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Robustness Analysis of Feedback Linearisation with Robust State Estimation for a Nonlinear Missile Model

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***Abstract*— A nonlinear missile model with time-varying uncertain parameters is controlled with a simple feedback linearisation and time-scale separation design, with synthesis based on the nominal model and full state feedback. The closed-loop system is then represented as a linear fractional transformation (LFT). A robust H_∞ filter is designed for the controlled plant, to estimate unknown states. Robust stability of the closed-loop system is then verified by using a scaled linear differential inclusion (LDI) technique.**

I. INTRODUCTION

In the synthesis of a feedback linearising controller, it is common to assume that all the necessary states and controlled outputs are available to the controller. In practice, the available measurements will only be a subset of these. In aerospace systems, models are often highly nonlinear and contain several uncertainties. Therefore, robustness of the state observer and robust stability analysis of the closed-loop system is important. Even if inversion was performed exactly, the system may still be unstable due to the nonminimum phase nature of its (generally nonlinear) internal dynamics [12]. For a robust stability guarantee in the presence of uncertainties in the plant model, we cannot assume that inversion is performed exactly.

Robust stability analysis of systems controlled using feedback linearisation is generally based on simulations e.g. [2], [13], [5]. This also applies to linear parameter varying (LPV) systems e.g. [18]. A systematic, stochastic approach to the nonlinear dynamic inversion synthesis is the focus of [20]. It is possible with these approaches that there is some "worst-case" scenario that is missed, which is why we aim for an analysis technique that will give a robust stability guarantee for all allowed combinations of uncertain parameters. Another interesting approach is observer-based feedback linearisation designed to alleviate the estimated disturbance [2], [10]. These place some restrictive assumptions on the form of the system e.g. full-state linearisable, or uncertainty only on the input channel. We aim for a simple controller design. In [1], integral quadratic constraints (IQCs) are used to perform a robust stability analysis for this system, controlled using feedback linearisation and time scale separation. However, there is no observer i.e. full state feedback is assumed. In [17] the stability analysis assumes the fast subsystem inversion is performed exactly. The assumption that control deflections affect only the moments is carried through from controller synthesis to stability analysis. Also, there is no robustness guarantee, which is the main aim of this paper.

We derive a method for the synthesis of a robust linear time invariant (LTI) filter to estimate unknown states, by solving a system of linear matrix inequalities (LMIs), which is based on [19]. The filter is designed to minimise the L_2 gain from an external input to the estimation error. This method requires that the plant is stable. Therefore, we assume that we are designing a filter for the controlled plant, because the model in question is only marginally stable. As we do not assume *a priori* knowledge of the reference signal, we cannot use the H_2 method given in [19]. We therefore derive the H_∞ condition for a system with structured uncertainty. We also verify robust stability of the closed-loop system, including the filter, which is again a sufficient LMI condition using a linear differential inclusion (LDI) and scalings, based on a quasi-LPV/LFT form for the system.

This paper is organised as follows: In Section II we present a nonlinear missile model. In Section III we give a simple input-ouput linearising controller using time-scale separation. In Section IV, we move from a nonlinear to a quasi-LPV/LFT model, and derive an LMI condition for synthesis of a robust filter. In Section IV-B we show an LMI condition for giving a robust upper-bound on the L_2 gain of the closed-loop system. Nonlinear simulation with uncertainties and filter are given in Section V. Here we also give robust stability analysis using the LMI of Section IV-B. Conclusions are given in Section VI. Notation: $\|X\|$ means the maximum singular value of X. $F_l(X, \Delta)$ means the lower linear fractional transformation of X with Δ [21].

II. THE PLANT

The nonlinear missile model from [15], has states angle of attack and pitch rate α (rad) and q (rad/s), output normal acceleration η (m/s^2) and input tail fin deflection u (rad):

$$\dot{\alpha}(t) = K_1 M(t) C_z(\alpha(t), M(t), u(t)) \cos(\alpha) + q(t) \quad (1a)$$
$$\dot{q}(t) = K_2 M^2(t) C_m(\alpha(t), M(t), u(t)) \quad (1b)$$
$$\eta(t) = K_3 M^2(t) C_z(\alpha(t), M(t), u(t)) \quad (1c)$$

The Mach number $M(t)$ is treated as an exogenous variable. The term $\cos(\alpha) \approx 1$ for the operating range and is therefore neglected from hereon. The aerodynamic coefficients are

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given by:

$$\begin{aligned} C_z(\alpha, M, u) =& z_3|\alpha|^2\alpha + z_2|\alpha|\alpha \\ &+ z_1(2 - M/3)\alpha + z_0 u \\ C_m(\alpha, M, u) =& m_3|\alpha|^2\alpha + m_2|\alpha|\alpha \\ &+ m_1(-7 + 8M/3)\alpha + m_0 u \end{aligned} \tag{2}$$

Measurements available to the controller are q and η. Physical data is given in Table II.

