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Robust Model Predictive Control of Uncertain Linear Systems with Persistent Disturbances and Input Constraints

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Abstract— **This paper presents computationally attractive robust model predictive control approaches for the control of discrete-time linear systems with input constraints, structured parameter uncertainties and persistent disturbances. In order to ensure robust stability of constrained uncertain systems, constructive methods are proposed to compute robust positively invariant sets for stabilizing predictive controller. The proposed robust predictive control (RMPC) systems satisfy both recursive feasibility and input-to-state stability. In the controller design, the 0-step predictive controller with a simple structure is proposed. In order to deal with the RMPC problem with a fixed terminal set, the result is extended to the N-step predictive controller. Simulations results have demonstrated the efficacy of the proposed predictive control approaches.**

I. INTRODUCTION

Model predictive control (MPC), also known as receding horizon control, has been widely studied in the last two decades. Many significant results has been reported on the nominal systems. Readers may refer to the review papers [1], [2] and the references therein for more information. In practical applications, traditional MPC methods may fail due to the uncertainties and/or disturbances brought in. In this condition, robust model predictive control (RMPC) should be considered. Many existing papers studied RMPC for linear systems. The authors of [3] considered the linear time varying (LTV) systems. Both polytopic and structured uncertainties were considered. The work was improved by the authors of [4]. In [5], constrained linear system with structured norm-bounded uncertainties were considered. The author of [6] discussed the condition of unstructured uncertainties. There are also papers considering the existence of disturbances. In [7], constrained linear system with bounded disturbances was studied, where a tube based RMPC approach was proposed. Similar system was considered in [9].

It is noticed that existing papers studying RMPC for systems with both uncertainties and disturbances mainly belong to the category of output feedback RMPC, i.e., there are uncertainties in the system state and disturbance in the measured output [8]. It is a bit different from our work. In this paper, we consider RMPC for a certain type of linear systems, which contain both uncertainties and disturbances in the evolution of system state. Moreover, the uncertainties are supposed to be state-dependant while the disturbances are not. It is known that asymptotic stability can hardly be guaranteed if persistent disturbances exist. In this case, it is only possible to steer the system states to a robust positively invariant (RPI) set neighboring the origin. Computation of RPI sets with ellipsoidal forms are discussed in this paper. We will show that we can online compute the RPI set that contains the measured system states, which refers to the "0-step prediction" as discussed in this paper. It is proven that recursive feasibility and input-to-state stability can be guaranteed. Another contribution of this paper is that we develop a controller with N-step prediction based on the concept of robust one-step set. The predictive controller is able to steer the system states into a predetermined target set in N steps. It is shown that a family of approximate one-step sets can be computed off-line, while the online control law can easily be obtain with bisection searches and a convex optimization subject to linear matrix inequalities (LMIs).

Notations. In this paper, we use $\mathbf{H}(\cdot)$ to denote $(\cdot)+(\cdot)^T$, $\Delta f(k)$ denotes $f(k)-f(k-1)$, where k and $k-1$ are time instants.

II. PRELIMINARIES

Consider a discrete-time linear system

$$x^+ = f(x,u,w,d), \tag{1}$$

where x is the vector of system states, u is the system input, w is the uncertainty, and d is the vector of external disturbances. It holds $u \in \mathbb{U}$, $w \in \mathbb{W}$ and $d \in \mathbb{D}$, where $\mathbb{U}$, $\mathbb{W}$ and $\mathbb{D}$ are compact sets containing the origin.

Definition 1 (Robust Positively Invariant (RPI) Set): A set Γ is called a robust positively invariant (RPI) set for the closed-loop system corresponding to the control law $u = h(x)$, if $\forall x \in \Gamma$, it holds $f(x,h(x),w,d) \in \Gamma$ for $\forall w \in \mathbb{W}$ and $\forall d \in \mathbb{D}$.

