-\|x\|_{P_f}^2}{1-\|x\|_{P_f}^2}\right) = 0 \; \forall x \in \mathcal{X}_f^\circ$. Thus, $\tilde{F}(x^+) - \tilde{F}(x) \leq -\tilde{\ell}(x, Kx) \; \forall x \in \mathcal{X}_f^\circ$, and the CLF condition A4 is satisfied. As shown in [9] and [1], this finally implies $\tilde{J}^*(x^+) - \tilde{J}^*(x) \leq -\tilde{\ell}_0(x, u_0^*(x))$ with $x^+ = A_D x + B_D u_0^*(x)$, which proves that $\tilde{J}^*(x)$ is a Lyapunov function for the closed-loop system. ■

Hence, under the assumption of initial feasibility, the barrier function based MPC scheme asymptotically stabilizes the origin of system (1) for any barrier weighting parameter $\varepsilon > 0$. As discussed in Remark 2, the approach allows a trade-off between the size of the terminal set $\mathcal{X}_f \subset \mathcal{P}_K \subset \mathcal{X}_K$ and the impact of the terminal weight $x^T P_f x$ by adjusting the scalar parameter γ.

IV. A Barrier Function Based Continuous-Time Algorithm for Linear MPC

In this section, we propose a continuous-time algorithm that allows to implement the barrier function based linear MPC discussed in the previous section without the need of explicit on-line optimization. Instead, the control input is obtained as the output of a dynamical system that asymptotically tracks the optimal solution of the open-loop optimal control problem (4). For the sake of simplicity, we discuss in the following a more compact matrix based representation of (4) that is similar to (3). To this end, the barrier functions for the input and state constraints can be expressed as

$$B_c^{qp}(U,x) := -\sum_{i=1}^{q}\left(\ln\left(-G^iU + E^ix + w^i\right) + \ln\left(w^i\right) - \frac{G^iU - E^ix}{w^i}\right) , \quad (24)$$

where G, E, and w are defined analogously to the constraint matrices from the standard QP formulation (3) and M^i denotes the i^{th} row of a given matrix M. Again using the system dynamics (1) in a recursive fashion, the quadratic form $\|x_N\|_{P_f}^2$ can be expressed as a function of U and x:

$$\|x_N\|_{P_f}^2 = \varphi(U,x) := x^TDx + 2x^TLU + U^TNU \;, \quad (25)$$

where the matrices D, L, and N are given as

$$D := A_D^{N^T} P_f A_D^N \;, \quad L := A_D^{N^T} P_f \left[A_D^{N-1}B_D, \dots, B_D\right]$$
$$N := \left[A_D^{N-1}B_D, \dots, B_D\right]^T P_f \left[A_D^{N-1}B_D, \dots, B_D\right] .$$

As a consequence, the barrier function for the terminal set constraint (21) can be reformulated as

$$B_f^{qp}(U,x) := -\ln\Big(1 - \varphi(U,x)\Big) . \quad (26)$$

Therefore, the barrier function based open-loop optimal control problem (4) can be represented in the following compact QP-like form

$$\tilde{J}^*(x) = \min_U \tilde{J}(U,x) \quad \text{with} \quad (27)$$
$$\tilde{J}(U,x) = \frac{1}{2}U^THU + x^TFU + \varepsilon B_c^{qp}(U,x) + \varepsilon B_f^{qp}(U,x),$$

where we neglected the quadratic state-dependent term x^TYx for the sake of notational simplicity. Note that, concerning the optimal control input, the problems (27) and (4) are completely equivalent. Thus, by Theorem 1, the optimizer $\tilde{U}^*(x)$ that solves (27) yields an asymptotically stabilizing control law $u(x) = u_0^*(x) = \mathscr{P}(\tilde{U}^*(x))$.

Theorem 2 *Suppose $x_0 \in \mathcal{X}_N^\circ$ and $x(t)$ evolves in time according to $\dot{x} = Ax(t) + Bu(t)$, where $u(t)$ may be any (measurable) function of time that ensures $x(t) \in \mathcal{X}_N^\circ$. Then, for any initial condition $U_0 \in \mathcal{U}_N^\circ(x_0)$, the solution $U(t)$ of the dynamical system*

$$\dot{U} = -\left(\frac{\partial^2 \tilde{J}(U,x)}{\partial U^2}\right)^{-1}\left(\frac{\partial \tilde{J}(U,x)}{\partial U}^T + \frac{\partial^2 \tilde{J}(U,x)}{\partial x \partial U}\,\dot{x}\right) \quad (28)$$

satisfies $U(t) \in \mathcal{U}_N^\circ(x(t)) \; \forall t \geq t_0$ and converges asymptotically to the unique optimal solution $\tilde{U}^*(x(t))$ of (27) as $t \to \infty$. Furthermore, for the optimal initial condition $U_0 = \tilde{U}^*(x_0)$, the solution $U(t)$ tracks the optimal solution of (27) exactly, i.e., $U(t) \equiv \tilde{U}^*(x(t)) \; \forall t \geq t_0$.