The plant is augmented with a second-order actuator with input commanded tail fin deflection $u_c(t)$(rad) and output $u(t)$:

$$\begin{bmatrix} \dot{u}(t) \\ \ddot{u}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_a{}^2 & -2\xi_a\omega_a \end{bmatrix} \begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_a{}^2 \end{bmatrix} u_c(t) \tag{3}$$

The operating range is given by $|\alpha(t)| \le 20°$ and $1.5 \le M(t) \le 3$. The controller should achieve robust stability over the operating range, to uncertainty in the α and u dependent parts of C_m that can vary independently by $\pm 25\%$. Performance specifications are that the controller should track step commands η_c with maximum time constant 350ms, overshoot 10% and steady-state error 1%. The maximum tail fin deflection rate should meet $|\dot{u}(t)| \le 25°$/s for step command $\eta_c = 1g$.

III. Controller Synthesis

We follow the method in [1] for controller synthesis. The model is nonminimum phase, hence a time-scale separation technique is used. Neglecting the actuator, the plant is split into slow and fast subsystems. The u-dependent term in C_z is neglected in the slow subsystem.

Slow subsystem: The slow subsystem has one state α, input q_c (commanded value for pitch rate) and output η. Defining $\tilde{C}_z := C_z(\alpha, M, 0)$, $e_\eta := \eta - \eta_c$ and following standard input-output linearising controller synthesis [12], the slow subsystem controller is

$$q_c = -K_1 M \tilde{C}_z + \left(K_3 M^2 \frac{\partial C_z}{\partial \alpha} \right)^{-1} (-k_1 e_\eta + \dot{\eta}_c) \tag{4}$$

which will, for the *approximate* slow subsystem, achieve asymptotic tracking of η_c with first-order dynamics. The term $\partial C_z(|\alpha|, M)/\partial\alpha$ has one positive root $|\alpha| = 73 - 69°$ for $M = 1.5 - 3$, which is far outside the operating range, hence the controller is well-defined.

Fast subsystem: The fast subsystem has one state q, input u_c and output q. Defining $\tilde{C}_m := C_m(\alpha, M, 0)$, $e_q := q - q_c$ an input-output linearising controller with second-order dynamics is given by

$$u_c = \frac{-\tilde{C}_m}{m_0} + \left(K_2 M^2 m_0\right)^{-1} \left(-k_2 q - k_3 \int e_q dt \right) \tag{5}$$

to achieve asymptotic tracking of q_c.

The controller gains k_1, k_2 and k_3 were tuned in [1] using a genetic algorithm, for particular values of Mach. Here, we use the gains calculated in [1] for the nominal Mach value $M = 2.25$: $k_1 = 4.69$, $k_2 = 18.3$, $k_3 = 211$.

IV. Filter Synthesis and Stability Analysis

A. Robust Filter Synthesis

In [19], LMI conditions are derived for synthesis of an LTI filter for a system with uncertainty that can be represented as an LFT. For a system with structured uncertainty, only the H_2 result is given in [19]. However, it is stated in [19] that the H_∞ result can be derived using the methods given in that paper, which is what we do here.

The controlled plant with actuator can be written in quasi-LPV form, treating a reference demand r as the external input to the system and measured output y_m:

$$\begin{bmatrix} \dot{x}(t) \\ y_m(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\theta(t)) & \mathcal{B} \\ \mathcal{C}(\theta(t)) & \mathcal{D} \end{bmatrix} \begin{bmatrix} x(t) \\ r(t) \end{bmatrix} =: M(\Delta(t)) \begin{bmatrix} x(t) \\ r(t) \end{bmatrix} \tag{6}$$

