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Incremental Stability of Planar Filippov Systems

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***Abstract*— We study the problem of proving incremental stability of a planar Filippov system. In particular, referring to systems that present an attractive sliding region on their discontinuity boundary, we will give a differential condition on such region able to guarantee incremental exponential stability of sliding mode trajectories. We will then derive conditions for the incremental stability of the whole system. The approach is based on using tools from contraction theory, extending their applicability to include discontinuous dynamical systems.**

I. INTRODUCTION

Incremental stability has been defined in [1] for a generic nonlinear dynamical system. It characterizes asymptotic convergence of trajectories with respect to one another rather than towards some attractor known a priori. Such idea of convergence is closely related to other tools and definitions explored in the dynamical systems and control literature (see [2], [3] and also [4], [5], [6], [7], [8] and references therein for more details).

Despite the many results available in the literature to study contraction and incremental stability of continuously differentiable systems, few results deal with the problem of investigating incremental exponential stability of piecewise smooth systems (PWS). Convergence properties, which can be related to incremental stability, have been considered in [9], [10] and [11] for PWA systems and in [12] for PWS continuous systems, while an extension of contraction to PWS continuous (but not always differentiable) systems was outlined in [13] and formalized in [14], [15]. To the best of our knowledge, studying incremental stability of Filippov systems with sliding mode solutions is instead a completely unexplored problem.

In this paper, we address this open research challenge by considering the case of planar Filippov systems. The approach is based on two steps. Firstly, we derive local conditions for contraction of the sliding vector field, guaranteeing that solutions on the sliding surface exponentially converge towards each other. Then, we give conditions for the sliding region to be attractive so that when both set of conditions are satisfied the whole system is exponentially incrementally stable. Two numerical examples are used to illustrate the theory.

We wish to emphasize that the focus on planar Filippov systems serves as a useful starting point to embark on the investigation of higher-dimensional Filippov systems. Indeed, the extension to higher dimensions of the ideas presented in this paper is currently under investigation.

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II. MATHEMATICAL PRELIMINARIES

Definition 1: [16] Let us consider a finite collection of disjoint, open and non-empty sets $\mathcal{S}_1, \dots \mathcal{S}_p$, such that $\mathcal{D} \subseteq \bigcup_{k=1}^{p} \bar{\mathcal{S}}_k \subseteq \mathbb{R}^n$ is a connected set, and that the intersection $\Sigma_{hk} := \bar{\mathcal{S}}_h \cap \bar{\mathcal{S}}_k$ is either a $\mathbb{R}^{n-1}$ lower dimensional manifold or it is the empty set. A dynamical system $\dot{x} = f(t,x)$, with $f : [t_0, +\infty) \times \mathcal{D} \mapsto \mathbb{R}^n$, is called a *piecewise smooth dynamical system* (PWS) when it is defined by a finite set of ODEs, that is, when

$$f(t,x) = F_k(t,x), \qquad x \in \mathcal{S}_k,\ k = 1,\dots,p, \tag{1}$$

with each vector field $F_k(t,x)$ being smooth in both the state x and the time t for any $x \in \mathcal{S}_k$. Furthermore, each $F_k(t,x)$ is continuously extended on the boundary $\partial\mathcal{S}_k$.

For PWS systems such as (1), different kinds of solutions can be considered (see [17] and references therein). In this paper we use the concept of *Filippov solutions* [18]. Notice that, as reported in [17], for the discontinuous system (1), a Filippov solution exists under the mild hypothesis of local boundedness. As stated in [18], p. 90, Filippov solutions show continuous dependence on the initial conditions.

A common class of PWS dynamical systems, which is particularly important in many applications [19], [20], [16], [21], is that of *bimodal PWS systems* that can be written in the form:

$$f(t,x) = \begin{cases} F_1(t,x) & \textit{if } H(x) \leq 0 \\ F_2(t,x) & \textit{if } H(x) > 0 \end{cases}, \tag{2}$$

where $F_1(t,x)$ and $F_2(t,x)$ are two smooth vector fields and $H(x)$ is a smooth scalar function. $H(x) = 0$ with $\frac{\partial}{\partial x}H(x) \neq 0$ defining the smooth discontinuity manifold Σ in $\mathbb{R}^n$.

