

A Drawdown Formula for Stock Trading Via Linear Feedback in a Market Governed by Brownian Motion

Shirzad Malekpour¹ and B. Ross Barmish²

Abstract—Control of *drawdown* is one of the greatest concerns to both stock traders and portfolio managers. That is, one typically monitors “drops in wealth” over time from highs to subsequent lows, and investors often shy away from funds with a past history of large drawdowns. With this motivation in mind, this paper addresses the analysis of drawdown when a feedback control is employed in stock trading in an idealized market with prices governed by Geometric Brownian Motion. We begin with a result in the applied probability literature which is applicable to cases involving buy-and-hold. Subsequently, after modifying this result via an Ito correction to account for geometric compounding of the daily stock price, we consider the effect on drawdown when a simple pure-gain feedback control is used to vary the investment $I(t)$ over time. That is, letting $V(t)$ denoting the trader’s account value at time $t \geq 0$, when a feedback control $I(t) = KV(t)$ is used to modify the amount invested, the buy-and-hold result no longer applies. Our first result is a formula for the expected value for the maximum drawdown in logarithmic wealth $\log(V(t))$. This formula is given in terms of the feedback gain K , the price drift μ , the price volatility σ and terminal time T . Subsequently, using a fundamental relationship between logarithmic and percentage drawdowns, we obtain an estimate for the expected value of the maximum percentage drawdown of $V(t)$. This paper also includes an analysis of the asymptotic behavior of this drawdown estimate as $T \rightarrow \infty$ and Monte Carlo simulations aimed at validation of our estimates.

I. INTRODUCTION

The takeoff point for this paper is the fact that the notion of “drawdown” is very important to stock traders and fund managers in financial markets. That is, when tracking the value of a time-varying portfolio $V(t)$, risk aversion dictates that drops from peaks to later valleys should not be too large. Careful monitoring of drawdown in $V(t)$ is one of the most important aspects of risk management; e.g., see [1]-[3]. A large drawdown is associated with significant “tail risk” and large skewness of the probability distribution for wealth. For such situations, a classical mean-variance analysis as in the celebrated work of Markowitz [4] does not suffice.

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To provide further context for this paper, we note that over the last two decades, a significant body of research has resulted in the development of new risk measures for financial markets. Until 1999, perhaps the most widely used of these measures was the the well-known “Value at Risk” which is known as VaR, e.g. see [6]. In a seminal paper [5], the notion of a “coherent risk measure” is introduced. To complete this extremely brief overview of risk measures, we mention [6]-[9] which provide examples of other risk measures commonly used. Suffice it to say, the *drawdown measure* is one of these and has received considerable attention over the past ten years. In this regard, the important results on stock price drawdown in [1] strongly motivate the research reported in this paper.

A distinction between the stock price drawdown results cited above and our research involving trading is important to make: When we use feedback to continuously modify the amount invested $I(t)$, drawdown in the account value $V(t)$ can be dramatically different from drawdown in the stock price $p(t)$. They are only equivalent when $I(t)$ corresponds to buy-and-hold with full investment $I(t) \equiv V(t)$. In our case, when feedback is used, the result can be a highly skewed probability distributions for the account value; this effect is quantified in our earlier work [13]. Finally, context for this paper is provided by the emerging line of research on the use of feedback in trading; for example, see [14]-[24]. This paper addresses the issue of drawdown in this context. Although an investment may end up with a large gain at the terminal time, if there is a large drawdown along the way, the performance may be deemed as unsatisfactory.

The Starting Point: The starting point in this paper is an idealized market, fully described in Section 2, with stock price p , governed by the Geometric Brownian Motion (GBM)

$$\frac{dp}{p} = \mu dt + \sigma dW_t$$

where μ is the drift, σ is the volatility and W_t is a standard Wiener process. When a trader is brought into the picture with investment $I(t)$ for $0 \leq t \leq T$, the resulting stochastic increment for the account value becomes

$$dV = \frac{dp}{p}I + r(V - I)dt$$

where $r \geq 0$ is the risk-free rate of return and for simplicity, we also take r as margin rate. In this setting, the issue of drawdown, formally defined in Section 2, arises.

Our analysis of drawdown begins by modifying the results in [1] to address dp/p instead of dp . Subsequently, to

analyze the drawdown in the account value $V(t)$ with trading involved, we consider the case of a feedback controller which modifies the investment $I(t)$ using a feedback gain K applied to $V(t)$. That is, we consider the controller

$$I = KV$$

with $K > 0$ and note that such a *proportional-to-wealth* investment schemes of this sort are rather classical in financial markets; e.g., see [25]. Subsequently, we obtain a formula for drawdown in $V(t)$ which specializes to the existing result in the literature for the special case of buy-and-hold which is obtained when $K = 1$.