$x \in \mathbb{R}^n$, $y_m \in \mathbb{R}^{n_y}$, $r \in \mathbb{R}^{n_r}$, $\theta \in \mathbb{R}^{n_\theta}$, where x contains controller, plant and actuator states and $\theta(t)$ contains time-varying uncertain parameters and states in which the plant is nonlinear. Due to space constraints, we assume that the bound on the endogenous part of the quasi-LPV form is valid. It is generally necessary [16] to find a bound on the energy of the input r for which we can find a bound on the states that form the endogenous part of $\theta(t)$. This can be made computationally tractable by finding an ellipsoidal region that lies within a polytope defined by the bound on Δ (a condition for which is given in [7], [9]). With the system in quasi-LPV form, the uncertainties and nonlinear terms can be pulled out, and the system written in LFT form, i.e.

$$\begin{aligned} M(\Delta(t)) &= F_l(H, \Delta) \\ &= \begin{bmatrix} A & B_r \\ C_y & D_{yr} \end{bmatrix} + \begin{bmatrix} B_p \\ D_{yp} \end{bmatrix} \Delta(t)(I - D_{qp}\Delta(t))^{-1} \begin{bmatrix} C_q & 0 \end{bmatrix} \end{aligned} \tag{7}$$

where

$$H = \left[\begin{array}{cc:c} A & B_r & B_p \\ C_y & D_{yr} & D_{yp} \\ \hdashline C_q & 0 & D_{qp} \end{array} \right] \tag{8}$$

and $\Delta(t)$ is a block-diagonal matrix with n_θ blocks. Here, each diagonal block consists of a repeated scalar $\delta_i(t)$; the normalised variation in one of the elements of $\theta(t)$. The size of each block, k_i, depends on the degree of nonlinearity with which that scalar appears in the system equations. $\Delta(t)$ therefore represents structured uncertainty and is norm-bounded, with allowed values in the set (here $l = n_\theta$):

$$\begin{aligned} \mathbf{\Delta} := \{ &\mathrm{diag}(\delta_1 I_{k_1}, \ldots, \delta_l I_{k_l}) : \|\Delta\| \le \sigma^{-1}, \\ &\delta_i \in \mathbb{R},\ \sigma > 0 \} \subset \mathbb{R}^{n_p \times n_p} \end{aligned} \tag{9}$$

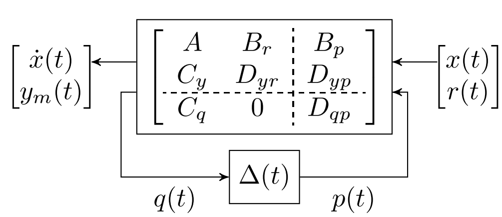

Fig. 1. Linear fractional representation of the controlled plant (6).

$$\dot{x}(t) = Ax(t) + B_r r(t) + B_p p(t) \tag{10a}$$
$$y_m(t) = C_y x(t) + D_{yr} r(t) + D_{yp} p(t) \tag{10b}$$
$$q(t) = C_q x(t) + D_{qp} p(t) \tag{10c}$$
$$p(t) = \Delta(t) q(t) \tag{10d}$$

with p, $q \in \mathbb{R}^{n_p}$. We want to estimate

$$z(t) := Lx(t), \quad L \in \mathbb{R}^{n_z \times n}$$

with a full-order LTI filter of the form

$$\dot{\hat{x}}(t) = A_f \hat{x}(t) + B_f y_m(t) \tag{11a}$$
$$\hat{z}(t) = L_f \hat{x}(t) \tag{11b}$$

where $A_f \in \mathbb{R}^{n \times n}$, $B_f \in \mathbb{R}^{n \times n_y}$ and $L_f \in \mathbb{R}^{n_z \times n}$ are the state-space matrices of the filter, to be found. Substitute (10b), in (11a):

$$\dot{\hat{x}} = A_f \hat{x}(t) + B_f \left[C_y x(t) + D_{yr} r(t) + D_{yp} p(t) \right] \tag{12}$$

Define the augmented state vector $\mu := \begin{bmatrix} x^T & \hat{x}^T \end{bmatrix}^T \in \mathbb{R}^{2n}$, then:

$$\dot{\mu} = \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ B_f C_y & A_f \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_r \\ B_f D_{yr} \end{bmatrix} r + \begin{bmatrix} B_p \\ B_f D_{yp} \end{bmatrix} p$$
$$=: \bar{A}\mu + \bar{B}r + \bar{L}p \tag{13}$$

Define the output of the augmented system as the estimation error:

$$e_z := z - \hat{z} = Lx - L_f \hat{x} = \begin{bmatrix} L & -L_f \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} =: \bar{C}\mu \tag{14}$$