In this paper we will restrict our attention to the class of systems that can exhibit sliding mode solutions: a specific Filippov solution characterized by the fact that the evolution of the PWS dynamical system belongs to (or "slides along") the discontinuity manifold $H(x) = 0$. More precisely, we give the following definition [16]:

Definition 2: The *sliding region* for a system of the form (2) is given by the set:

$$\hat{\Sigma} = \{x \in \mathbb{R}^n : H(x) = 0, \mathcal{L}_{F_1}H(x) \cdot \mathcal{L}_{F_2}H(x) < 0\},$$

where $\mathcal{L}_{F_i}H(x) := \frac{\partial}{\partial x}H(x)F_i(t,x)$ is the *Lie derivative* of $H(x)$ with respect to the vector field $F_i(t,x)$, that is the component of $F_i(t,x)$ normal to the discontinuity manifold at the point x.

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The equations of the sliding flow can be written using *Filippov's convex method* [18] as:

$$F_s(t,x) = \frac{(1-\beta(t,x))}{2} F_1(t,x) + \frac{(1+\beta(t,x))}{2} F_2(t,x), \tag{3}$$

with $\beta(t,x) \in [-1,1]$ given by:

$$\beta(t,x) = \frac{\mathcal{L}_{F_1}(H) + \mathcal{L}_{F_2}(H)}{\mathcal{L}_{F_1}(H) - \mathcal{L}_{F_2}(H)}; \tag{4}$$

and the sliding region can also be defined as the set [16]:

$$\hat{\Sigma} = \{x \in \mathbb{R}^n : H(x) = 0, -1 \leq \beta(t,x) \leq 1\},$$

while its boundary is given by the set:

$$\partial\hat{\Sigma} = \{x \in \mathbb{R}^n : H(x) = 0, \beta(t,x) = \pm 1\},$$

where tangency of one vector field or the other occurs when $\beta(t,x) = \pm 1$. Notice that a sliding mode solution satisfies the definition of Filippov solution given before.

In what follows, given an n-dimensional vector x, we will denote with $|x|$ its generic norm.

Incremental stability [1] can be defined as follows:

Definition 3: A dynamical system of the form $\dot{x} = f(t,x), \quad x(t_0) = x_0$ is said to be *incrementally asymptotically stable* (δAS) in an invariant connected set $\mathcal{D} \subseteq \mathbb{R}^n$ if there exists a function ς of class $\mathcal{KL}$ [22] such that for all $\xi, \zeta \in \mathcal{D}$ and all $t \geq t_0$ the trajectories $x(t) = \varphi(t-t_0, t_0, \xi)$ and $y(t) = \varphi(t-t_0, t_0, \zeta)$, starting respectively from the two initial conditions ξ and ζ, satisfy:

$$|x(t) - y(t)| \leq \varsigma(|\xi - \zeta|, t) \qquad \forall t \geq t_0, \tag{5}$$

Furthermore, if there exist constants $K, c > 0$ such that the following holds

$$|x(t) - y(t)| \leq K e^{-c(t-t_0)} |\xi - \zeta| \qquad \forall t \geq t_0, \tag{6}$$

the system is said to be *incrementally exponentially stable* (δES). Due to the equivalence of norms in finite dimensional spaces and using the properties of $\mathcal{KL}$ functions [22], it is immediate to verify that the above definition is independent from the specific vector norm being used. Notice that in the case of $\mathcal{D} = \mathbb{R}^n$ in (5) (or (6)), then incremental asymptotic (or exponential) stability holds globally (δGAS, or δGES). Notice also that the definition of incremental asymptotic stability δAS considered here is not the same definition of local incremental asymptotic stability given in [1].

Before introducing the notion of contraction, we give first two preliminary definitions [8].

Definition 4: Let $K > 0$ be an arbitrary positive real number. A subset $\mathcal{C} \subset \mathbb{R}^n$ is *K-reachable* if, for any two points x_0 and y_0 in $\mathcal{C}$ there is some continuously differentiable curve $\gamma : [0,1] \mapsto \mathcal{C}$ such that:

1) $\gamma(0) = x_0$;
2) $\gamma(1) = y_0$;
3) $|\gamma'(r)| \leq K\,|y_0 - x_0|$, $\forall r \in [0,1]$.

For convex sets $\mathcal{C}$, we may pick $\gamma(r) = x_0 + r(y_0 - x_0)$, so $\gamma'(r) = y_0 - x_0$ and we can take $K = 1$. Thus, convex sets are 1-reachable, and it is easy to show that the converse holds as well. Also in this case, due to norms equivalence, the above definition does not depend on the particular choice of the norm $|\cdot|$.