II. IDEALIZED MARKET AND DRAWDOWN DEFINITIONS

The results in this paper apply to a so-called “idealized market” characterized by the Geometric Brownian Motion model above and some basic assumptions about continuous trading, perfect liquidity, price-taking, lack of transaction costs and adequacy of resources. These assumptions are briefly summarized and interpreted as follows: As far as continuity, perfect liquidity and price-taking is concerned, it is assumed that the trader can continuously update the investment $I(t)$ to whatever desired level at the current market price $p(t)$. This implies that no gap between the bid and ask prices are experienced as shares are acquired or sold.

The assumptions above, particularly that of price-taking, basically implies that the trader is “small enough” so that the stock price is not changed during the course of a transaction. In contrast, for the case of a large hedge fund buying or selling millions of shares at a time, we note that assumptions of this sort are not readily justifiable. For example, if millions of shares are being purchased at time $t = t^*$, a lack of liquidity could mean that the “last” shares are more costly to acquire than the “first” shares. Finally it should be noted that continuity and liquidity assumptions of the sort described above pervade many of the fundamental papers on markets; e.g., see [26] and [27]. In addition, the reader is referred to [14]-[16] where idealized markets are described in detail.

The remaining two assumptions, lack of transaction costs and adequacy of resources, are straightforward to explain: For active traders who are frequently placing orders of reasonable size, say several thousand dollars per trade, nowadays, commissions are sufficiently low so as to be, more or less, a non-issue. In fact, in many cases, traders pay a lump sum for an unlimited number of trades. Our final assumption is that the trader has adequate collateral to make a trade when $I(t) > V(t)$. In practice, when such trades are made, margin interest and collateral requirements may be involved. As previously stated, in the sequel, we take r to be the margin interest rate which will apply when $K > 1$.

A. Preservation of Geometric Brownian Motion

In an idealized GBM market with linear feedback $I = KV$, it is easy to show that the price $p(t)$ induces a GBM on the account value $V(t)$. To see this, we begin with a closed loop stochastic differential equation for the account value.

In view of the discussion above, the stochastic increment for V satisfies

$$\begin{aligned} dV &= \frac{dp}{p}I - r(I - V)dt = \frac{dp}{p}KV - r(K - 1)Vdt \\ &= [K\mu - r(K - 1)]Vdt + K\sigma VdW_t. \end{aligned}$$

Hence, when we divide by V , the induced account value follows a Geometric Brownian Motion

$$\frac{dV}{V} = \mu' dt + \sigma' dW_t$$

with modified drift μ' and modified volatility σ' given by

$$\mu' \doteq K\mu - r(K - 1); \quad \sigma' \doteq K\sigma.$$

It is also important to note that *logarithmic wealth* $\log(V(t))$ is also a quantity which is monitored by traders. In this case, via straightforward application of Ito’s Lemma, for example see [28], we modify the formulae above and obtain a standard Brownian Motion with increment

$$d(\log V) = \mu_* dt + \sigma_* dW_t$$

where $\sigma_* \doteq \sigma' = K\sigma$ and

$$\mu_* \doteq K\mu - r(K - 1) - \frac{1}{2}K^2\sigma^2.$$

B. Drawdown Definitions

Since $V(t)$ is a GBM process, its sample paths are continuous, and, the *maximum absolute drawdown* is defined by

$$D_{max}(V) \doteq \max_{0 \leq s \leq t \leq T} V(s) - V(t).$$

When we replace V by $\log(V)$ above, we obtain $D_{max}(\log(V))$, the logarithmic wealth version of absolute drawdown. The *maximum percentage drawdown of V* is similarly defined as

$$d_{max}(V) \doteq \max_{0 \leq s \leq t \leq T} \frac{V(s) - V(t)}{V(s)}.$$

Since increments dV are obtained as percentages of the current account value $V(t)$ at time t , the denominator in the definition of d_{max} cannot vanish. Hence, this quantity is well defined. Since $V(t)$ cannot become zero, $D_{max}(\log(V))$ is also well defined.

Absolute Versus Percentage Drawdown: When a maximum absolute drawdown occurs, it is not necessarily the case that this corresponds to a maximum percentage drawdown and vice versa. For example consider the function $V(t)$ shown in Figure 1. It has two major drops: a drop from $V(1) = 0.2$ to $V(2) = 0.05$ and another drop from $V(10) = 10$ to $V(11) = 5$. From the plot, we obtain $D_{max}(V) = 5$ and $d_{max}(V) = 0.75$. Moreover, these drawdowns are different as far as their times of occurrence are concerned.