Then, defined in terms of μ, (10c) becomes:

$$q = \begin{bmatrix} C_q & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + D_{qp} p =: \bar{E}\mu + D_{qp} p \tag{15}$$

and (10d), as before (augmenting with the filter does not change the p, q relation). We can represent the augmented system as an LFT, analogously to the controlled plant, however as $\mathcal{B}$ and $\mathcal{D}$ are known constant matrices, we can represent the system in the following LFT:

$$\begin{aligned} \bar{A}_\Delta &:= F_l \left(\begin{bmatrix} \bar{A} & \bar{L} \\ \bar{E} & D_{qp} \end{bmatrix}, \Delta(t) \right) \\ &= \bar{A} + \bar{L}\Delta(t)(I - D_{qp}\Delta(t))^{-1}\bar{E} \end{aligned} \tag{16}$$

This means that (13), (14), (15), (10d) are assumed to be equivalent to:

$$\dot{\mu}(t) = \bar{A}_\Delta \mu(t) + \bar{B} r(t) \tag{17a}$$
$$e_z(t) = \bar{C}\eta(t) \tag{17b}$$

We would like to find the state-space filter matrices A_f, B_f and L_f that minimise, in a H_∞ sense, the L_2 gain from r to e_z.

Theorem 1: For the LDI (17) and a given $\sigma > 0$ and assuming the energy of the input is such that the bound on the endogenous part of $\theta(t)$ is valid, minimise β^2 subject to the LMIs $P_0 > 0$, $P_1 - P_0 > 0$ and (18) (below), then a robust LTI filter is given by state-space matrices $A_f = P_3^{-1} M_A P_3^{-1}$, $B_f = P_3^{-1} M_B$, $L_f = M_L P_3^{-1}$ where $P_3 = P_0^{1/2}$ and β is an upper bound on the L_2 gain from input r to estimation error e_z. Defining $P = \begin{bmatrix} P_1 & P_3 \\ \star & I_n \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$, $V(\mu) = \mu^T P \mu$ is a Lyapunov function that proves it. Also, the LDI is well-posed. The variables in (18) are P_1, P_0, $M_A \in \mathbb{R}^{n \times n}$, $M_B \in \mathbb{R}^{n \times n_y}$, $M_L \in \mathbb{R}^{n_z \times n}$ and $\beta \in \mathbb{R}$.

$$X := \begin{bmatrix} X_1 & X_3 \\ \star & X_2 \end{bmatrix} < 0 \tag{18}$$

where

$$X_1 := \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \star & \Psi_{22} & \Psi_{23} \\ \star & \star & \Psi_{33} \end{bmatrix}$$

$$\begin{aligned} \Psi_{11} &= P_1 A + M_B C_y + A^T P_1 + C_y^T M_B^T + C_q^T S C_q \\ \Psi_{12} &= M_A + A^T P_0 + C_y^T M_B^T \\ \Psi_{13} &= P_1 B_p + M_B D_{yp} + C_q^T S D_{qp} + C_q^T G \\ \Psi_{22} &= M_A + M_A^T, \quad \Psi_{23} = P_0 B_p + M_B D_{yp} \\ \Psi_{33} &= D_{qp}^T S D_{qp} + D_{qp}^T G - G D_{qp} - \sigma^2 S \end{aligned}$$

$$X_2 := \begin{bmatrix} -\beta^2 I_{n_r} & 0 \\ 0 & -I_{n_z} \end{bmatrix}$$

$$X_3 := \begin{bmatrix} P_1 B_r + M_B D_{yr} & L^T \\ P_0 B_r + M_B D_{yr} & -M_L^T \\ 0 & 0 \end{bmatrix}$$

Proof: By standard control system theory [4], if $\exists\, 0 < P \in \mathbb{R}^{2n \times 2n}$ and $\beta > 0$ such that

$$\begin{bmatrix} \bar{A}_\Delta^T P + P\bar{A}_\Delta + \bar{C}^T \bar{C} & P\bar{B} \\ \star & -\beta^2 I \end{bmatrix} < 0, \ \forall\, \Delta(t) \in \mathbf{\Delta} \tag{19}$$

then the induced L_2 gain from reference r to estimation error e_z for the augmented system (17) is less than β for all permitted values of Δ.