Definition 2 (Robust One-step Set $\widetilde{\mathcal{Q}}(\Omega)$): The robust one-step set $\widetilde{\mathcal{Q}}(\Omega)$ is the set of all states which can be steered to Ω by an admissible input, for all allowable disturbances and uncertainties. That is

$$\widetilde{\mathcal{Q}}(\Omega) := \{x \in \mathbb{R}^n | \exists u \in \mathbb{U}, f(x,u,w,d) \in \Omega, \forall w \in \mathbb{W}, \forall d \in \mathbb{D}\}.$$

For closed-loop systems, $\widetilde{\mathcal{Q}}^h(\Omega)$ is used to denote the robust one-step set corresponding to the control law $u = h(x)$. It can be described as

$$\widetilde{\mathcal{Q}}^h(\Omega) := \{x \in \mathbb{R}^n | f(x,h(x),w,d) \in \Omega, \forall w \in \mathbb{W}, \forall d \in \mathbb{D}\}.$$

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III. Computation of Ellipsoidal RPI Sets

In this paper, we consider the discrete-time linear system with the following form

$$x^+ = \widetilde{A}x + \widetilde{B}u + Ed, \tag{2}$$

where $\widetilde{A} := A + MTN_A$, $\widetilde{B} := B + MTN_B$, and A, B, E, M, N_A, N_B are constant matrices, T is Lebesgue measurable and satisfies $T^T T \leq I$. We define $w := MTN_A + MTN_B$ hereafter. In the following, we will develop a method to compute ellipsoidal RPI sets for system (2).

Assumption 1: The system states of (2) are measurable at each time. System input $u(k)$ satisfies $u(k) \in \mathbb{U}$, where $\mathbb{U} := \{u : |u_t| \leq u_{t,\max}\}$, u_t is the tth element of the input vector and $1 \leq t \leq m$. For the disturbances, it is assumed that $d(k) \in \mathbb{D}$, with $\mathbb{D} := \{d | d^T d \leq \gamma^2\}$, where γ is a known positive constant.

Lemma 1 ([11]): Let matrices $L = L^T$, S, N and $\Delta(t)$ be real matrices of appropriate dimensions, the inequality

$$L + S\Delta(t)N + N^T\Delta^T(t)S^T < 0 \tag{3}$$

holds for all $\Delta^T(t)\Delta(t) \leq I$ if and only if for some positive scalar $\epsilon > 0$,

$$L + \epsilon SS^T + \epsilon^{-1}N^T N < 0. \tag{4}$$

Theorem 1: For the discrete-time linear system (2), if there exist a positive definite matrices X, a general matricx Y and positive scalars λ_1, λ_2 and η, such that it holds

$$\begin{bmatrix} -\lambda_1 X & 0 & 0 & \Xi_1^T & \Xi_2^T & 0 \\ \star & -\lambda_2 I & 0 & E^T & 0 & 0 \\ \star & \star & -1+\lambda_1+\lambda_2\gamma^2 & 0 & 0 & 0 \\ \star & \star & \star & -X & 0 & \eta M \\ \star & \star & \star & \star & -\eta I & 0 \\ \star & \star & \star & \star & \star & -\eta I \end{bmatrix} \leq 0, \tag{5}$$

$$\begin{bmatrix} Z & Y \\ \star & X \end{bmatrix} \geq 0, \; Z_{tt} \leq u_{t,\max}^2, \tag{6}$$

where $\Xi_1 := AX + BY$, $\Xi_2 := N_A X + N_B Y$, Z_{tt} is the tth diagonal element of matrix Z, then the set $\{x | x^T P x \leq \xi\}$ is a RPI set corresponding to the feedback control law $h(x) = Kx$, where ξ is a positive constant, $P := \xi X^{-1}$, and $K := YX^{-1}$.

Proof: Please see the Appendix A. ■

Remark 1: Inequality (5) is a bilinear matrix inequality (BMI) due to (1,1) element of the matrix. Some existing solvers (e.g., PENBMI [16]) can solve it.