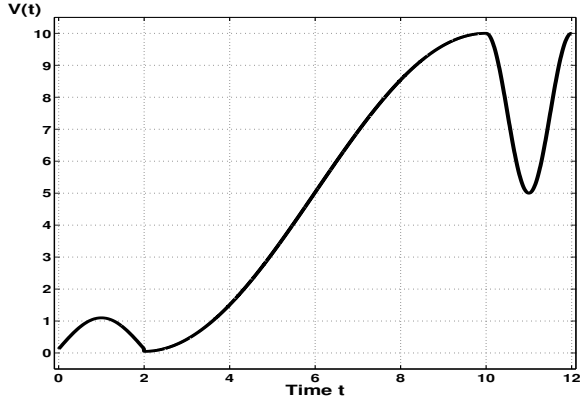


Fig. 1. Maximum Drawdown Versus Maximum Percentage Drawdown

III. MAIN RESULT

In this section, we determine the expected maximum percentage drawdown on wealth using a formula we obtain for logarithmic wealth. When the GBM model for the price generically has non-zero volatility σ , common sense reasoning dictates that a “gambler’s ruin” type of situation presents itself. That is, when the terminal time T is sufficiently large, it becomes exceedingly likely that somewhere along the way, a drawdown which approaches 100% will occur. Furthermore, common sense also dictates that the larger the ratio μ/σ , the longer we expect it to take before one experiences a “bad run” with $V(t)$ tending to zero. The results presented in this section are seen to confirm this intuitive reasoning.

With the above considerations in mind, our plan is as follows: We present a preliminary lemma which connects $d_{max}(V(t))$ to $D_{max}(\log(V(t)))$ along sample paths of the closed loop system. Subsequently, we bring the Q -functions, introduced in [1], into our analysis in two ways: First, we modify these functions so that they apply in the feedback case rather than buy-and-hold. Second, we bring percentage drawdown into the picture with the help of the preliminary lemma to follow.

A. Preliminary Lemma

Note that the lemma below can actually be given in a much more general form than what is used for our analysis. If $V(t)$ is any positive continuous on $[0, T]$ rather than GBM, the same proof can be used.

Preliminary Lemma: *Given any sample path for the account value $V(t)$, it follows that*

$$d_{max}(V) = 1 - e^{-D_{max}(\log(V))}.$$

Proof: Recalling that the sample path $V(t)$ is continuous and non-vanishing on $[0, T]$, $\log(V(t))$ is continuous and hence, $D_{max}(\log(V))$ is well defined as a maximum rather than a supremum. Now, for a sample path $V(t)$, let (s^*, t^*) be any pair which achieves $D_{max}(\log(V))$; i.e.,

$$D_{max}(\log(V)) = \log(V(s^*)) - \log(V(t^*)) = \log\left(\frac{V(s^*)}{V(t^*)}\right).$$

Now, since \log function is increasing, the same pair (s^*, t^*) maximizes $\frac{V(s)}{V(t)}$ and therefore minimizes $\frac{V(t)}{V(s)}$. In turn, this is equivalent to the pair (s^*, t^*) maximizing

$$1 - \frac{V(t)}{V(s)} = \frac{V(s) - V(t)}{V(s)}.$$

The last step is to recognize that

$$\begin{aligned} d_{max}(V) &= 1 - \frac{V(t^*)}{V(s^*)} = 1 - e^{-\log\left(\frac{V(s^*)}{V(t^*)}\right)} \\ &= 1 - e^{-D_{max}(\log(V))}. \quad \square \end{aligned}$$

B. Previous Work: Introduction to Q Functions

In [1] and [12], a pair of real-valued functions $Q_p(x)$ and $Q_n(x)$ are introduced in the analysis of absolute drawdown for the Brownian Motion $dp = \mu dt + \sigma dW_t$. These two functions involve rather complicated integrals which are numerically computed and stored as table of values. In Figure 2, plots are given which summarize the data describing these two numerical functions.