Conditions on the matrix measure [23] of the Jacobian of a continuously differentiable system have been given in the literature in order to assess incremental exponential stability [8]. For further details we remind to the referred articles due to space limitation.

III. Incremental Stability of Planar Filippov Systems

Here we derive a sufficient condition for incremental stability of a bimodal PWS dynamical system. The approach consists in two different items: (i) proving contraction of the system within the sliding region and (ii) ensuring attractivity of such sliding region.

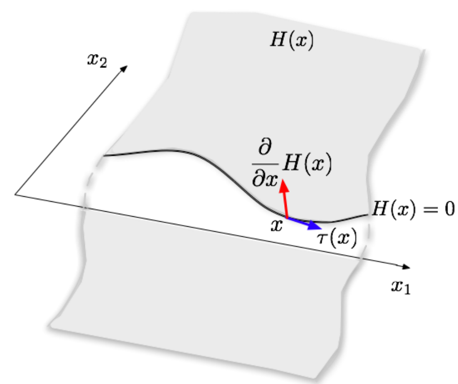


Fig. 1. Graphical representation of the manifold $H(x) = 0$.

A. Contracting Sliding Vector Fields

For a planar Filippov system we give a convergence result related to the sliding mode trajectories by considering an infinitesimal contraction condition on the discontinuity boundary.

Theorem 1: Let us consider a PWS system (2) with $x \in \mathbb{R}^2$. Suppose that the vector fields $x \mapsto F_1(t,x)$ and $x \mapsto F_2(t,x)$ are continuously differentiable and that the function $x \mapsto H(x)$ is twice continuously differentiable in an open set containing the region $\hat{\Sigma}$. Suppose also that there exist a $\hat{\mathcal{D}} \subseteq \hat{\Sigma}$, invariant connected set with respect to the topology induced on Σ, and a constant $c > 0$ such that the following condition holds:

$$\frac{\partial}{\partial x} m(t,x) \cdot \tau(x) \leq -c, \qquad \forall t \geq t_0, \forall x \in \hat{\mathcal{D}}, \tag{7}$$

where $m : [t_0, +\infty) \times \mathcal{O}\Sigma \mapsto \mathbb{R}$ is defined as:

$$m(t,x) = \left(\frac{1-\beta(t,x)}{2}\tau^T(x) F_1(t,x) + \frac{1+\beta(t,x)}{2}\tau^T(x) F_2(t,x)\right), \tag{8}$$

with $\mathcal{O}\Sigma \subseteq \mathbb{R}^n$ an open set containing Σ, and $\tau(x)$ is given:

$$\tau(x) = \begin{cases} \frac{1}{\left\|\begin{bmatrix}1 & -\frac{\frac{\partial H}{\partial x_1}(x)}{\frac{\partial H}{\partial x_2}(x)}\end{bmatrix}\right\|_2} \begin{bmatrix}1 & -\frac{\frac{\partial H}{\partial x_1}(x)}{\frac{\partial H}{\partial x_2}(x)}\end{bmatrix}^T & \text{if } \frac{\partial H}{\partial x_2}(x) \neq 0 \\ \begin{bmatrix}0 & 1\end{bmatrix}^T & \text{if } \frac{\partial H}{\partial x_2}(x) = 0 \end{cases}, \tag{9}$$

with $\beta(t,x)$ given by (4). Then, the sliding mode trajectories converge exponentially[1] towards each other in $\hat{\mathcal{D}}$. Furthermore, if there exists a $K > 0$ such that $|\tau(x)| \leq K$ for all $x \in \hat{\mathcal{D}}$, then the system is incrementally exponentially stable in $\hat{\mathcal{D}}$.

A schematic representation of the manifold $H(x) = 0$ and the tangent vector $\tau(x)$ is given in Fig. 1. Notice that expressions (7) and (9) are differential conditions which can be easily evaluated.