Remark 2: It is possible to compute the RPI set with the maximal or minimal size, when an appropriate objective function is selected.

IV. Nominal Prediction Based RMPC

A. 0-Step Prediction

In this section, we will discuss RMPC based on nominal prediction for the discrete-time linear system (2). Denote $\bar{x}$ as the state vector of nominal system $\bar{x}^+ = A\bar{x} + Bu$. We use $\bar{x}(k+i|k)$, $u(k+i|k)$, $i \geq 1$ to represent the predicted nominal system states and inputs for time k respectively, and use $\bar{x}(k)$, $u(k)$ to represent $\bar{x}(k|k)$ and $u(k|k)$ respectively for simplicity. Also, it should be noted that $\bar{x}(k)$ equals $x(k)$, which is the measured system state at time k. Consider the infinite nominal cost function at time k

$$\bar{J}_\infty(k) := \sum_{i=0}^{\infty} \ell(k+i|k),$$

where $\ell(k+i|k)$ is the stage cost and is defined as

$$\ell(k+i|k) := \bar{x}^T(k+i|k)Q\bar{x}(k+i|k) + u^T(k+i|k)Ru(k+i|k),$$

with Q and R positive definite matrices. If it holds

$$\begin{aligned} &\bar{x}^T(k+i+1|k)P\bar{x}(k+i+1|k) \\ &-\bar{x}^T(k+i|k)P\bar{x}(k+i|k) < \ell(k+i|k), \end{aligned} \tag{7}$$

then changing i in the above inequality from 0 to ∞ and summing them together, one has

$$\begin{aligned} \bar{J}_\infty(k) &< \bar{x}^T(k)P\bar{x}(k) - \bar{x}^T(\infty|k)P\bar{x}(\infty|k) \\ &\leq \bar{x}^T(k)P\bar{x}(k). \end{aligned} \tag{8}$$

At time k, $\bar{x}^T(k)P\bar{x}(k)$ is an upper bound of the infinite nominal cost function, and is thus minimized. In order to minimize $\bar{x}^T(k)P\bar{x}(k)$, the following problem is considered

$$\min \; \xi, \quad \bar{x}^T(k)P\bar{x}(k) \leq \xi. \tag{9}$$

Inequality $\bar{x}^T(k)P\bar{x}(k) \leq \xi$ can be written as LMI

$$\begin{bmatrix} 1 & \bar{x}^T(k) \\ \star & X \end{bmatrix} \geq 0, \tag{10}$$

where $X := \xi P^{-1}$. In order to satisfy inequality (7), the feedback control law $u(k+i|k) = K\bar{x}(k+i|k)$, $i \geq 0$ is considered. It can be shown that the following LMI guarantees (7) [3]

$$\begin{bmatrix} -X & XA^T + Y^T B^T & XQ & Y^T R \\ \star & -X & 0 & 0 \\ \star & \star & -\xi Q & 0 \\ \star & \star & \star & -\xi R \end{bmatrix} < 0. \tag{11}$$

Furthermore, the robust invariant property of the set $\{\bar{x} | \bar{x}^T P \bar{x} \leq \xi\}$ is added, which guarantees recursive feasibility. At time k, the following optimization problem is solved

$$\textbf{OP1}: \quad \min_{\xi, X, Y, \lambda_1, \lambda_2, \eta} \; \xi$$

subject to (5), (6), (10) and (11).

Lemma 2 ([10]): If the system $x^+ = f(x,d)$ admits an ISS-Lyapunov function $V(x)$, then it is input-to-state stable w.r.t. the disturbance d.

Lemma 3 ([13]): For a quadratic function $Q(x) = x^T P x$, there exist a finite Lipschitz constant $\mathcal{L}_Q > 0$ such that $|Q(x_1) - Q(x_2)| \leq \mathcal{L}_Q \|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{X}$.