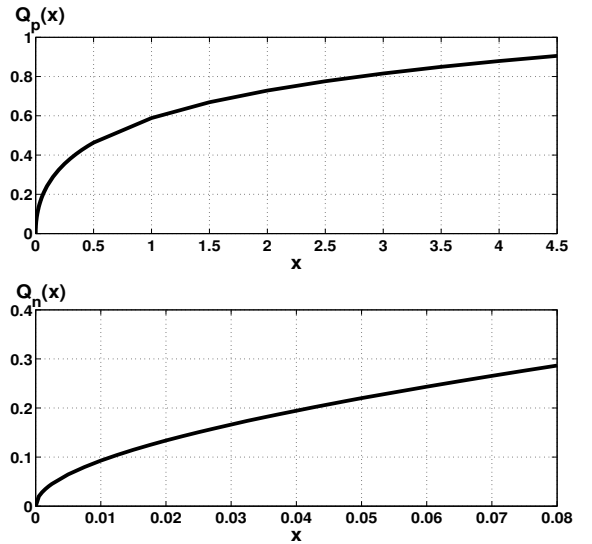


Fig. 2. Plot of the Q Functions

For the case when only the drawdown in price is concerned, as discussed in Section 1, this corresponds to buy-and-hold in our formulation with one share of stock, $K = 1$ and investment $I(t) = V(t) = p(t)$. Hence, for this special case, the results in [1] tell us that

$$\mathbb{E}(D_{max}(V)) = \begin{cases} \frac{2\sigma^2}{\mu} Q_p\left(\frac{\mu^2 T}{2\sigma^2}\right) & \text{if } \mu > 0; \\ 1.2533\sigma\sqrt{T} & \text{if } \mu = 0; \\ \frac{-2\sigma^2}{\mu} Q_n\left(\frac{\mu^2 T}{2\sigma^2}\right) & \text{if } \mu < 0. \end{cases}$$

In addition to handling arbitrary feedback gains, we want to address the case when prices follow the more realistic percentage change model in dp/p rather an absolute change model given above for p . These issues are addressed below.

C. Main Result

In the theorem below, for simplicity of notation, we assume margin and interest rate $r = 0$. The more general case is discussed immediately after the theorem. We also provide some remarks about the drawdown formula obtained and address the asymptotic case when $T \rightarrow \infty$.

Theorem: For the feedback control $I = KV$ in the idealized GBM market with dynamics $dp/p = \mu dt + \sigma dW_t$ and $r = 0$, the maximum absolute drawdown of logarithmic wealth has expected value

$$\mathbb{E}(D_{max}(\log V)) = \begin{cases} \frac{2K\sigma^2}{\mu - \frac{1}{2}K\sigma^2} Q_p \left(\frac{(\mu - \frac{1}{2}K\sigma^2)^2 T}{2\sigma^2} \right) & \text{if } K < \frac{2\mu}{\sigma^2}; \\ 1.2533K\sigma\sqrt{T} & \text{if } K = \frac{2\mu}{\sigma^2}; \\ -\frac{2K\sigma^2}{\mu - \frac{1}{2}K\sigma^2} Q_n \left(\frac{(\mu - \frac{1}{2}K\sigma^2)^2 T}{2\sigma^2} \right) & \text{if } K > \frac{2\mu}{\sigma^2} \end{cases}$$

with corresponding maximum percentage drawdown satisfying the condition

$$\mathbb{E}(d_{max}(V)) \leq 1 - e^{-\mathbb{E}(D_{max}(\log(V)))}.$$

Proof: Recalling the analysis given in Section 2, the stochastic differential equation governing $\log V$ is a GBM with drift and volatility given by

$$\mu_* = K\mu - \frac{1}{2}K^2\sigma^2; \quad \sigma_* = K\sigma$$

respectively. Now, substitution of μ_* for μ , σ_* for σ and $\log V$ for V in $\mathbb{E}(D_{max}(V))$ above, we obtain

$$\mathbb{E}(D_{max}(\log(V))) = \begin{cases} \frac{2\sigma_*^2}{\mu_*} Q_p \left(\frac{\mu_*^2 T}{2\sigma_*^2} \right) & \text{if } \mu_* > 0; \\ 1.2533\sigma_*\sqrt{T} & \text{if } \mu_* = 0; \\ -\frac{2\sigma_*^2}{\mu_*} Q_n \left(\frac{\mu_*^2 T}{2\sigma_*^2} \right) & \text{if } \mu_* < 0. \end{cases}$$

Replacing μ_* and σ_* with their corresponding values in terms of μ , σ and $K > 0$, a straightforward calculation leads to the formula given for $\mathbb{E}(D_{max}(\log(V)))$. To complete the proof, we note the following: For an arbitrary sample path $V(t)$, recalling the Preliminary Lemma above, we have

$$d_{max}(V) = 1 - e^{-D_{max}(\log(V))}.$$

Now, since $\log V$ is concave, upon taking the expectation and using the Jensen's inequality, we obtain

$$\mathbb{E}(d_{max}(V)) \leq 1 - e^{-\mathbb{E}(D_{max}(\log(V)))}. \quad \square$$