By Schur complement [4], (19) is equivalent to $-\beta^2 I < 0$ (which is obvious), together with

$$T_1 + T_2\Delta(I - T_4\Delta)^{-1} T_3 + T_3^T (I - T_4\Delta)^{-T} \Delta^T T_2^T < 0 \tag{20}$$

where

$$\begin{aligned} T_1 &:= \bar{A}_\Delta^T P + P\bar{A}_\Delta + \bar{C}^T\bar{C} + \frac{1}{\beta^2} P\bar{B}\bar{B}^T P \\ T_2 &:= P\bar{L},\ T_3 := \bar{E},\ T_4 := D_{qp} \end{aligned} \tag{21}$$

We now associate with $\mathbf{\Delta}$ the subspaces of block-diagonal scaling matrices (representing relationships for real, structured uncertainty):

$$\begin{aligned} \mathbf{S} &:= \left\{ \operatorname{diag}(S_1, \ldots, S_l) : 0 < S_i \in \mathbb{R}^{k_i \times k_i} \right\} \subset \mathbb{R}^{n_p \times n_p} \\ \mathbf{G} &:= \left\{ \operatorname{diag}(G_1, \ldots, G_l) : G_i = -G_i^T \in \mathbb{R}^{k_i \times k_i} \right\} \subset \mathbb{R}^{n_p \times n_p} \end{aligned}$$

Then [8], [9] (20) holds and is well-posed $\forall\, \Delta(t) \in \mathbf{\Delta}$, i.e. $\det(I - T_4\Delta) \neq 0$, if $\exists\, S \in \mathbf{S}$ and $G \in \mathbf{G}$ such that

$$\begin{bmatrix} T_1 + T_3^T S T_3 & T_2 + T_3^T S T_4 + T_3^T G \\ \star & T_4^T S T_4 + T_4^T G - G T_4 - \sigma^2 S \end{bmatrix} < 0 \quad (22)$$

This is a sufficient condition and therefore introduces conservatism.

The augmented system matrices, as defined in (13), (14) and (15), are substituted into (22), with P partitioned as $\begin{bmatrix} P_1 & P_3 \\ \star & P_2 \end{bmatrix}$. The filter matrices are not fixed and therefore, without loss of generality [6], we can assume that $P_2 = I_n$. This results in a nonaffine matrix inequality $Z < 0$, with variables P_1, P_3, S, G, A_f, B_f, L_f and β. The goal is to find the filter matrices that minimise β, subject to $Z < 0$, $P > 0$, $\beta > 0$, $S \in \mathbf{S}$ and $G \in \mathbf{G}$.

Making Z affine requires a nonlinear change of variables and the use of Schur complement. First, define $P_0 := P_3 P_3^T$, so $P_3 = P_0^{1/2}$. By Schur complement on P, we require $P_0 > 0$ and $P_1 - P_0 > 0$. Next, we define the new variables $M_A := P_3 A_f P_3^T$, $M_B := P_3 B_f$ and $M_L := L_f P_3^T$. Then, we pre and post multiply Z by J^T and J respectively, where $J := \text{diag}\left(I_n, P_3^T, I_{n_p}\right)$. This results in the new matrix inequality

$$\bar{Z} := J^T Z J = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ \star & Z_{22} & Z_{23} \\ \star & \star & Z_{33} \end{bmatrix} < 0 \quad (23)$$

where

$$Z_{11} = P_1 A + M_B C_y + A^T P_1 + C_y^T M_B^T + L^T L + \dots$$

$$+ C_q^T S C_q + \beta^{-2} (P_1 B_r + M_B D_{yr})(B_r^T P_1 + D_{yr}^T M_B^T)$$

$$Z_{12} = M_A + A^T P_0 + C_y^T M_B^T - L^T M_L + \dots$$

$$\dots + \beta^{-2} (P_1 B_r + M_B D_{yr})(B_r^T P_0 + D_{yr}^T M_B^T)$$

$$Z_{13} = P_1 B_p + M_B D_{yp} + C_q^T S D_{qp} + C_q^T G$$

$$Z_{22} = M_A + M_A^T + M_L^T M_L \dots$$

$$\dots + \beta^{-2} (P_0 B_r + M_B D_{yr})(B_r^T P_0 + D_{yr}^T M_B^T)$$

$$Z_{23} = P_0 B_p + M_B D_{yp} \quad \text{and} \quad Z_{33} = \Psi_{33}$$

It can be seen that $\bar{Z}$ is still not affine in the variables, so we use Schur complement. We have $\bar{Z} = X_1 - X_3 X_2^{-1} X_3^T < 0$, with X_1, X_2 and X_3 defined as in (18) and $X_2 < 0$, hence by Schur complement $\bar{Z} < 0$ is equivalent to $\begin{bmatrix} X_1 & X_3 \\ \star & X_2 \end{bmatrix} < 0$. ■

B. Robust L_2 Gain

In this section, we present a method for robust stability analysis, by solving a system of LMIs. The proof is straightforward to derive from the literature [4], [9], [7]. In Section V, we will apply this analysis to the closed-loop system formed by the plant, controller, actuator and filter.