Lemma 4 ([14]): Let f be a function $f(x,u) : \mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}^c$. Then f is an uniformly continuous function in x for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$ iff there exists a $\mathcal{K}_\infty$-function σ such that $\|f(x_1,u) - f(x_2,u)\| \leq \sigma(\|x_1 - x_2\|)$, holds for $\forall x_1, x_2 \in \mathbb{X}, \forall u \in \mathbb{U}$.

Theorem 2: If the problem **OP1** has a solution at the beginning, then it will always be feasible. Furthermore, the discrete-time linear system (2) is input-to-state stable.

Proof: Please see Appendix B. ■

Remark 3: When the disturbances and uncertainties disappear, asymptotic stability can be guaranteed.

In the proposed control strategies **OP1**, a RPI set is obtained at each time instant. It behaves like giving the system a varying target set, with the current system states always be contained in it. Thus we call such control strategy "0-step prediction" RMPC.

B. N-step Prediction

In this subsection, we consider the condition that the discrete-time linear system (2) has a predetermined RPI set as the terminal set. Denote the terminal set as Ω_T. The control objective is to steer the system states into Ω_T in N steps, that is $x(k+N|k) \in \Omega_T$. Note that if $x(k+N-1|k)$ belongs to the robust one-step set of Ω_T, denoted as $\widetilde{\mathcal{Q}}^{h_{N-1}}(\Omega_T)$, i.e., $\widetilde{\mathcal{Q}}^{h_{N-1}}(\Omega_T)$ is the set of all states that can be steered to Ω_T in one step by applying the control law h_{N-1}, then $x(k+N|k) \in \Omega_T$ can be guaranteed. Generally speaking, computation of the exact robust one-step set corresponding to a feedback control law is not easy work. An approximating one with a user defined shape (e.g., ellipsoidal or polytopic) can be used instead. Thus in order to guarantee $x(k+N|k) \in \Omega_T$, we require that $x(k+N-1|k) \in \mathbb{X}_{N-1}$, where $\mathbb{X}_{N-1} \subseteq \widetilde{\mathcal{Q}}^{h_{N-1}}(\Omega_T)$ is an approximate robust one-step set. Similarly, constraint $x(k+N-2|k) \in \mathbb{X}_{N-2}$ is added with $\mathbb{X}_{N-2} \subseteq \widetilde{\mathcal{Q}}^{h_{N-2}}(\mathbb{X}_{N-1})$, in order to make sure $x(k+N-1|k) \in \mathbb{X}_{N-1}$. Following this idea, one has the constraints $x(k+i|k) \in \mathbb{X}_i$, where $0 \leq i \leq N-1$, $\mathbb{X}_i \subseteq \widetilde{\mathcal{Q}}^{h_i}(\mathbb{X}_{i+1})$ and $\mathbb{X}_N := \Omega_T$.

In this paper, we consider the set sequence $\{\mathbb{X}_i\}$, $0 \leq i \leq N$ as ellipsoidal ones. Denote $\mathbb{X}_i$ as $\{x|x^T X_i^{-1} x \leq 1\}$. After obtaining the set $\mathbb{X}_i$, we can compute its approximate robust one-step set $\mathbb{X}_{i-1}$. The following theorem presents the method of computation.

Theorem 3: For the system (2) with bounded disturbance, assume that the set $\mathbb{X}_i$ is known, which is described as $\{x|x^T X_i^{-1} x \leq 1\}$. If there exist a positive matrix X_{i-1}, a general matrix Y_{i-1}, and positive scalars η, λ_1, λ_2 such that the following constrained optimization problem is feasible,

$$\min_{X_{i-1}, Y_{i-1}, \lambda_1, \lambda_2, \eta} -\log\det(X_{i-1}) \tag{12}$$