Proof: Firstly, observe that, as the function $x \mapsto H(x)$ with $x \in \Sigma$ has a well defined gradient by assumption, the smooth manifold Σ can be viewed as the range of a curve $\psi_s(s)$ with s being the curvilinear abscissa parametrizing the curve itself. So, we will say that point $\psi_s(s_1)$ precedes point $\psi_s(s_2)$ if $s_1 < s_2$. Since $\Sigma = \psi_s(s)$ is a 1-dimensional smooth manifold we can define $\tau(\psi_s(s))$ as its tangent unit vector and evaluate the projection of the sliding vector field $F_s(t,x)$ defined in (3) on the manifold. We obtain the equivalent 1-dimension dynamical system:

$$\dot{s} = \tilde{f}_s(t,s), \tag{10}$$

with

$$\tilde{f}_s(t,s) := f_s(t,\psi_s(s)) = \tau^T(\psi_s(s)) \cdot F_s(t,\psi_s(s)). \tag{11}$$

We now introduce on Σ the metric $d(s_1,s_2) = |s_2 - s_1|$, where s_1 and s_2 is a generic pair of values of the abscissa s. Notice that such metric is analogue to the distances in $\mathbb{R}$ where the natural topology is used and so, due to the definition of curvilinear abscissa, the value $d(s_1,s_2)$ represents the length, say $L_{\psi_s}(s_1,s_2)$, of the curve between points $\psi_s(s_1)$ and $\psi_s(s_2)$. Such norm is, equivalently, any $p-$norm in $\mathbb{R}$. For the sake of clarity, we divide the proof in three steps.

Step 1. We prove that if (10) is such that condition $\frac{\partial}{\partial s}\tilde{f}_s(t,s) \leq -c$ holds, with $c > 0$ and with s such that $\psi_s(s) \in \hat{\mathcal{D}}$, then we have:

$$\begin{aligned} &|\psi_s(s_2(t)) - \psi_s(s_1(t))|_2 \leq \\ &Ke^{-c(t-t_0)}|\psi_s(s_2(t_0)) - \psi_s(s_1(t_0))|_2 \qquad \forall t \geq t_0. \end{aligned} \tag{12}$$

Supposing without loss of generality that $s_1 < s_2$ and defining $\bar{s} = \frac{s_2 - s_1}{2}$, it is immediate to verify that, since system (10) is strictly decreasing with maximum slope $-c$, we have:

$$\tilde{f}_s(t,s_1) \geq -c(s_1 - \bar{s}) + \tilde{f}_s(t,\bar{s}); \tag{13}$$

$$\tilde{f}_s(t,s_2) \leq -c(s_2 - \bar{s}) + \tilde{f}_s(t,\bar{s}). \tag{14}$$

[1]Here we mean that the trajectories converge with respect to the topology induced on Σ.

So, due to the dynamic of the difference between the two flows, the following inequality can be written:

$$\frac{d}{dt}(s_2(t) - s_1(t)) = \tilde{f}_s(t,s_2) - \tilde{f}_s(t,s_1) \leq -c(s_2(t) - s_1(t)),$$

integrating (over the time interval $[t_0, +\infty)$ since $\hat{\mathcal{D}}$ is invariant) the above expression, we obtain:

$$|s_2(t) - s_1(t)|_2 \leq e^{-c(t-t_0)}|s_2(t_0) - s_1(t_0)|_2. \tag{15}$$

The above inequality holds for the time interval $[t_0, +\infty)$ and shows the exponential convergence of the trajectories with respect to the topology induced on Σ. Remembering that the difference between the curvilinear abscissa is the length of the curve ψ_s between the two points, in order to show the incremental stability we can write the above inequality as:

$$L_{\psi_s}(s_1(t), s_2(t)) \leq e^{-c(t-t_0)} L_{\psi_s}(s_1(t_0), s_2(t_0)).$$

Now, because of the the smoothness of manifold Σ, the derivative of the curve $\frac{d}{dr}\psi_r(r)$ is bounded for any equivalent representation $\psi_r(r) \sim \psi_s(s)$. Taking into account the definition of the length of a curve, if $|\tau(x)| \leq K$ we can write:

$$\begin{aligned} L_{\psi_s}(s_1(t), s_2(t)) &\leq e^{-c(t-t_0)} L_{\psi_s}(s_1(t_0), s_2(t_0)) \\ &\leq Ke^{-c(t-t_0)}|\psi_s(s_2(t_0)) - \psi_s(s_1(t_0))|_2, \end{aligned}$$

Taking into account that for the length of a curve it holds:

$$|\psi_s(s_2) - \psi_s(s_1)|_2 \leq L_{\psi_s}(s_1, s_2),$$

condition (12) is verified.