Proof: Consider the positive semi-definite Lyapunov function candidate $W(U,x) = \frac{1}{2}\frac{\partial \tilde{J}(U,x)}{\partial U}^T \frac{\partial \tilde{J}(U,x)}{\partial U}$ which satisfies $W(U,x) \geq 0$, $W(U,x) = 0 \Leftrightarrow \nabla_U \tilde{J}(U,x) = 0$. For any given $x \in \mathcal{X}_N^\circ$, the function $W(U,x) : \mathcal{U}_N^\circ(x) \to \mathbb{R}$ is well-defined and $W(U,x) \to \infty$ whenever $U \to \partial\mathcal{U}_N^\circ(x)$. The derivative of W along U and x is given by

$$\dot{W} = \frac{\partial \tilde{J}(U,x)}{\partial U}\left(\frac{\partial^2 \tilde{J}(U,x)}{\partial U^2}\dot{U} + \frac{\partial^2 \tilde{J}(U,x)}{\partial x \partial U}\dot{x}\right) . \quad (29)$$

Inserting $\dot{U} = f(U,x)$ from Eq. (28), yields

$$\dot{W} = -\frac{\partial \tilde{J}(U,x)}{\partial U}^T \frac{\partial \tilde{J}(U,x)}{\partial U} = -2W(U,x) \leq 0 \; . \quad (30)$$

Since $W(U_0,x_0) := W_0 < \infty$, we have $W(U(t),x(t)) \leq W_0$ for $t \geq t_0$ and thus $U(t)$ is feasible for $t \geq t_0$, i.e., $U(t) \in \mathcal{U}_N^\circ(x(t))$, because for $x(t) \in \mathcal{X}_N^\circ$, $U(t)$ exists and is feasible if and only if $W(U(t),x(t)) < \infty$. Moreover, $W(U,x)$ converges to zero exponentially fast. Furthermore, $\dot{W} = 0 \Leftrightarrow \nabla_U \tilde{J}(U,x) = 0$, and therefore $\nabla_U \tilde{J}(U,x) \to 0$ as $t \to \infty$. Since (27) is strongly convex, this implies $U(t) \to U^*(x(t))$ whenever $t \to \infty$. Finally, for $U_0 = \tilde{U}^*(x_0)$, it holds that $W(U,x) \equiv 0$ and hence $U(t) \equiv \tilde{U}^*(x(t)) \; \forall t \geq t_0$. ■

Note that we make explicit use of the continuous-time system model $\dot{x} = Ax + Bu$ in the formulation of the dynamical system $\dot{U} = f(U,x)$ in (28). In order to preserve the sample and hold character of the discrete-time

MPC feedback, the control input needs to be piecewise constant, i.e., $u(x(t)) = \text{const}$ for $t \in [t_k, t_k + T_s)$. As a result, we propose the following algorithm that allows to implement discrete-time linear MPC based on the discussed continuous-time system $\dot{U} = f(U, x)$.

Algorithm 1 *(Barrier function based continuation MPC)*

Initialization for $t = t_0$:
(i) set $U(0) = U_0 \in \mathcal{U}_N(x_0)$ for $x_0 \in \mathcal{X}_N$
(ii) set $u_s = \mathscr{P}(U(0))$
Integration for $t > t_0$:
∟ apply $u = u_s$ to the plant;
∟ measure the current state $x(t)$ and obtain $U(t)$ by integrating (28) with $\dot{x} = Ax + Bu_s$;
∟ *Sampling* whenever $t = t_k = kT_s$, $k \in \mathbb{N}_{>0}$:
  ∟ set $u_s = \mathscr{P}(U(t_k))$.

