

# Comparison of Reachability Methods for Uncertain Linear Time-Invariant Systems

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**Abstract**—In this paper, we evaluate a number of methods for computing reachable sets using on one hand approximations and on the other hand invariants. We address systems of the form  $\dot{x}(t) = Ax(t) + Bu(t)$  with uncertain but bounded input function  $u(t)$ . We introduce our implementation based on zonotopes and use SpaceEx for support functions. Then, we go through our LMI-based implementation of ellipsoidal invariants and the canonical decomposition for large scale systems. Performance and accuracy of each method are evaluated using academic and practical examples.

## I. INTRODUCTION

Reachable sets play a fundamental role in control design and safety check of uncertain systems. However, the exact computation of the reachable set for systems of more than two or three dimensions is practically undecidable. This warrants the use of approximation approaches to compute a guaranteed enclosure of the reachable set. Approximate reachability methods of uncertain linear systems are generally based on a recursion for a stepwise computation of successive reachable sets and a preliminary choice of geometric abstractions, like boxes, ellipsoids [17], polytopes [9], zonotopes [7], [5], [3], [8], [6] or support functions [12], [13], [14]. The performance of these approaches and the quality of the approximation depend crucially on the choice of parameters, like the time step, the time horizon, the number of generators for zonotopes or directions for support functions. Furthermore the complexity of these approaches increases with the dimension of the dynamical system making them in general infeasible for large scale systems [10]. An attractive alternative to reachability analysis is the computation of invariant sets [18], since invariants are at the same time an overapproximation of reachable sets. Furthermore, among the various families of invariants, ellipsoidal invariants are known to exist for stable linear systems, even with (bounded) inputs. Additionally, ellipsoids have the advantage to be appropriately formulated as a solution of parametrized linear matrix inequalities (LMI) [16], [15].

In this paper, we use different approaches to compute the reachable sets of uncertain linear systems. Our objective is to compare the different methods in terms of accuracy of the approximate sets and their representation form and also their applicability for large scale systems. We apply first approximate reachability. We begin with our approach based on zonotopes [4], [11]. Then we use the verification Toolbox

SpaceEx [14] and its implementation based on support functions. Afterwards, we suggest an LMI-based implementation of ellipsoidal invariants extended with a decomposition based invariance framework for large systems. As case studies, we consider first two academic examples. The exact reachable set, if it is known, as well as those obtained from the above-mentioned methods are subsequently compared. We investigate then a controlled cooperative platoon of vehicles as a practical example. The objective thereby is to compare the minimum safe distances between vehicles issued from each method.

The rest of the paper is organized as follows. We begin with preliminaries about reachability of uncertain linear systems in Section 2. Then, in Section 3, we outline our approach based on zonotopes and the approximate method using support functions within SpaceEx. Section 4 gives a brief description of our LMI-based ellipsoidal invariants and the Jordan decomposition method to reduce the complexity of computing invariants for high dimensional systems. We give then in Section 5 details about our case studies and experimental results before presenting our conclusions.

## II. BASIC DEFINITIONS

A linear time-invariant system is a dynamic system described by the following differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u(t) \in U, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $A \in \mathbb{R}^{n \times n}$  is the constant system matrix,  $B \in \mathbb{R}^{n \times m}$  is the constant input matrix, and  $u(t) \in \mathbb{R}^m$  the time-varying uncertain input. The input is unknown but bounded for all times by  $u(t) \in U$ , where  $U \subset \mathbb{R}^m$  is a given compact convex set. Given a set of initial states  $X \subseteq \mathbb{R}^n$ , the set of reachable states from  $X$  at time  $\tau$  is defined by

$$\mathcal{R}_\tau(X) = \{x(\tau) \mid \dot{x}(t) = Ax(t) + Bu(t), u(t) \in U, \forall t, x(0) \in X\}. \quad (2)$$

The set of states reachable from  $X$  in a time interval  $[t_1, t_2]$  is defined by

$$\mathcal{R}_{[t_1, t_2]}(X) = \cup_{t \in [t_1, t_2]} \mathcal{R}_t(X). \quad (3)$$

In the present work, we compare methods to compute overapproximations of  $\mathcal{R}_{[0, \infty]}(X_0)$  and  $\mathcal{R}_{[0, T]}(X_0)$  for a given set of initial states  $X_0$  and a time horizon  $T \in \mathbb{R}$ .