To summarise, we have shown that condition

$$\frac{\partial}{\partial s}\tilde{f}_s(t,s) \leq -c, \tag{16}$$

with $c > 0$ and s such that $\psi_s(s) \in \hat{\mathcal{D}}$ implies incremental exponential stability of the sliding vector field given by (12).

Step 2. We now show that condition (16) is equivalent to hypotheses (7)-(9). To do this, we first recall [24] that two regular curves, $\psi_r(r)$ and $\psi_s(s)$, are equivalent, $\psi_r(r) \sim \psi_s(s)$, if there exists an invertible diffeomorphism $s = T(r)$ (i.e. a continuously differentiable and invertible function) with $\frac{d}{dr}T(r) > 0$ for all s. In this way it is possible to write $\psi_r(T^{-1}(s)) = \psi_s(s)$. In particular, if s parametrizes the curve as its curvilinear abscissa it also holds:

$$\frac{d}{dr}T(r) = \left|\frac{d}{dr}\psi_r(r)\right| \tag{17}$$

$$\frac{d}{ds}T^{-1}(s) = \frac{1}{\left|\frac{d}{dr}\psi_r(T^{-1}(s))\right|}. \tag{18}$$

So, condition (16) is:

$$\begin{aligned} \frac{\partial}{\partial s}\tilde{f}_s(t,s) &= \frac{\partial}{\partial s}f_s(t,\psi_s(s)) = \frac{\partial}{\partial s}f_s(t,\psi_r(T^{-1}(s))) = \\ &= \left.\frac{\partial}{\partial x}f_s(t,x)\right|_{\psi_r(r)} \cdot \frac{1}{\left|\frac{d}{dr}\psi_r(r)\right|}\frac{d}{dr}\psi_r(r) \leq -c. \end{aligned} \tag{19}$$

If we now define the tangent unit vector of the curve as:

$$\tau(x)|_{\psi_r(r)} := \frac{1}{\left|\frac{d}{dr}\psi_r(r)\right|}\frac{d}{dr}\psi_r(r),$$

condition (19) becomes formally equivalent to condition (7).

Step 3. We show that the tangent unit vector $\tau(x)$ can be expressed as in (9). Indeed, we can consider that the smooth curve defined implicitly by $H(x) = 0$ can be parametrized as a graph of a function in the two following ways:

$$\psi_r(r) = \begin{cases} x_1 = r; \\ x_2 = h(r); \end{cases} , \qquad (20)$$

or

$$\psi_r(r) = \begin{cases} x_1 = h'(r); \\ x_2 = r; \end{cases} . \qquad (21)$$

Although functions $h(\cdot)$ and $h'(\cdot)$ are in general not known explicitly, the Implicit Function Theorem [25] ensures that they exist locally for each point on the set defined by the equation $H(x) = 0$. Now, deriving (20) by r we obtain $\frac{dx_1(r)}{dr} = 1$ and $\frac{dx_2(r)}{dr} = -\frac{\partial H}{\partial x_1}/\frac{\partial H}{\partial x_2}$ from the Implicit Function Theorem. Analogously, deriving (21) we obtain $\frac{dx_1(r)}{dr} = -\frac{\partial H}{\partial x_2}/\frac{\partial H}{\partial x_1}$ and $\frac{dx_2(r)}{dr} = 1$. Normalizing such derivatives to unitary module we have, respectively, the two expressions in (9). ■

Remark 1: Notice that if the set $\hat{\mathcal{D}}$ is bounded, then the smoothness of manifold Σ trivially guarantees that there exists a K such that $|\tau(x)| \leq K$.

For a two-dimensional PWS system, it is also possible to generalize Theorem 1 as follows.