subject to

$$\begin{bmatrix} -\lambda_1 X_{i-1} & 0 & 0 & \Xi_4^T & \Xi_5^T & 0 \\ \star & -\lambda_2 I & 0 & E^T & 0 & 0 \\ \star & \star & -1+\lambda_1+\lambda_2\gamma^2 & 0 & 0 & 0 \\ \star & \star & \star & -X_i & 0 & \eta M \\ \star & \star & \star & \star & -\eta I & 0 \\ \star & \star & \star & \star & \star & -\eta I \end{bmatrix} \leq 0, \tag{13}$$

$$\begin{bmatrix} Z & Y_{i-1} \\ \star & X_{i-1} \end{bmatrix} \geq 0, \; Z_{tt} \leq u_{t,\max}^2, \; 1 \leq t \leq m, \tag{14}$$

$$\begin{bmatrix} -X_{i-1} & \Xi_1^T & XQ & Y^T R \\ \star & -X_{i-1} & 0 & 0 \\ \star & \star & -Q & 0 \\ \star & \star & \star & -R \end{bmatrix} < 0, \tag{15}$$

where $\Xi_4 := AX_{i-1} + BY_{i-1}$, $\Xi_5 := N_A X_{i-1} + N_B Y_{i-1}$, then the set $\mathbb{X}_{i-1} = \{x|x^T X_{i-1}^{-1} x \leq 1\}$ is an approximate robust one-step set of $\mathbb{X}_i$, corresponding to the control law $h_{i-1} := K_{i-1}x$ and $K_{i-1} := Y_{i-1}X_{i-1}^{-1}$.

Proof: Please see Appendix C. ■

It is assumed that $\mathbb{X}_N$ (the terminal set) is predetermined. Following Theorem 3, the set $\mathbb{X}_{N-1} \cdots \mathbb{X}_0$ and their corresponding state feedback gains $K_{N-1} \cdots K_0$ can be obtained interatively. Let $\mathbb{S} := \mathbb{X}_0 \bigcup \mathbb{X}_1 \bigcup \cdots \bigcup \mathbb{X}_N$. For $\forall x \in \mathbb{S}$, the states of discrete-time linear system (2) can be robustly steered to the terminal set $\mathbb{X}_N$ in N steps. This holds true since the controller $u = K_{i-1}x$ guarantees that for $\forall x \in \mathbb{X}_{i-1} \setminus \mathbb{X}_i$, i.e., x belongs to $\mathbb{X}_{i-1}$ but stays outside of $\mathbb{X}_i$, it holds $x^+ \in \mathbb{X}_i$. However, using the pre-computed controllers directly may lose the performance of the closed-loop system. In the following, we will discuss the online computation that minimizes an upper bound of the nominal infinite cost function.

At sampling time instant k, a bisection search is carried out to find the index i such that $x(k) \in \mathbb{X}_{i-1} \setminus \mathbb{X}_i$. Input $u(k)$ should fulfill the objective of steering the system states into $\mathbb{X}_i$ at time $k+1$. Choose the predicted input sequence as $u(k+j|k) = K_i \bar{x}(k+j|k)$, where $j \geq 1$. Since LMI (15) implies

$$\Delta \, \bar{x}^T(k+j+1|k) X_i^{-1} \bar{x}(k+j+1|k) < -\ell(k+i|k), \tag{16}$$

it holds

$$\begin{aligned} \bar{J}_{1\infty}(k) &:= \sum_{j=1}^{\infty} \ell(k+i|k) \\ &< \bar{x}^T(k+1|k) X_i^{-1} \bar{x}(k+1|k). \end{aligned} \tag{17}$$