## III. NUMERICAL RECURSIVE SET-VALUED APPROXIMATION TECHNIQUES

One of the major challenges of reachability analysis of hybrid systems resides in computing a tight approximation

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of the reachable set within discrete modes. In this context, we propose to use the numerical recursive set-valued method for non autonomous linear systems as merely one of the many alternatives. For a set of states  $X \subseteq \mathbb{R}^n$ , a set of inputs  $U \subseteq \mathbb{R}^m$ , and a time  $t$ , the map

$$R(t, X, U) = e^{tA}X \oplus \int_0^t e^{(t-s)A}BU ds \quad (4)$$

where  $\oplus$  is the Minkowski sum, generates the reachable states from  $X$  at time  $t$ . Choosing the time partition  $0 \leq \dots \leq t_k \leq t_{k+1} \leq \dots \leq T = rN$ , where  $r$  is the time step, the following recursion scheme for computing successive reachable sets can be derived from (4).

$$R_k = e^{rA}R_{k-1} \oplus V. \quad (5)$$

where  $V$  is the input contribution.

The first set in this sequence is given by  $R_0 = \mathcal{R}_{[0,r]}(X_0)$ . The inflation of the overapproximation error during computation depends crucially on the accuracy of the approximation of both sets  $R_0$  and  $V$ . This is, in turn, highly dependent on the geometry used to approximate sets of reachable states. Typical choices are ellipsoids, polytopes, zonotopes and support functions.

#### A. Zonotopes

In this subsection, we review our approach and its implementation based on zonotopes for the computation of reachable sets. We assume the input to be piecewise constant, but not necessarily identical in successive time intervals  $[0, r], [r, 2r], \dots, [kr, (k+1)r], \dots$ . This assumption is practically justified firstly because hardware implementation on microcontroller often requires zero-hold inputs and secondly because the struggle to get tight approximation is strongly coupled with the choice of a small time steps. Consequently, we deduce the following recursive scheme:

$$R_k = A_d R_{k-1} \oplus B_d U, \quad A_d = e^{rA}, \quad B_d = \int_0^r e^{(r-s)A} B ds. \quad (6)$$

A zonotope  $Z$  of order  $p/n$  is defined by its center  $c \in \mathbb{R}^n$  and its generators  $\langle g_1, \dots, g_p \rangle$  and is hence practically represented as a matrix  $(c, \langle g_1, \dots, g_p \rangle)$  with  $c, g_i$  as columns:

$$Z = \{x \in \mathbb{R}^n \mid x = c + \sum_{i=1}^p \alpha_i g_i, -1 \leq \alpha_i \leq 1\}.$$

Zonotopes are therefore closed under linear transformation and Minkowski sum.

1) *Convex hull of  $Z$  and  $e^{rA}Z$* : In general, the convex hull of two zonotopes is not a zonotope. In our computation, a zonotope overapproximation  $K$  of the convex hull  $CH(Z, e^{rA}Z)$  is required. In [3] the following approximation was proposed:

$$K = \left( \frac{c+e^{rA}c}{2}, \left\langle \frac{g_1+e^{rA}g_1}{2}, \dots, \frac{g_p+e^{rA}g_p}{2}, \frac{c-e^{rA}c}{2}, \frac{g_1-e^{rA}g_1}{2}, \dots, \frac{g_p-e^{rA}g_p}{2} \right\rangle \right) \quad (7)$$

This operation is involved in the computation of an approximate for  $R_0$ .

The Minkowski sum, which is frequently used in the iterative computation, causes a rapid increase of the number of

generators of the resulting zonotope enclosure. In order to control the complexity, the order of the computed zonotope is reduced to a maximum allowed order  $q$ . The reduction operation proposed in [3] reduces the order by one. In our implementation, this operation is repeated until the order equals  $q$ .