Theorem 2: Consider a two-dimensional PWS system of the form (2). Suppose that the vector fields $x \mapsto F_1(t,x)$ and $x \mapsto F_2(t,x)$ are continuously differentiable and that the function $x \mapsto H(x)$ is (i) well defined (i.e. $\frac{\partial}{\partial x}H(x) \neq 0$) for all x such that $H(x) = 0$, and (ii) is twice continuously differentiable in an open set containing the region $\hat{\Sigma}$. Let $\hat{\mathcal{D}} \subseteq \hat{\Sigma}$ and define $\tau(x)$ and $\beta(t,x)$ as in (9) and (4) respectively. Also, define

$$J_s(t,x) := \frac{\partial}{\partial x}m(t,x) \cdot \tau(x). \qquad (22)$$

Then, for a given scalar $c > 0$, for all $t \in [t_0, T)$ such that trajectories starting in $\hat{\mathcal{D}}$ remain in $\hat{\mathcal{D}}$, we have:

i. if $J_s(t,x) \leq -c$ for all $x \in \hat{\mathcal{D}}$ and for all $t \in [t_0, T)$, then the sliding mode trajectories in $\hat{\mathcal{D}}$ converge exponentially towards each other;
ii. if $J_s(t,x) \geq c$ for all $x \in \hat{\mathcal{D}}$ and for all $t \in [t_0, T)$, then the sliding mode trajectories in $\hat{\mathcal{D}}$ diverge exponentially from each other;
iii. if $J_s(t,x) = 0$ for all $x \in \hat{\mathcal{D}}$ and for all $t \in [t_0, T)$, then the sliding mode trajectories in $\hat{\mathcal{D}}$ keep their distance constant.

Proof: Item i. has been proved in Theorem 1 (equation (15)). To prove items ii. and iii. we can follow the same steps as those in the proof of Theorem 1. In particular, for item ii. we have that conditions (13)-(14) are replaced by:

$$\tilde{f}_s(t,s_1) \leq c(s_1 - \bar{s}) + \tilde{f}_s(t,\bar{s});$$

$$\tilde{f}_s(t,s_2) \geq c(s_2 - \bar{s}) + \tilde{f}_s(t,\bar{s}),$$

and so, the dynamic of the error trajectory becomes:

$$\frac{d}{dt}(s_2(t) - s_1(t)) = \tilde{f}_s(t,s_2) - \tilde{f}_s(t,s_1) \geq 2c(s_2(t) - s_1(t)).$$

To prove item iii. conditions (13)-(14) must be replaced instead with:

$$\tilde{f}_s(t,s_1) = \tilde{f}_s(t,s_2) = \bar{f}_s,$$

with $\bar{f}_s$ being a constant value. The error dynamics in this case are:

$$\frac{d}{dt}(s_2(t) - s_1(t)) = 0.$$

Both for items ii. and iii. the rest of the proof then follows as the rest of the proof in Theorem 1. ■

Taking into account Theorem 2 it is possible to classify a connected region $\hat{\mathcal{D}} \subseteq \hat{\Sigma}$ by looking at the function $J_s(t,x)$. In particular if case i. is verified, we will term $\hat{\mathcal{D}}$ an *infinitesimally contracting sliding region*; if case ii. is verified, we will term $\hat{\mathcal{D}}$ an *infinitesimally stretching sliding region*; while, if case iii. is verified, $\hat{\mathcal{D}}$ will be termed an *infinitesimally neutral sliding region*. Finally, if $J_s(t,x)$ change sign in $\hat{\mathcal{D}}$, the set will be termed *indifferent sliding region*.

B. Incremental stability of a planar Filippov system

We can now use the theorems given above to derive conditions for incremental stability of a planar PWS system.

Theorem 3: Let us consider the PWS system (2) with $x \in \mathbb{R}^2$, $F_1(t,x)$ and $F_2(t,x)$ being two smooth vector fields and $H(x)$ hyperplane of equation $H(x) = h^T(x - x_h)$, with $h, x_h \in \mathbb{R}^2$. Let $\mathcal{S}_1 = \{x : H(x) < 0\}$, $\mathcal{S}_2 = \{x : H(x) > 0\}$ be the two regions in which the state space is partitioned by the switching manifold $\Sigma := \{x : H(x) = 0\}$. If there exists a convex invariant region $\mathcal{C} \subseteq \mathbb{R}^2$ such that $\Sigma_{\mathcal{C}} = \mathcal{C} \cap \Sigma \neq \emptyset$, and if the two following conditions hold:

i. there exist two scalars $\lambda_1 > 0$ and $\lambda_2 < 0$ such that $h^T F_1(t,x) \geq \lambda_1$ for all $x \in \mathcal{C} \cap \mathcal{S}_1$ and $h^T F_2(t,x) \leq \lambda_2$ for all $x \in \mathcal{C} \cap \mathcal{S}_2$;
ii. $J_s(t,x) \leq -c$ for all $x \in \Sigma_{\mathcal{C}}$, with $c > 0$ and with $J_s(t,x)$ defined as in (22);

then the PWS system is δAS.