Theorem 3 *Let the cost function $\tilde{J}(U, x)$ and the barrier functions in (27) satisfy the conditions in Theorem 1. Furthermore, let assume that feasibility of the state trajectory at discrete sampling points also implies feasibility of the intermediate continuous-time trajectories, i.e., $x(t_k) \in \mathcal{X}_N^\circ$ and $x(t_{k+1}) \in \mathcal{X}_N^\circ \Rightarrow x(t) \in \mathcal{X}_N^\circ \ \forall \ t \in [t_k, t_{k+1}]$. Then, for the optimal initial condition $U_0 = \tilde{U}^*(x_0)$, Algorithm 1 asymptotically stabilizes the origin of the closed-loop system (1) while guaranteeing strict satisfaction of all input and state constraints. Moreover, for any feasible initialization $U_0 \in \mathcal{U}_N^\circ(x_0)$, the state of the closed-loop system converges asymptotically to the origin, again under strict constraint satisfaction.*

Proof: In the case $U_0 = \tilde{U}^*(x_0)$, $x_0 \in \mathcal{X}_N^\circ$, we Theorem 2 implies that $U(t)$ tracks the optimal solution of problem (27), and therefore of problem (4), and $u(t) = u_s = \mathscr{P}(U(t_k))$ guarantees $x(t) \in \mathcal{X}_N^\circ$ due to intermediate feasibility of the continuous-time trajectories. In other words, Algorithm 1 is equivalent to the standard MPC implementation based on (4), and both strict satisfaction of constraints and asymptotic stability of the closed loop based on the discrete-time plant model (1) follow directly from Theorem 1.
In the case $U_0 \neq \tilde{U}^*(x_0)$, $U_0 \in \mathcal{U}_N^\circ(x_0)$, $x_0 \in \mathcal{X}_N^\circ$, we know by Theorem 2 that $\tilde{U}(x(t))$ is always strictly feasible, i.e. $\tilde{U}(x(t)) \in \mathcal{U}_N^\circ(x(t))$ and $x(t) \in \mathcal{X}_N^\circ$ for $t \geq t_0$. Moreover, we know from Theorem 2 that $\tilde{U}(x(t)) = \tilde{U}^*(x(t)) + R(t)$ with $R(t)$ bounded and $R(t) \to 0$. Therefore, Algorithm 1 generates an input sequence of the form $\tilde{u}_0(k) = \tilde{u}_0^*(k) + r(k)$ with $r(k)$ bounded and $r(k) \to 0$ for $k \to \infty$. Hence, the discrete-time closed-loop system is given by

$$x(k+1) = A_D x(k) + B_D u_0^*(k) + B_D r(k) \tag{31}$$

with $x(k) := x(t_k)$. Thus, for any $k \geq 0$, and $1 \leq l < \infty$ we have $\|x(k+l)\| \leq \|A_D^l x(k) + \sum_{j=0}^{l-1} A_D^j B_D u_0^*(k+l-j-1)\| + \|\sum_{j=0}^{l-1} A_D^j B_D r(k+l-j-1)\|$ Since i) $x(k+l) \in \mathcal{X}_N^\circ \ \forall \ k, l$, ii) the input sequence $u_0^*(\tilde{k})$ asymptotically stabilizes the nominal discrete time system, and iii) $r(\tilde{k}) \to 0$ as $\tilde{k} \to \infty$, we see that for any $l \geq 0$ and any constant $\bar{c} > 0$ there exists an index $\bar{k}$ such that $\|x(k+l)\| \leq \bar{c} \ \forall \ k \geq \bar{k}$, or equivalently $\forall \ \bar{c} > 0 \ \exists \ \tilde{k}_l \ : \ \|x(\tilde{k})\| \leq \bar{c} \ \forall \ \tilde{k} \geq \tilde{k}_l$ with $\tilde{k}_l = \bar{k} + l$ for any finite $l \geq 1$. Hence, we have that $\lim_{\tilde{k} \to \infty} \|x(\tilde{k})\| = 0$, which proves convergence of the system state to the origin for $U_0 \neq \tilde{U}^*(x_0)$, $U_0 \in \mathcal{U}_N^\circ(x_0)$, $x_0 \in \mathcal{X}_N^\circ$. Note that this result can also be interpreted as an asymptotic gain property [12], which shows that the closed-loop system is in some form input-to-state stable with respect to the particular class of disturbances $r(k)$ described above. ■

Remark 3 *Note that the intermediate feasibility of the continuous-time state trajectories is a rather strong assumption that is not needed by standard discrete-time MPC schemes, in which the satisfaction of constraints between sampling points is not considered at all. We, however, need this technical assumption in order to guarantee that both the system (28) and the corresponding Lyapunov function $W(U(t), x(t))$ are well-defined for all times. From a practical point of view, intermediate constraint violations during the runtime of the algorithm can be handled by setting the corresponding barrier function to a very large value.*

Note that Algorithm 1 represents an MPC scheme in which the (optimal) control input is given as the output of a dynamical system and that no iterative optimization algorithm is required in the on-line implementation, see Fig. 1. Clearly, the output function $h(.)$ is given by the projection $\mathscr{P}(.)$.