2) *Order reduction operation*: Many approaches have been proposed to reduce the order of zonotopes. In general, a zonotope  $G = (c, \langle g_1, \dots, g_{bn} \rangle)$  of order  $b$  can be reduced to a zonotope of order  $q$  by replacing the  $n(b-q+1)$  less significant generators with their interval hull. To decide about the choice of generators, different sorting methods can be used. We opt for the method proposed in [3] which reduces the order by applying the following sorting criterion. Let  $G = (c, \langle g_1, \dots, g_{(q+1)n} \rangle)$  be a zonotope satisfying

$$\|g_1\|_1 - \|g_1\|_\infty \leq \dots \leq \|g_{(q+1)n}\|_1 - \|g_{(q+1)n}\|_\infty.$$

We construct the zonotope  $\widehat{G} = (c, \langle Q, g_{2n+1}, \dots, g_{(q+1)n} \rangle)$  where  $Q \in \mathbb{R}^{n \times n}$  is the diagonal matrix that satisfies

$$Q_{ii} = \sum_{j=1}^{2n} |g_j^i|, \quad i = 1, \dots, n$$

$g_j^i$  is the  $i^{\text{th}}$  component of  $g_j$ . The obtained zonotope has hence the order  $q$  and satisfies  $G \subseteq \widehat{G}$ .

3) *Approximation of  $R_0$  with a zonotope*: The initialization step consists in computing  $R_0$ . Based on the following inclusion,

$$\cup_{t \in [0, r]} R(t, R, U) \subseteq (\cup_{t \in [0, r]} e^{tA}R) \oplus BU \subseteq S \oplus BU \quad (8)$$

a two steps method was proposed in [3] to compute a zonotope inclusion for  $R_0$ . First the set  $S$  is approximated by  $CH(Z, e^{rA}Z)$ , which in turn is approximated by  $K$  given in (7). Then  $K$  is bloated with a ball of radius  $\alpha_r$  to assure the inclusion of all reachable states within the time interval  $[0, r]$ . This  $\alpha_r$  is an upper bound of the distance between the set  $K$  and the exact reachable set of the homogeneous system  $\dot{x} = Ax$ .

$$S \subseteq K \oplus \blacksquare(\alpha_r) \quad \text{with } \alpha_r = (e^{r\|A\|} - 1 - r\|A\|) \sup_{x \in I} \|x\|$$

#### B. Support functions

The support function  $\rho_C(l)$  of a convex set  $C \subseteq \mathbb{R}^n$  in a direction  $l \in \mathbb{R}^n$  is the function defined by:

$$\begin{aligned} \rho_C : \mathbb{R}^n &\rightarrow \mathbb{R} \cup \{\pm\infty\} \\ l &\mapsto \sup_{x \in C} l \cdot x \end{aligned} \quad (9)$$

Additionally, support functions offer interesting geometrical properties which simplify their implementation. Let  $C_1, C_2$  be two compact sets in  $\mathbb{R}^n$ ,  $\lambda \geq 0$  a real number and  $M \in \mathbb{R}^{n \times n}$  a matrix, then

$$\begin{aligned} 1) \rho_C(\lambda l) &= \lambda \rho_C(l), \quad 2) \rho_{MC}(l) = \rho_C(M^T l), \\ 3) \rho_{C_1 \oplus C_2} &= \rho_{C_1} + \rho_{C_2}, \quad 4) \rho_{CH(C_1, C_2)}(l) \leq \max(\rho_{C_1}(l), \rho_{C_2}(l)). \end{aligned}$$

The recursive implementation scheme is quite similar to the scheme adopted for zonotopes. We have just to replace sets by their support functions:

$$\rho_{\mathcal{R}_t}(l_i) = \rho_{X_0}([e^{tA}]^T l_i) + \int_0^t \rho_U([e^{\tau A}]^T l_i) d\tau, \quad i = 1, \dots, p \quad (10)$$