D. Non-Zero Margin and Interest Rates

For non-zero margin and interest rates, via a lengthy but straightforward computation, we can readily modify the results in the theorem. We illustrate for $K > 1$, the case when margin arises. Indeed, letting

$$\begin{aligned} \mu_* &\doteq K\mu - r(K-1) - \frac{1}{2}K^2\sigma^2; \\ K_p &\doteq \frac{(\mu - r) + \sqrt{(\mu - r)^2 + 2r\sigma^2}}{\sigma^2}, \end{aligned}$$

with a simple calculation, the small- K regime which corresponds to $\mu_* > 0$ or equivalently $K \in [1, K_p]$,

$$\begin{aligned} \mathbb{E}(D_{max}(\log(V))) &= \frac{2K^2\sigma^2}{K\mu - r(K-1) - \frac{1}{2}K^2\sigma^2} \\ &\times Q_p \left(\frac{(K\mu - r(K-1) - \frac{1}{2}K^2\sigma^2)^2 T}{2K^2\sigma^2} \right). \end{aligned}$$

In the second regime obtained with $\mu_* = 0$ or $K = K_p$,

$$\mathbb{E}(D_{max}(\log(V))) = 1.2533K\sigma\sqrt{T}.$$

Finally, for the third regime when $\mu_* > 0$ or $K > K_p$,

$$\begin{aligned} \mathbb{E}(D_{max}(\log(V))) &= \frac{-2K^2\sigma^2}{K\mu - r(K-1) - \frac{1}{2}K^2\sigma^2} \\ &\times Q_n \left(\frac{(K\mu - r(K-1) - \frac{1}{2}K^2\sigma^2)^2 T}{2K^2\sigma^2} \right). \end{aligned}$$

E. Remarks and Asymptotic Behavior

In the drawdown formulae above, the three regimes for the feedback gain K can be linked to a signal-to-noise type ratio μ/σ for the underlying price process. For a given feedback gain K , the smaller this ratio, the more drawdown we expect to see. For fixed μ and σ , if we allow K to increase, we expect to see larger and larger drawdowns.

It is instructive to consider the asymptotic versions of the results given in the theorem which are obtained as $T \rightarrow \infty$. Analogous to the arguments used in the proof of the theorem, we can readily modify the asymptotic analysis of functions $Q_p(x)$ and $Q_n(x)$ and apply these results to the feedback control problem being considered here. Namely, beginning with the asymptotic estimates

$$Q_p(x) \approx \frac{1}{4} \log x + 0.4988; \quad Q_n(x) \approx x + \frac{1}{2},$$

for x suitably large, via a lengthy but straightforward calculation along the lines given in the proof of the theorem, as $T \rightarrow \infty$, the quantity $\mathbb{E}(D_{max}(\log(V)))$ is estimated in three regimes as follows: In the first regime, obtained with $K < 2\mu/\sigma^2$,

$$\begin{aligned} \mathbb{E}(D_{max}(\log(V))) &\approx \frac{4K\sigma^2}{2\mu - K\sigma^2} \\ &\times (0.63519 + 0.5 \log T + \log \frac{\mu - 0.5K\sigma^2}{\sigma}). \end{aligned}$$

In the second regime, with critical value $K = 2\mu/\sigma^2$,

$$\mathbb{E}(D_{max}(\log(V))) \approx 1.2533K\sigma\sqrt{T}.$$

Finally, in the third regime, obtained with $K > 2\mu/\sigma^2$,

$$\mathbb{E}(D_{max}(\log(V))) \approx -(\mu K - 0.5K^2\sigma^2)T - \frac{K\sigma^2}{\mu - 0.5K\sigma^2}.$$

For all three regimes of K , using the formulae in the theorem, as expected, we see

$$\lim_{T \rightarrow \infty} \mathbb{E}(D_{max}(\log(V))) = \infty.$$

However, the specific value of K has a significant impact on the rate of convergence for this limit. For the small- K

regime, a rate of $\frac{1}{T}$ is obtained from the formulae above. For the critical value case, the rate is $e^{-\sqrt{T}}$ and finally, for the large- K regime, the rate becomes e^{-T} .

IV. NUMERICAL EXAMPLES WITH SIMULATIONS

In this section, we provide plots of the drawdown functions indicating how risk evolves as a function of the duration of the trade T . We recall that when we work with percentage, the theorem in Section 3 provides an upper bound which arises when Jensen’s Inequality is invoked. Hence, it is natural to ask whether the upper bound we obtain is “tight.” We conducted Monte Carlo simulations in order to compare our upper bound with a Monte Carlo estimate of true drawdown.