The system under consideration is again an LDI, similarly to the case for filter synthesis, with $\mathbf{\Delta}$, $\mathbf{S}$ and $\mathbf{G}$ defined analogously. The external input is $r \in \mathbb{R}^{n_r}$, output for performance analysis is $e \in \mathbb{R}^{n_e}$ and states $x \in \mathbb{R}^n$:

$$\dot{x}(t) = Ax(t) + B_r r(t) + B_p p(t) \quad (25a)$$

$$e(t) = C_e x(t) + D_{er} r(t) + D_{ep} p(t) \quad (25b)$$

$$q(t) = C_q x(t) + D_{qr} r(t) + D_{qp} p(t) \quad (25c)$$

$$p(t) = \Delta(t) q(t) \quad (25d)$$

where $q, p \in \mathbb{R}^{n_p}$.

Theorem 2: For a given $\sigma > 0$ and assuming the energy of the input is such that the bound on the endogenous part of $\theta(t)$ is valid, if $\exists\, P > 0$, $S \in \mathbf{S}$, $G \in \mathbf{G}$ and $\gamma > 0$, such that LMI (26) holds, then the LFT system (25) has a finite L_2 gain from input r to output e, with upper bound γ, $\forall\, \Delta \in \mathbf{\Delta}$. Moreover, the LDI is well-posed, i.e. $\det\left(I - D_{qp}\Delta(t)\right) \neq 0$.

$$\Pi := \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ \star & \Pi_{22} & \Pi_{23} \\ \star & \star & \Pi_{33} \end{bmatrix} < 0 \quad (26)$$

where

$$\Pi_{11} = PA + A^T P + C_e^T C_e + C_q^T S C_q$$

$$\Pi_{12} = P B_r + C_e^T D_{er} + C_q^T S D_{qr}$$

$$\Pi_{13} = P B_p + C_e^T D_{ep} + C_q^T S D_{qp} + C_q^T G$$

$$\Pi_{22} = D_{er}^T D_{er} - \gamma^2 I_{n_r} + D_{qr}^T S D_{qr}$$

$$\Pi_{23} = D_{er}^T D_{ep} + D_{qr}^T S D_{qp} + D_{qr}^T G$$

$$\Pi_{33} = D_{ep}^T D_{ep} + D_{qp}^T S D_{qp} - \sigma^2 S + D_{qp}^T G - G D_{qp}$$

Given that we only have a sufficient condition for stability, we may find that the LMIs are not feasible with $\sigma = 1$. This means that we cannot find, by this method, a single quadratic Lyapunov function that guarantees stability over all allowed values of $\Delta(t)$. By increasing σ iteratively, we may be able to satisfy the LMIs, at the cost of reducing the bound on Δ.

V. Simulation & Results

Filter: We aim for a filter that will give an estimate of α, using pitch rate q as the measured output from the plant. The controlled plant with actuator can be written in quasi-LPV form, treating the reference demand as the external input to the system, $r := \begin{bmatrix} \eta_c & \dot{\eta}_c \end{bmatrix}^T$ and measured output $y_m := q$. In order to do this, we define $|\alpha(t)| \in [0, 0.349\text{rad}]$, $C_m = d_1(t)\tilde{C}_m(\alpha(t), M(t)) + d_2(t) m_0 u(t)$, with d_1 and $d_2 \in [0.75, 1.25]$. Then $x := \begin{bmatrix} \alpha & q & u & \dot{u} & e_q \end{bmatrix}^T$ and

$\theta(t) = \begin{bmatrix} |\alpha| & M & d_1 & d_2 \end{bmatrix}^T$. We create the LFT using the free MATLAB toolbox [14] and the LFT is normalised such that $\sigma = 1 \Leftrightarrow \|\Delta\| = 1$. We obtain an LFT with $\Delta(t) \in \mathbb{R}^{17\times 17}$: 4 in $\delta_\alpha(t)$, 11 in $\delta_M(t)$, 1 in $\delta_{d1}(t)$ and 1 in $\delta_{d2}(t)$, where these are the normalised variations in the elements of $\theta(t)$ about their nominal values.