Thereby, the quality of the solution depends strongly on the number  $p$  of the support directions  $l_i$  as well as on the choice of their directions or their distribution in the state space  $\mathbb{R}^n$ . Built on a variable step algorithm, SpaceEx computes an approximation  $\Omega_k$  of the set  $\mathcal{R}_{[t_k, t_{k+1}]}(X)$ , where  $t_k = \sum_{i=0}^{k-1} \delta_i$  and  $\delta_i$  are different time steps. In fact, based on the property

$$\mathcal{R}_{[t_k, t_{k+1}]}(X) = e^{t_k A} \mathcal{R}_{[0, \delta_k]}(X) \oplus \mathcal{R}_{t_k}(\{0\}) \quad (11)$$

and considering the following overapproximations  $\Psi_\delta(U)$ ,  $\Psi_k$  and  $\Omega_{[0, \delta]}(X_0, U)$  such that

$$\begin{aligned} \mathcal{R}_\delta(\{0\}) &\subseteq \Psi_\delta(U), \quad \mathcal{R}_{t_k}(\{0\}) \subseteq \Psi_k, \\ \mathcal{R}_{[0, \delta]}(X_0) &\subseteq \Omega_{[0, \delta]}(X_0, U), \end{aligned} \quad (12)$$

the sequence  $\Omega_k$  is computed using following recursions:

$$\begin{aligned} \Psi_{k+1} &= \Psi_k \oplus e^{t_k A} \Psi_{\delta_k}(U), \\ \Omega_k &= e^{t_k A} \Omega_{[0, \delta_k]}(X_0, U) \oplus \Psi_k \end{aligned} \quad (13)$$

with  $\Psi_0 = 0$  and  $\Omega_{[0, \delta]}$ ,  $\Psi_\delta$  approximated as given by lemma 2 and lemma 3 in [14].

#### IV. INVARIANT SETS

An *invariant* is a subset of the state space  $\mathcal{I} \subseteq \mathbb{R}^n$ , such that for all  $x_0 \in \mathcal{I}$ , and all admissible input functions  $u : \mathbb{R} \rightarrow U$ , the solution to (1) with  $x(0) = x_0$  satisfies  $x(t) \in \mathcal{I}$  for all  $t \geq 0$ . Intuitively, the system remains trapped in the invariant for all future times.

To see the significance of invariants in the context of reachability analysis, assume a set of initial states  $X \subseteq \mathbb{R}^n$ . If we know an invariant  $\mathcal{I}$  with  $X \subseteq \mathcal{I}$ , then certainly  $\mathcal{I}$  is an overapproximation of the states reachable from  $X$ . Of course, the state space  $\mathbb{R}^n$  is itself an invariant, but it does not answer any question about reachable states. In this section, we focus on techniques for computing bounded invariants close to reachable sets.

One advantage of invariants over iterative methods presented in Sec. III, is that invariants cover unbounded time horizon, without any extra cost. A second advantage is that invariants in general have a compact representation. For example an invariant ellipsoid is represented by a single  $n$ -by- $n$  matrix. Whereas, iterative methods produce a large number of sets, often with growing complexity. Each of these sets has to be taken into account in order to enclose all reachable states.

On the other side, the main advantage of iterative methods over invariants resides in their accuracy. In fact, for the systems considered in this paper, the overapproximations converge to the exact reachable set when the time step is decreased towards zero<sup>1</sup>. The best one can say in this respect about the invariants presented in Sec. IV-B is that their boundary touches the reachable set at  $2k$  points, where  $k$  is the number of real eigenvalues of the system matrix. To sum it up, invariants are a useful complement to iterative reachability algorithms.

<sup>1</sup>If we ignore round-off and floating point errors.

#### A. LMIs and invariant ellipsoids

A (zero-centered) ellipsoid is a subset of  $\mathbb{R}^n$  defined by

$$\mathcal{E} := \{x \in \mathbb{R}^n \mid x^T \cdot P \cdot x \leq 1\}. \quad (14)$$

The matrix  $P$  is called the shape matrix of  $\mathcal{E}$ . It is known that stable autonomous LTI systems have quadratic Lyapunov functions, and that the level sets of quadratic Lyapunov functions are invariant ellipsoids. Since the uncertain input of the LTI system (1) is assumed to be bounded, we can conclude that this system has an ellipsoidal invariant.