Indeed, we considered a stock with GBM model with time t measured in years, $T = 5$, annualized drift $\mu = 0.25$ and annualized volatility $\sigma = \sqrt{0.5} \approx 0.7071$. These values were picked intentionally so that the so-called critical value of K in the theorem is given by $K^* \doteq \frac{2\mu}{\sigma^2} = 1$.

For our simulations, one value of K was used for each of the three regimes in the theorem. More specifically, we took $K = 0.1$, $K = 1$ and $K = 2$ in our computations. For these three cases, the Monte Carlo estimates of the $\mathbb{E}(d_{max}(V))$ along with the upper bounds are shown in Figure 3. For each value of T , our Monte Carlo estimate for the expected value of d_{max} was generated using 5000 sample paths. Our estimates appear to have converged quite well in that we do not see significant changes above the one thousand sample path level. In the three simulations below, we see that the “error” between the upper bound in the theorem versus the Monte Carlo estimate is different in each of the three regimes for K . In all cases, the error is at most a few percent.

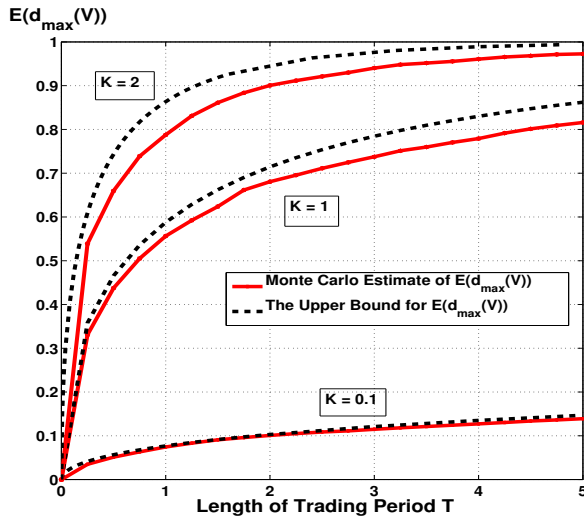


Fig. 3. Bound on $\mathbb{E}(d_{max}(V))$ for Different Values of K

Other Feedback Based Strategies: The drawdown concepts presented here not only apply to the linear feedback $I = KV$ but also to practically any other linear feedback control law which one might imagine. To illustrate, we

consider another known strategy which uses a combination of two linear feedbacks, one long trade in combination with one short trade. This is the so-called Simultaneous Long-Short (SLS) feedback law which has been studied in detail in [13]-[16]. Whereas this feedback control law in these papers is a mapping on the trading gain $g(t)$, so as to maintain consistency with the formulation considered here, we study a version of SLS which operates on $V(t)$.

This strategy can be viewed as the superposition of two independent trades as follows: The trader holds both a long investment $I_L(t) > 0$ and short investment position $I_S(t) < 0$ at the same time. In practice, these two positions can be “netted out” so that the overall investment is given by $I(t) = I_L(t) + I_S(t)$. Now, for feedback gain $K > 0$, these two investment components are given by $I_L(t) = KV_L(t)$ and $I_S(t) = -KV_S(t)$ with initial conditions $V_L(0) = V_S(0) = V_0/2$ and resulting account value evolving over time as $V(t) = V_L(t) + V_S(t)$.

For this SLS system, the results in the literature, for example, see [13] and [14], tell us that for all but the degenerate case when $\mu = 0$, the positive expectation condition $\mathbb{E}(V(T)) > V_0$ is guaranteed. In addition, the mean, variance and skewness of the probability density function of $V(T)$ are increasing functions of K . With this context in mind, we now analyze the drawdown of this scheme.

Since $I_L(t)$ and $I_S(t)$ move in opposite directions, it is natural to expect that the drawdown of V will be smaller than the drawdowns of V_L and V_S separately. Hence, it is also natural to conjecture that the SLS drawdown is always lower than the one obtained in the simulation for the pure long case $I = KV$ above. Furthermore, as time goes on, a large “run-up” in one of these positions is exceedingly probable. Hence we expect to see the drawdown for SLS behave much the same as the pure long case. That is, we conjecture that $\mathbb{E}(d_{max}(V))$ should go to 1 as time T tends to infinity. Finally, in view of the reasoning given above, we expect to see the rate of convergence for this SLS case to be slower than what we obtained with a purely long controller.