Proof: Condition i. ensures that the flow $x(t) = \phi(t - t_0, t_0, \xi)$ reaches in a finite time the discontinuity manifold $\Sigma_{\mathcal{C}}$ for any initial condition $x(t_0) = \xi \in \mathcal{C}$. Indeed, suppose without loss of generality that $\xi \in \mathcal{C} \cap \mathcal{S}_1$. The time derivative of $H(x(t))$ is $\dot{H}(x(t)) = \mathcal{L}_{F_1}H(t,x) = h^T F_1(t,x) \geq \lambda_1$ and so, integrating this expression and considering that $\mathcal{C}$ is invariant and that $H(x)$ is monotone on the direction h, we have that the flow reaches the set $\Sigma_{\mathcal{C}}$ at a time instant $t_s \leq -H(\xi)/\lambda_1$. A similar result follows for any $\xi \in \mathcal{C} \cap \mathcal{S}_2$. Then, all trajectories reach the sliding region in finite time. Moreover, from Theorem 1, condition ii. implies exponential convergence among trajectories in the sliding region, and therefore the theorem is proven. ■

Notice that condition i. in Theorem 3 can be replaced by other possible conditions able to guarantee convergence of all trajectories towards the sliding region. Due to space limitation we direct the reader to the literature for more detailed information about other possible conditions (see for example [18],[16], [26]).

IV. Numerical Examples

Here we give two numerical examples in order to illustrate the theoretical derivations. In particular, in the two examples we study the stability of the equilibrium points by looking at the incremental stability of the sets where these points belong to. Both the examples are given in terms of piecewise linear systems for the sake of clarity. Obviously, piecewise nonlinear systems can equally be considered.

A. Contracting Sliding Region

We study the bimodal PWS system:

$$\dot{x} = \begin{cases} Ax + B & \textit{if } Cx \le 0 \\ Ax - B & \textit{if } Cx > 0 \end{cases},$$

with:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad B = [1\ 1]^T, \quad C = [0\ 1].$$

It is easy to verity that the vector field of this system is continuously differentiable everywhere but on $\{Cx = 0\}$ and that the function $H(x) = Cx$ is twice continuously differentiable with a well defined gradient. Applying equation (4) we have that:

$$\beta(x) = \frac{CAx}{CB} = x_1 - x_2,$$

and since $-1 \le \beta(x) \le 1$ and taking into account that on the discontinuity manifold $x_2 = 0$, we have that $\hat{\Sigma} = \{x : -1 \le x_1 \le 1, x_2 = 0\}$. Furthermore we have that:

$$\frac{\partial}{\partial x} H(x) = C = [0\ 1], \quad \tau(x) = [1\ 0]^T.$$

Now, evaluating expression (22) with respect of this system, we obtain after some manipulations that:

$$J_s(x) = -1.$$

So, due to Theorem 2, region $\hat{\Sigma}$ is a contracting sliding region. Furthermore, since the system present an equilibrium point at the origin (see [17] for further details on equilibrium point of Filippov systems) all sliding trajectories converge towards each other and onto the trivial trajectory $x = 0$. Fig. 2 displays the evolution of the error between the two trajectories evolving from the initial points $x_0^{(1)} = [0.5\ 0.5]^T$ and $x_0^{(2)} = [-0.5\ \ -0.5]^T$ outside the sliding manifold. When both the flow reach the sliding region (at $t \simeq 0.6$) they converge exponentially towards each others, as it is possible to note from the picture.

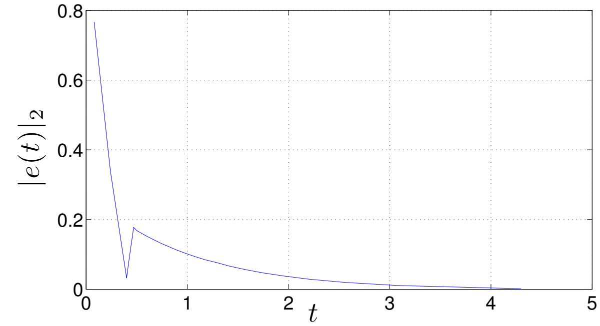


Fig. 2. Evolution of trajectories: norm of the error

B. Neutral Sliding Region

In this example we study the well known spring-damping mass with Coulomb friction described by:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 - \frac{F_c}{m} sgn(x_2) \end{cases},$$