Nagumo's Theorem (see the overview by Blanchini [18], Sec. 3) provides a criterion for finding invariants. Intuitively, the idea is that a set is invariant if all trajectories, that start on the boundary of the set, do not immediately leave this set.

In [16], [15] this criterion for invariance is adapted to ellipsoidal sets and further translated into a parametrized linear matrix inequality (LMI). The resulting LMI

$$\left( \begin{array}{c|c|c} A^T P + PA + \alpha P & PB & 0 \\ \hline & -\tau_1 & \\ \hline B^T P & \ddots & 0 \\ \hline & & -\tau_m \\ \hline 0 & 0 & -\alpha + \sum_{i=1}^m \tau_i \end{array} \right) \leq 0 \quad (15)$$

is a constraint over the matrix variable  $P$  in (14) and further variables  $\alpha, \tau_1, \dots, \tau_m$  which are introduced by the so-called S-procedure. The resulting equation (15) is, however, not an LMI in  $\alpha$ . Hence,  $\alpha$  needs to be fixed before automated solvers can be used to find optimal solutions. This is the major disadvantage of this method, since the feasibility of the LMI and the size of the invariant depend on  $\alpha$ . Especially for high-dimensional systems it is tedious to find good values for  $\alpha$ .

Eventually, if a matrix  $P$  satisfies (15), then  $P$  is the shape matrix of an ellipsoidal invariant. The inverse, however, is not true. The LMI constraint is in general a stronger condition. But this is not a limiting factor in practice.

Finding the optimal ellipsoidal invariant<sup>2</sup> is done by minimizing or maximizing an appropriate objective function subject to the LMI constraint (15). This is a convex optimization problem which can be efficiently solved with freely available tools, such as CVX [19], [20].

#### B. Invariants based on decomposition

An alternative approach to derive invariants is to first transform the LTI system (1) into its canonical normal form. This is done by rewriting the differential equation in new coordinates  $y(t) = Q^{-1}x(t)$ . In the canonical normal form, the system matrix  $Q^{-1}AQ$  is as close to diagonal form as possible, separating in this way truly coupled and decoupled state variables<sup>3</sup>:

$$\dot{y}(t) = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix} y(t) + Q^{-1}Bu(t), u(t) \in U. \quad (16)$$

<sup>2</sup>With respect to a measure, such as the volume. See [15] for alternative optimality criteria, and details on the optimization problem.

<sup>3</sup>See [15], Sec. 2.1 for more details.

Variables corresponding to one diagonal block  $B_i$  comprise an independent subsystem. These subsystems are then handled separately. In general, the subsystems are only one- and two-dimensional systems. For one-dimensional subsystems it is quite easy to derive an invariant interval directly. Higher-dimensional subsystems are handled using the LMI approach discussed in the previous subsection. In this way, all state variables of the system in canonical form are bounded by appropriate invariants. Hence, the intersection of all subsystem-invariants is an invariant for the whole system and is also bounded. By applying a back-transformation into the original coordinates, we get a bounded invariant for (1). In general a decomposition-based invariant can be written in the form

$$\bigcap_{i=1\dots k} \{x \mid x^T P_i x \leq 1\} \quad (17)$$

with positive semidefinite matrices  $P_i$ .

A benefit of the decomposition method is its robustness for higher dimensional examples. Transformation to real canonical form is numerically stable and fast. Optimizations are applied in general only to one-, and two-dimensional subsystems for which it is easy to find satisfying values for the parameter  $\alpha$ .

## V. CASE STUDIES AND RESULTS

### Case study 1

We consider the simple two-dimensional example

$$\dot{x}(t) = \begin{pmatrix} -4 & -3 \\ 2 & 1 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} u(t) \quad (18)$$

with  $u(t) \in [-1, 1]^2$ . We aim to find the states reachable from the origin using the invariant method and approximation methods based on zonotopes and on support functions. For the invariant technique, we have not to impose restriction on the time horizon. For both approximate methods, however, the time horizon must be fixed. But both methods are able to detect whether an invariant is reached or not within this time horizon. For the sake of comparison, this example was designed such that the decomposition invariant and the set of the states reachable from the origin are identical. This becomes obvious, when transforming (18) in its canonical normal form. We define new coordinates  $y = \begin{pmatrix} -2 & -3 \\ -1 & -1 \end{pmatrix} x$  in which the differential equation becomes

$$\dot{y}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} y(t) + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} u(t). \quad (19)$$