V. Numerical Case Studies

In the following, we illustrate the behavior of the proposed algorithm briefly on an academic numerical example. Considered is a double integrator system with the discrete-time system model

$$x(t+1) = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} T_s^2 \\ T_s \end{bmatrix} u(t) \ , \tag{32}$$

where the discretization time is chosen to be $T_s = 0.1$ s. The linear MPC open-loop optimal control problem is formulated over a prediction horizon of $N = 10$, using a quadratic cost function with the weight matrices $Q = \text{diag}(\,[1,\ 0.1]\,)$, $R = 1$ and the input and state constraints $|u| \leq 1$, $|x_1| \leq 2.8$, and $|x_2| \leq 0.8$. Based on the design procedure discussed in the Sections III and IV, we implemented Algorithm 1 in MATLAB and tested the closed-loop behavior for different initial conditions and a fixed barrier weighting parameter $\varepsilon = 0.01$. The results are illustrated in Fig. 3, Fig. 4, and Fig. 5 together with some comments.

VI. Conclusion

In this paper, we presented a novel continuous-time algorithm for linear model predictive control with guaranteed stability. The algorithm is realized as a dynamical system and allows to asymptotically track the solution of an unconstrained optimization problem that depends on a time-varying parameter. In combination with an newly proposed barrier function based approach for MPC, we established a novel MPC scheme which guarantees asymptotic stability of the

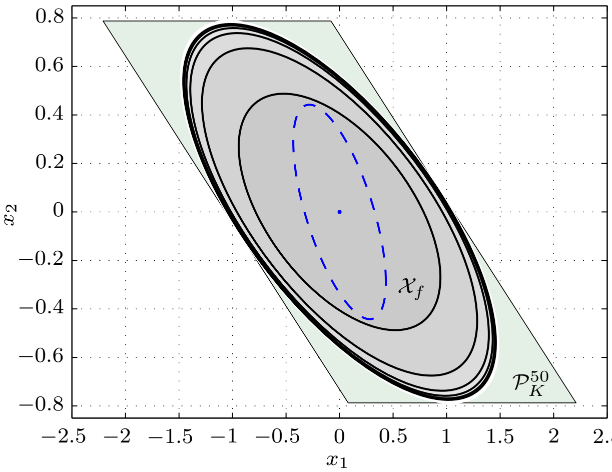

Fig. 3. The resulting ellipsoidal terminal sets for $\bar{\gamma} \in \{2, 10, 15, 20, 25, 50\}$. The dashed blue ellipsoid indicates the terminal set when using the approach from [9]. Also illustrated: the polytope $\mathcal{P}_K$ for $\bar{\gamma} = 50$.

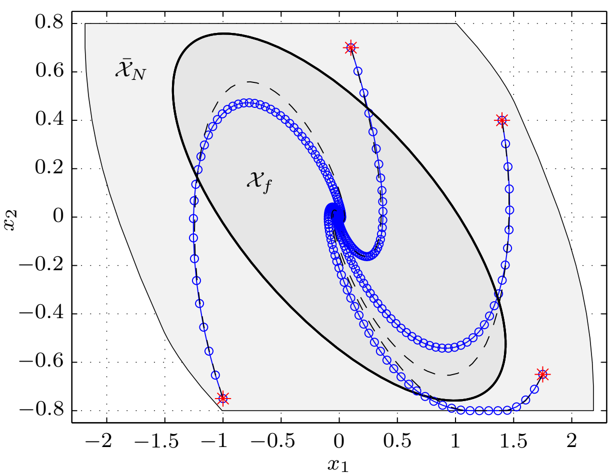

Fig. 4. Closed-loop behavior for standard MPC (−−) and the proposed continuous-time algorithm (—○—) for $\bar{\gamma} = 15$ and $U_0 = \tilde{U}^*(x_0)$. The set $\bar{\mathcal{X}}_N$ is an inner polytopic approximation of the controllable set $\mathcal{X}_N$ for the ellipsoidal terminal set $\mathcal{X}_f$.

closed loop if the algorithm is optimally initialized as well as convergence to the origin for a feasible but suboptimal initialization.