Indeed, a Monte Carlo simulation for the SLS controller was carried out for $K = 2$ using the same parameters as in the pure-long simulation above. In Figure 4, we see that the result is consistent with the conjectures given above. For T small, we see $\mathbb{E}(d_{max}(V))$ for the SLS case to be significantly below that obtained for the pure long case with asymptotic behavior as predicted.

V. CONCLUSION AND FURTHER RESEARCH

In the proof of the theorem in Section 3, we used Jensen’s inequality. The results in [1] suggest an avenue of analysis which avoids introducing this inequality but may be computationally prohibitive. That is, $\log(V)$ is a standard Brownian motion, modifying the theory in [1] should be possible to construct the probability density function for $D_{max}(\log(V))$.

As a practical matter, however, in lieu of finding an exact representation of the probability density function, it

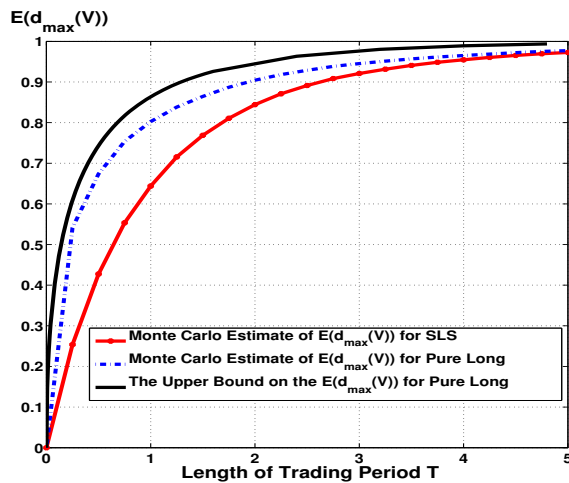


Fig. 4. Estimates of $\mathbb{E}(d_{max}(V))$, SLS and Pure Long for $K = 2$

appears much easier to carry out a Monte Carlo simulation and obtain a histogram to represent the density function for $D_{max}(\log(V))$ or even $d_{max}(V)$. This is illustrated in Figure 5 where the histogram for $d_{max}(V)$ is provided for the trading scenario described by the case $\mu = 0.5, \sigma = 0.5, T = 0.4$ and $K = 1$. The expected value of the maximum percentage drawdown for V is estimated to be $\mathbb{E}(d_{max}(V)) \approx 0.2558$. In contrast, the upper bound given in the theorem in Section 3 is approximately 0.27.

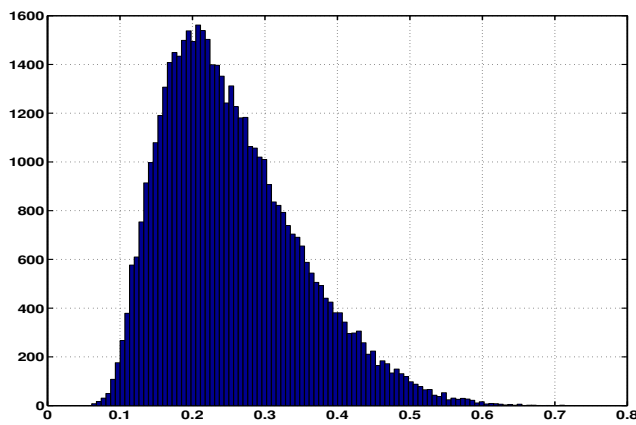


Fig. 5. Histograms of $d_{max}(V)$

Finally, to conclude this paper, we refer back to the discussion in the introduction, involving the ongoing line of research on risk measures. Analogous to the information provided by a conditional value at risk measure, in the case of drawdown, a conservative investor might want to know how large the downside can be if $d_{max}(V) \geq \mathbb{E}(d_{max}(V))$ is experienced. Hence, one can define a conditional version of the percentage measure. That is, along a sample path $V(t)$, let

$$\mathcal{D}_{max}(V) \doteq \mathbb{E}(d_{max}(V) | d_{max}(V) \geq \mathbb{E}(d_{max}(V))).$$

Finally, it would also be of interest to study the extent to

which various measures of drawdown are compatible with the theory of coherent risk measures taking off from [5].