Hence, the resulting system consists of two one-dimensional completely decoupled systems, i.e.  $y_1(t), y_2(t)$  are independent from each other, and they have no common inputs. Consequently, the reachable states must form a paralleloptope. This is of course an exceptional case. In general, the presented type of invariants cannot capture the reachable states exactly. Figure 1 shows the results of each method. The recursive scheme from Sec. III, based on the assumption of piecewise continuous input, produces after 10 iterations with the time step  $0.5s$  the innermost green dotted zonotope, which has 20 generators. The blue dotted zonotope has only 10 generators. This is the zonotopic approximation obtained after applying the order reduction algorithm (cf. III-A.2).

We remark that this is done at the cost of the approximation accuracy.

The light-blue filled polytope is produced by SpaceEx for the same time horizon  $T = 5s$ . However, a smaller time step was necessary, here  $0.05s$  was used. Otherwise the overapproximation will be overly pessimistic. The polytope has 16 facets. The normal directions of these facets are uniformly distributed vectors around the unit circle. We note that the accuracy of this overapproximation can be enhanced substantially by choosing small time steps. Furthermore, we note that increasing the number of directions leads to minor enhancements in the accuracy of the approximation.

The red ellipsoid is an invariant derived via LMIs and there-

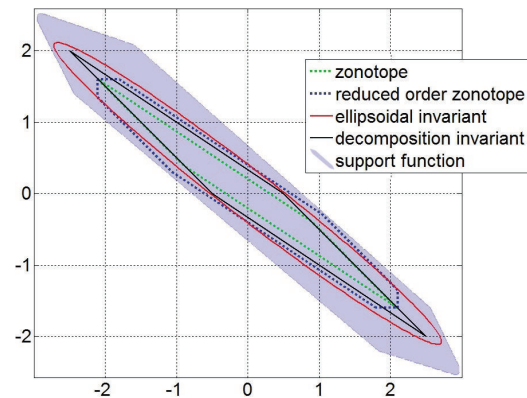


Fig. 1: Results of case study 1.

fore a guaranteed overapproximation of the states reachable from zero with unbounded time horizon. The decomposition method yields the black paralleloptope invariant which, as mentioned before, coincides with the exact reachable set. As a first observation, we note that the green dotted zonotope is smaller than the black paralleloptope. This means that the time horizon  $T = 5s$  chosen for the iterative computations was too small to cover all reachable states. That indicates, moreover, that within this time both approximation algorithms do not reach a fixpoint. SpaceEx needs a small time step as well as a large number of directions. Although, the exact reachable set, corresponding to the black paralleloptope, has only four facets. The zonotope approximation, on the other hand, produces accurate results even for quite big time steps, which is due to the implicit assumption of piecewise-constant inputs. But it suffers from the growing number of generators, due to the Minkowski sum involved more than one time in the recursion scheme (6).

Figure 2 shows the invariants and the maximal reachable sets resulting from both approximate methods after  $T = 10s$  with the time step  $r = 0.001s$ . To get a tight approximation, we used 128 uniformly distributed directions for the support functions and we did not limit a maximal order for zonotopes. We note that both sets are identical and exactly confined within the decomposition invariant, which as mentioned before coincides in this case with the exact reachable set.

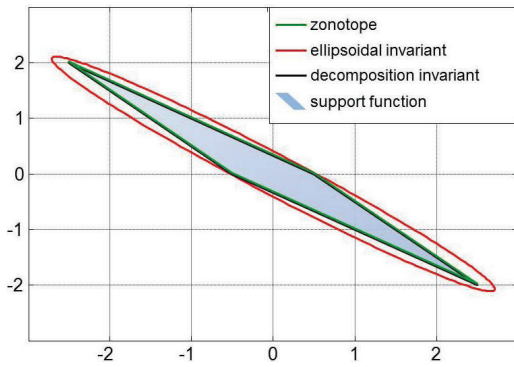


Fig. 2: Results of case study 1 with  $T=10s$ ,  $r=0.001s$  for the approximate methods.