Theorem 1 is used to obtain the filter state-space matrices. We find that the LMIs are not feasible with $\sigma^2 = 1$, which means we cannot find a filter that guarantees a robust L_2 gain from r to e_z for all allowed values of $\theta(t)$. We therefore increase σ^2 iteratively, until we find the smallest value for which the LMIs are feasible. The smallest value is $\sigma^2 = 1.36$, which implies $\|\Delta\| = 0.86$, with a corresponding decrease in the range of $\theta(t)$ over which robust stability is guaranteed. The filter matrices obtained are:

$$A_f = \begin{bmatrix} -8245 & 3127 & 1492 & -155.5 & 43700 \\ 4003 & -1692 & -797.0 & 58.93 & -21200 \\ 1190 & -429.6 & -313.9 & 85.77 & -6308 \\ 182.3 & -126.5 & -51.79 & -18.87 & -960.1 \\ 43590 & -16500 & -7873 & 821.6 & -231100 \end{bmatrix}$$

$$B_f = \begin{bmatrix} -35150 & 17050 & 5074 & 773.4 & 185900 \end{bmatrix}^T$$

$$L_f = \begin{bmatrix} 0.9901 & -0.2175 & 0.2883 & 0.1374 & 0.1906 \end{bmatrix}$$

Robust Stability & Performance: Results for the L_2 performance analysis using Theorem 2 are given in Table I. The output for performance analysis is $e := \eta - \eta_c$. We minimise γ, for a given value of σ^2. As with the filter synthesis, this is a linear objective with LMI constraints and is solved using the MATLAB toolbox [3]. Our goal is to minimise σ^2 whilst still being able to find a finite L_2 gain (much like the approach in [11] for robust stability analysis). We find that the LMIs are not feasible with $\sigma^2 = 1$. The smallest value for which we can obtain a result is $\sigma^2 = 1.42$, which implies a robust L_2 gain, not over all allowed values of $\theta(t)$, but for $M \in [1.62, 2.88]$, $|\alpha| \in [1.61, 18.4°]$ and d_1, d_2 represent $\pm 21\%$ on the α and u dependent parts of C_m, respectively. We obtain $\gamma = 198$, which of course does not guarantee good robust performance in tracking η_c. By increasing σ^2 we can obtain better values of γ, at the cost of further reducing the region over which we can give a robustness guarantee. We note that increasing σ^2 beyond about 10 does not reduce γ significantly.

TABLE I

ROBUST STABILITY

σ^2	$\|\Delta\|$	γ
1	1	infeasible
1.42	0.84	198
2	0.71	4.57
10	0.32	1.77
100	0.1	1.47
1000	0.032	1.42

Nonlinear simulation: The simulation results are given in figs. 2 to 5 for a series of constant step demands in η_c. This is the same series of steps carried out in [15] and [1]. There are four simulations shown: case 1 is the nominal model without filter; cases 2 to 4 include the filter. All simulations were done at $M = 3$ (constant) and included the actuator. Case 2 is the nominal model with the filter (no uncertainty on C_m, i.e. $d_1 = d_2 = 1$). Cases 3 and 4 include independent time-varying uncertainties $d_1(t)$ and $d_2(t)$. For case 3 $d_1 = 1+0.25sin(2\pi t/4.5+0.1)$, $d_2 = 1+0.25sin(2\pi t/4.5+0.2)$. For case 4 $d_1 = 1 + 0.25sin(2\pi t/4.5 + 0.3)$, $d_2 = 1 + 0.25sin(2\pi/9 - 0.8)$. These were chosen such that for case 3 the initial values of the uncertainties are small, whereas for case 4 the initial values are larger. It can be seen that for case 4, the performance for the initial $30g$ step is significantly poorer.