The underlying long-term goal of our research is to study and design optimization-based control algorithms that guarantee stability of the closed loop when considering the algorithms as they are actually implemented in practice. Hence, possible future research could include to find appropriate discretization and integration schemes for the proposed continuous-time algorithm, to remove the intermediate feasibility assumption of continuous-time trajectories, or to provide a theoretical justification for the practical approach of setting the barrier function of violated constraints to a large value, e.g., by using approximate barrier function methods. Moreover, it would be interesting to incorporate polytopic terminal sets or to eliminate the sampling of the control input in order to fully exploit the fact that the solution of the discrete-time

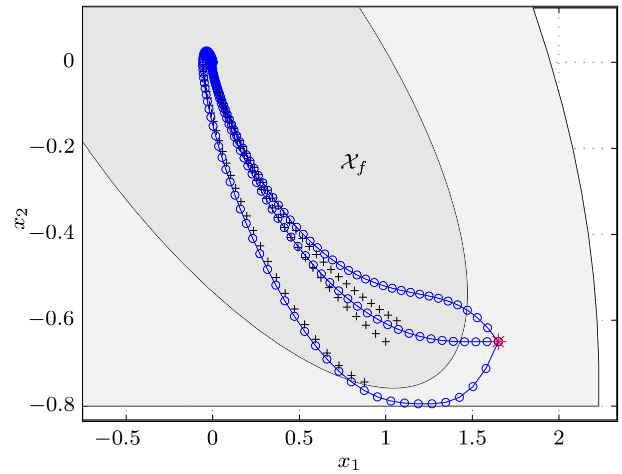

Fig. 5. Closed-loop behavior for Algorithm 1 for $x_0 = [1.65, -0.65]^T$ and the initial conditions $U_0 = \tilde{U}^*(x_0)$, $U_0 = 0$, and $U_0 = -\frac{1}{2}\tilde{U}^*(x_0)$ which all satisfy $U_0 \in \mathcal{U}_N^\circ(x_0)$. Note that the predictions $x_N(t_k)$ ('+') always satisfy the terminal set constraint $x_N \in \mathcal{X}_f$ and that the state converges to the origin also for suboptimal initialization.

open-loop optimal control problem is available at any instant of time. Further open issues are the comparison against an interior point method with only one iteration as well as extensions of the proposed control scheme to nonlinear systems [5].

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References

[1] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[2] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos, "The explicit linear quadratic regulator for constrained systems," *Automatica*, vol. 38, pp. 3–20, 2002.

[3] P. Tøndel, T. A. Johansen, and A. Bemporad, "An algorithm for multi-parametric quadratic programming and explicit MPC solutions," *Automatica*, vol. 39, pp. 489–497, 2003.

[4] T. Ohtsuka, "A continuation/GMRES method for fast computation of nonlinear receding horizon control," *Automatica*, vol. 40, pp. 563–574, 2004.

[5] M. Diehl, R. Findeisen, F. Allgöwer, H. Bock, and J. Schlöder, "Nominal stability of real-time iteration scheme for nonlinear model predictive control," *IEEE proceedings – Control Theory and Applications*, vol. 152, no. 3, pp. 296–308, 2005.

[6] D. DeHaan and M. Guay, "A Real-Time Framework for Model Predictive Control of Continuous-Time Nonlinear Systems," *IEEE Trans. on Automatic Control*, vol. 52, no. 11, pp. 2047–2057, 2007.

[7] M. N. Zeilinger and C. N. Jones, "Real-Time Suboptimal Model Predictive Control Using a Combination of Explicit MPC and Online Optimization," *IEEE Trans. on Automatic Control*, vol. 56, no. 7, pp. 1524–1534, 2011.

[8] S. L. Richter and R. A. DeCarlo, "Continuation Methods – Theory and Applications," *IEEE Trans. on Automatic Control*, vol. 28, no. 6, pp. 660–665, 1983.

[9] A. G. Wills and W. P. Heath, "Barrier function based model predictive control," *Automatica*, vol. 40, pp. 1415–1422, 2004.

[10] J. A. Rossiter, B. Kouvaritakis, and M. J. Rice, "A numerically robust state-space approach to stable-predictive control strategies," *Automatica*, vol. 34, no. 1, pp. 65–73, 1998.

[11] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, Cambridge, 2004.

[12] Z.-P. Jiang and J. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, pp. 857–869, 2001.