### Case study 2

Let us briefly consider another simple example:

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} u(t) \quad (20)$$

with  $u(t) \in [-1, 1]^2$  and an initial set close to the origin. Here, the system states are coupled by joined inputs. This leads to the rather big invariants (the blue box and the green ellipsoid in Fig. 3). The maximal reachable sets obtained with zonotopes (red) or with support functions (light-blue) are both computed with the time step  $0.001s$  and for a period of time equal to  $10s$ . For support functions, we used 128 uniformly distributed directions.

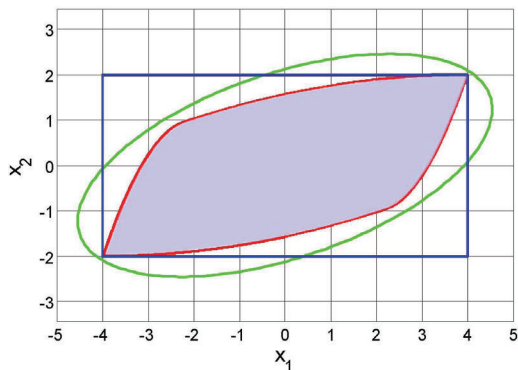


Fig. 3: Results for case study 2: Reachable states approximated by zonotopes (red) and support functions (light-blue), box invariant (blue), ellipsoidal invariant (green).

### Case study 3: WLAN cooperative platoon of vehicles

In this section, we investigate the WLAN cooperative platoon described in [4], [11]. It consists in a closed loop feedforward controlled three autonomous vehicles platoon with a manually driven leader ahead. Its dynamics is given by (1) with the state vector  $x = [e_1, \dot{e}_1, a_1, e_2, \dot{e}_2, a_2, e_3, \dot{e}_3, a_3]$ , the acceleration of the leader as input  $u = a_L$  and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.6050 & 4.8680 & -3.5754 & -0.8198 & 0.4270 & -0.0450 & -0.1942 & 0.3626 & -0.0946 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0.8718 & 3.8140 & -0.0754 & 1.1936 & 3.6258 & -3.2396 & -0.5950 & 0.1294 & -0.0796 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The initial set is assumed to be close to the origin. We aim for this example to determine the minimum allowable safe distances inside the platoon by computing the invariant and the reachable sets. We apply thereby the already introduced techniques. The comparison here has a more practical aspect than in the previous examples. We intend hence to compare the safe distances obtained by each method and to consider the minimum of them as the absolute safe distance. For both approximate methods, we

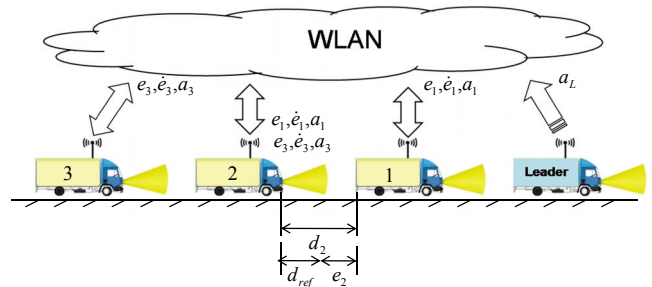


Fig. 4: WLAN cooperative platoon with three vehicles and a leader ahead.

use the time step  $0.01s$  and the time horizon  $T = 20s$ . In fact within this time horizon, we are sure that at least the approximation algorithm based on zonotopes converges to a fixpoint. Furthermore, we use 128 uniformly distributed directions for support functions. The number of generators is controlled using the aforementioned order reduction. We note that a maximum zonotope order of  $q = 40$ , produces the same result as shown in Fig.5a. The LMI method from Sec. IV-A does not work smoothly with the system. Extensive search for the parameter  $\alpha$  is necessary to make the optimization problem feasible at all. Among the feasible ones, only few values yield really useful invariants (see Table I).