The approximation made in the slow subsystem control synthesis, produces a steady-state error, which is quite noticeable for the larger step commands. This error essentially comes from the fact that the slow subsystem command for q_c is not the correct value in order to achieve $\dot{e}_\eta = -ke_\eta$. This means that there is an equilibrium $\dot{e}_\eta = 0$ when $e_\eta \neq 0$. This error is present even for the nominal case and is given by (for constant Mach):

$$e_{\eta,ss} = -\frac{K_1 K_3 M^3 z_0}{k_1 m_0} \frac{\partial C_z}{\partial \alpha}(|\alpha_{ss}|, M)\tilde{C}_m(\alpha_{ss}, M)$$

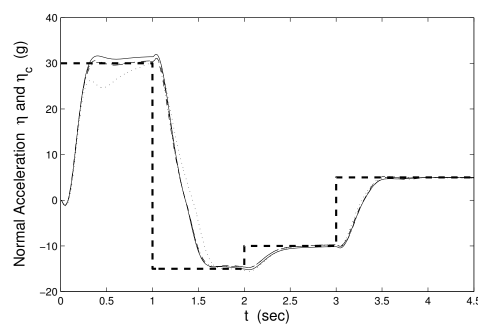

Fig. 2. Normal acceleration η for case 1 (solid line), case 2 ('- -'), case 3 ('-.') and case 4 ('..'), for the series of step commands (thick dashed line).

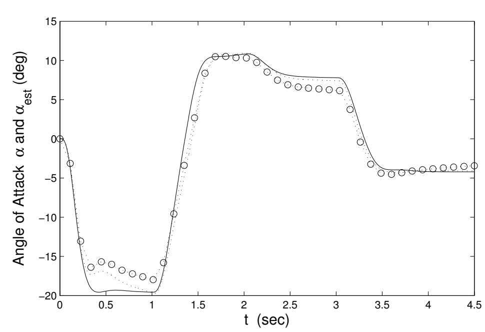

Fig. 3. Angle of attack for case 1 α (solid line), case 4 $\alpha, \hat{\alpha}$ ('..', '.o.').

VI. CONCLUSIONS

The benefit of the methods presented here is that they require similar LFT system representations for the robust

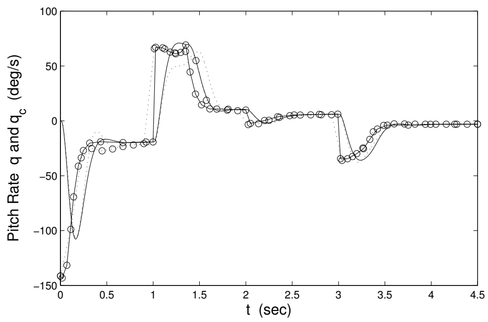

Fig. 4. Pitch rate q and q_c for case 1 (solid line, '-o-') and for case 2 ('..','.o.').

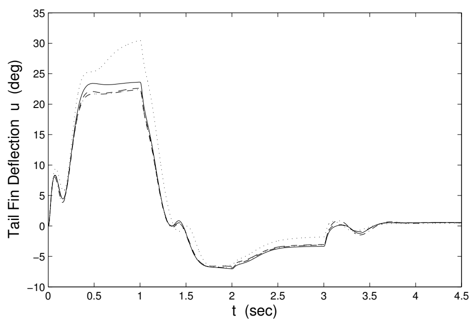

Fig. 5. Tail fin deflection for case 1 (solid line), case 2 ('- -'), case 3 ('-.') and case 4 ('..').

filter synthesis and for the robust performance analysis. We have shown that these methods can be applied successfully to a nonlinear, uncertain system with a simple feedback linearisation controller. Also, we have found that the filter performs well in all simulations, even in the presence of time-varying uncertainty.

Future work will focus on 1) design of robust observers for better state estimation (the filter does not calculate the residual in the output) and 2) reducing conservatism in the analysis of the overall system, by IQC or sum-of-squares techniques.

TABLE II
PHYSICAL DATA

P_0	46602 Pa	(static pressure at 20,000 ft)
v	315.9 m/s	(speed of sound at 20,000 ft)
S	0.04 m^2	(surface area)
m	204.02 kg	(mass)
d	0.23 m	(diameter)
I_y	247.44 kg/m^2	(pitch moment of inertia)
K_1	$0.7P_0S/mv$	
K_2	$0.7P_0Sd/I_y$	
K_3	$0.7P_0S/m$	
z_3	19.3470 rad^{-3}	
z_2	-31.0084 rad^{-2}	
z_1	-9.7174 rad^{-1}	
z_0	-1.9481 rad^{-1}	
m_3	40.4847 rad^{-3}	
m_2	-64.1657 rad^{-2}	
m_1	2.9221 rad^{-1}	
m_0	-11.8029 rad^{-1}	
ξ_a	0.7	(actuator damping ratio)
ω_a	150 rad/s	(actuator undamped natural frequency)

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