REFERENCES

- [1] M. Ismail, A. Atiya, A. Pratap, Y. Abu-Mostafa, "On the Maximum Drawdown of a Brownian Motion," *Journal of Applied Probability*, vol. 41, pp. 147–161, 2004.
- [2] S. Grossman, and Z. Zhou, "Optimal Investment Strategies for Controlling Drawdowns," *Journal of Mathematical Finance*, vol. 3, pp. 241–276, 1993.
- [3] B. T. Hayes, "Maximum Drawdowns of Hedge Funds with Serial Correlation," *Jour. of Alternative Investments*, vol. 8, pp. 26–38, 2006.
- [4] H. M. Markowitz, "Portfolio Selection: Efficient Diversification of Investments," John Wiley and Sons, 1959.
- [5] P. Artzner, F. Delbaen, J. Eber, D. Heath, "Coherent Measures of Risk," *Journal of Mathematical Finance*, vol. 9, pp. 203–228, 1999.
- [6] P. Jorion, "Value at Risk: The New Benchmark for Managing Financial Risk," McGraw-Hill, 2006.
- [7] Z. Bodie, A. Kane, A. Marcus, "Investments," McGraw-Hill, 2010.
- [8] J. A. Batten, J. Batten, T. A. Fetherston, "Financial Risk and Financial Risk Management," *Research in International Business and Finance Series*, 2002.
- [9] K. Horcher, "Essentials of Financial Risk Management," John Wiley and Sons, 2005.
- [10] M. Ismail, A. Atiya, A. Pratap, Y. Abu-Mostafa, "The Maximum Drawdown of the Brownian Motion," *Proceedings of the IEEE Conference on Financial Engineering*, Hong Kong, March 2003.
- [11] M. Ismail, A. Atiya, "An Analysis of the Maximum Drawdown Risk Measure," *Risk Magazine*, October 2004.
- [12] Q-function. <http://www.cs.rpi.edu/magdon/data/Qfunctions.html>.
- [13] S. Malekpour and B. R. Barmish, "How Useful are Mean-Variance Considerations in Stock Trading via Feedback Control?," *Proceedings of the IEEE Conf. on Decision and Control*, Maui, December 2012.
- [14] B. R. Barmish and J. A. Primbs, "On Arbitrage Possibilities Via Linear Feedback in an Idealized Brownian Motion Stock Market," *Proceedings of the IEEE Conference on Decision and Control*, pp. 2889–2894, Orlando, December 2011.
- [15] B. R. Barmish and J. A. Primbs, "On Market-Neutral Stock Trading Arbitrage Via Linear Feedback," *Proceedings of the American Control Conference*, pp. 3693–3698, Montreal, Canada, June 2012.
- [16] B. R. Barmish, "On Performance Limits of Feedback Control-Based Stock Trading Strategies," *Proceedings of the American Control Conference*, pp. 3874–3879, San Francisco, July 2011.
- [17] B. R. Barmish, "On Trading of Equities: A Robust Control Paradigm," *Proc. IFAC World Congress*, pp. 1621–1626, Seoul, Korea, July 2008.
- [18] G. C. Calafiore, "Multi-Period Portfolio Optimization with Linear Control Policies," *Automatica*, vol. 44, pp. 2463–2473, 2008.
- [19] G. C. Calafiore, "An Affine Control Method for Optimal Dynamic Asset Allocation with Transaction Costs," *SIAM Journal of Control and Optimization*, vol. 48, pp. 2254–2274, 2009.
- [20] G. C. Calafiore and B. Monastero, "Data-Driven Asset Allocation with Guaranteed Short-fall Probability," *Proceedings of the American Control Conference*, pp. 3687–3692, Montreal, Canada, June 2012.
- [21] S. Iwarere and B. R. Barmish, "A Confidence Interval Triggering Method for Stock Trading via Feedback Control," *Proceedings of the American Control Conference*, pp. 6910–6916, Baltimore, July 2010.
- [22] P. Meindl and J. A. Primbs, "Dynamic Hedging with Stochastic Volatility Using Receding Horizon Control," *Proceedings of Financial Engineering Applications*, pp. 142–147, Cambridge, November 2004.
- [23] S. Mudchanatongsuk, J. A. Primbs, and W. Wong, "Optimal Pairs Trading: A Stochastic Control Approach," *Proceedings of the American Control Conference*, pp. 1035–1039, Seattle, June 2008.
- [24] J. A. Primbs, "LQR and Receding Horizon Approaches to Multi-Dimensional Option Hedging under Transaction Costs," *Proceedings of the American Control Conference*, pp. 6891–6896, Baltimore, 2010.
- [25] J. L. Kelly, JR., "A New Interpretation of Information Rate," *IRE Trans. on Information Theory*, vol. 2, pp. 185–189, 1956.
- [26] F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, vol. 81, pp. 637–659, 1973.
- [27] R. C. Merton, "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory*, vol. 3, pp. 373–413, 1971.
- [28] B. Oksendal, "Stochastic Differential Equations: An Introduction with Applications," Springer, 2010.