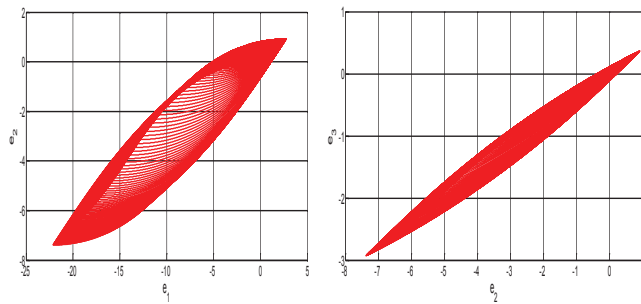
Computing invariants for the decomposed system in real canonical form, which has three 1-by-1 and another three 2-by-2 blocks, works fine. However, the resulting invariants are as expected rather poor since the input here is only one-dimensional (see last column of Table II).

The bounds for inter-vehicle distances  $x_1$ ,  $x_4$ , and  $x_7$  derived from different methods are given in Table II.

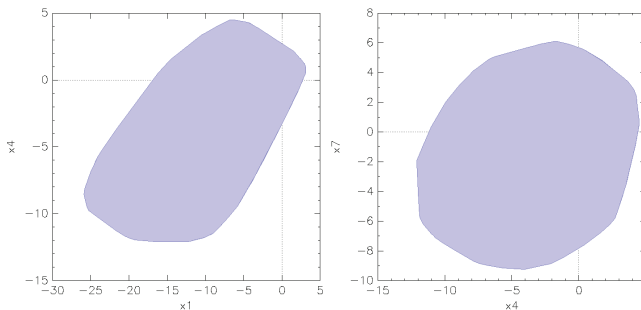
	LMI			Decomposition
	$\alpha = 0.0970$	$\alpha = 0.3077$	$\alpha = 0.6321$	
$x_1 (m)$	-30 ... 8	-26 ... 4	-94 ... 71	-43 ... 20
$x_4 (m)$	-11 ... 3	-10 ... 2	-35 ... 27	-20 ... 12
$x_7 (m)$	-4 ... 1	-4 ... 1	-14 ... 11	-12 ... 9

TABLE I: Bounds on the state variables derived from invariants. LMI method applied to the real canonical form with three different choices for the parameter  $\alpha$ .

Table II shows that the approximate method based on zonotopes gives the minimum safe distances. The results of the support function approximation and LMI for  $\alpha = 0.3077$  are almost equally good.



(a) Zonotopes



(b) Support functions

Fig. 5: Different 2D projections of the reachable sets where the minimum safe distances between the vehicles are given by their outer contours ( $x_1 = e_1, x_4 = e_2, x_7 = e_3$ ),  $r = 0.01s$  and  $T = 20s$ .

	Zonotopes	Support functions	LMI	Decomposition
$x_1$ (m)	-23	-26	-26	-43
$x_4$ (m)	-8	-13	-10	-20
$x_7$ (m)	-3	-9	-4	-12

TABLE II: Minimum safe distances obtained by different methods.

## VI. CONCLUSIONS

We presented a comparative study of different reachability approaches for linear uncertain systems. We used approximate recursive methods based on zonotopes and support functions as well as set invariance methods based on ellipsoids. We addressed the strengths and weaknesses, the applicability and limitations of each method by using particular academic examples as well as a cooperative platoon of vehicles as a practical case study. The zonotopic approximation turned out to be the tightest. But it suffers at the same time from growing complexity due to the increasing number of generators. Order reduction techniques can be used to control this complexity. However, this can significantly worsen the accuracy of the approximation. Support functions offer an efficient alternative to the memory-consuming zonotopes. But the accuracy of the approximation depends crucially on the number of directions as well as the time step chosen for the computation. The LMI invariant methods can be laborious in practice because of the search for optimal parameters. Invariants based on the decomposition are easier to compute but not well suited for examples such as the platoon. In fact, their accuracy declines drastically if the

state variables are strongly coupled by common inputs. That happens, in particular, for systems with few inputs. Both types of invariants are, however, time unbounded and a guaranteed enclosure of the reachable states.

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