where we indicate with x_1 the position of the mass and with x_2 its velocity. For this system we have:

$$\beta(x) = \frac{-kx_1 - bx_2}{F_c},$$

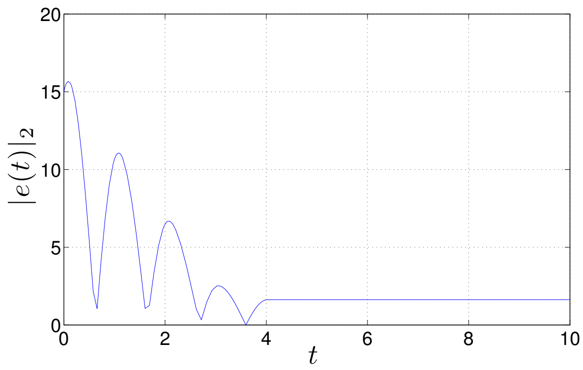
and, since on the discontinuity manifold $x_2 = 0$, and considering that $-1 \le \beta(x) \le 1$, the sliding region is the set $\hat{\Sigma} = \left\{x : -\frac{F_c}{k} \le x_1 \le \frac{F_c}{k}, x_2 = 0\right\}$. For this system we have:

$$\frac{\partial}{\partial x} H(x) = [0\ 1], \quad \tau(x) = [1\ 0]^T.$$

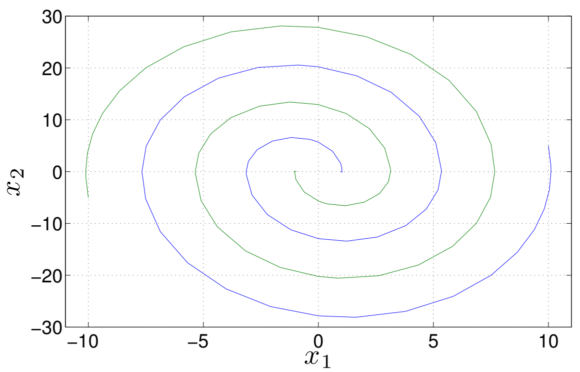
Therefore, from (22) we have $J_s(x) = 0$. Hence, $\hat{\Sigma}$ is a neutral sliding region and, for this reason, the error between sliding trajectories remains constant. Furthermore, since for this system it is easy to notice that the sliding trajectory $x = 0$ is a solution of the Filippov map, then the set $\hat{\Sigma}$ is an equilibrium set of the system. Indeed, such equilibrium set is associated in the physical model to the phenomenon of sticking [27]. Here we consider a numerical simulation with the following parameter values:$m = 1$, $b = 0.1$, $k = 10$, $F_c = 10$, starting from the initial conditions $x_0^{(1)} = [1\ 1]^T$ and $x_0^{(2)} = [-1\ \ -1]^T$. Fig. 3(a) shows the error between the trajectories, while Fig. 3(b) shows the state-space. It is possible to notice that when the flow reaches the sliding region, it sticks on the set. For this reason, as we expect from Theorem 2, the incremental error remains constant. The results of this example are consistent with the case of spring-damping mass with Coulomb friction analyzed in [28]. However, it is worth mentioning here that in [28] the specific case of stability of mechanical systems with friction is studied while, although the results are currently limited to the case of two-dimensional systems, a more general form of the Filippov systems is considered in the current paper.

V. Conclusion

In this paper we studied the problem of finding conditions for the incremental stability of 2-dimensional Filippov



(a)



(b)

Fig. 3. Evolution of trajectories: (a) norm of the error; (b) phase portrait.

systems. We approached the problem defining infinitesimal contraction of the sliding vector field and we showed that contraction on such region is able to guarantee exponential incremental stability of the sliding trajectories. If the contracting sliding surface is also attractive, the Filippov system is incrementally stable in any invariant region containing the sliding surface.

Notice that the mathematical tools used in this paper (i.e. Implicit Function Theorem, Jacobian of composed functions, local parametrization of manifolds on the tangent space) are defined for the generic $\mathbb{R}^n$ case. For this reason, the case of planar systems represents only a starting point, while the pressing open problem of studying incremental stability in higher dimensional Filippov systems is currently under investigation.

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