Noticing that $\bar{J}_\infty(k) = x^T(k)Qx(k) + u^T(k)Ru(k) + \bar{J}_{1\infty}$, one has $\bar{J}_\infty(k) < \bar{J}_1(k)$, where $\bar{J}_1(k) := x^T(k)Qx(k) + u^T(k)Ru(k) + \bar{x}^T(k+1|k)X_i^{-1}\bar{x}(k+1|k)$. Thus at time k, $\bar{J}_1(k)$ is optimized as an upper bound of the nominal infinite cost function. In the online computation, the following problem is considered

$$\min_{u(k)} \; \xi, \text{ subject to } \bar{J}_1(k) \leq \xi. \tag{18}$$

Resorting to the Schur complement, the $\bar{J}_1(k) \leq \xi$ can be formulated as

$$\begin{bmatrix} -\xi + x^T(k)Qx(k) & (Ax(k)+Bu(k))^T & u^T(k) \\ \star & -X_i & 0 \\ \star & \star & -R^{-1} \end{bmatrix} \leq 0. \tag{19}$$

Now consider another constraint $x(k+1) \in \mathbb{X}_i$, which holds if

$$x^T(k+1) X_i^{-1} x(k+1) \leq 1. \tag{20}$$

Taking into account $d^T(k)d(k) \leq \gamma^2$, (20) holds if and only if

$$x^T(k+1)X_i^{-1}x(k+1) - 1 - \lambda(d^T(k)d(k) - \gamma^2) \leq 0, \quad (21)$$

where λ is a positive scalar. Notice that

$$x(k+1) = \widetilde{A}(k)x(k) + \widetilde{B}(k)u(k) + Ed(k), \quad (22)$$

where $\widetilde{A}(k) = A + MT(k)N_A$ and $\widetilde{B}(k) = B + MT(k)N_B$. Substituting (22) into (21), one has

$$\begin{bmatrix} 1 & d^T(k) \end{bmatrix} \begin{bmatrix} \Xi_7 & \Xi_8 \\ \star & \Xi_9 \end{bmatrix} \begin{bmatrix} 1 \\ d(k) \end{bmatrix} \leq 0, \quad (23)$$

where $\Xi_7 := [\widetilde{A}(k)x(k) + \widetilde{B}(k)u(k)]^T X_i^{-1}[\widetilde{A}(k)x(k) + \widetilde{B}(k)u(k)] - 1 + \lambda\gamma^2$, $\Xi_8 := [\widetilde{A}(k)x(k) + \widetilde{B}(k)u(k)]^T X_i^{-1}E$ and $\Xi_9 := E^T X_i^{-1}E - \lambda$. It is easily known that (23) holds if

$$\begin{bmatrix} \Xi_7 & \Xi_8 \\ \star & \Xi_9 \end{bmatrix} \leq 0. \quad (24)$$

LMI (24) can be written as

$$\begin{bmatrix} -1+\lambda\gamma^2 & 0 \\ \star & -\lambda I \end{bmatrix} + \begin{bmatrix} (\widetilde{A}(k)x(k) + \widetilde{B}(k)u(k))^T \\ E^T \end{bmatrix} X_i^{-1} \begin{bmatrix} \widetilde{A}(k)x(k) + \widetilde{B}(k)u(k) & E \end{bmatrix} \leq 0. \quad (25)$$

Using the Schur complement, the above inequality is equivalent to

$$\begin{bmatrix} -1+\lambda\gamma^2 & 0 & (\widetilde{A}(k)x(k) + \widetilde{B}(k)u(k))^T \\ \star & -\lambda I & E^T \\ \star & \star & -X_i \end{bmatrix} \leq 0. \quad (26)$$

Referring the method used in Theorem 1, LMI (26) is satisfied if one has

$$\begin{bmatrix} -1+\lambda\gamma^2 & 0 & \Xi_{10}^T & \Xi_{11}^T & 0 \\ \star & -\lambda I & E^T & 0 & 0 \\ \star & \star & -X_i & 0 & \eta M \\ \star & \star & \star & -\eta I & 0 \\ \star & \star & \star & \star & -\eta I \end{bmatrix} \leq 0, \quad (27)$$

where $\Xi_{10} := Ax(k) + Bu(k)$, $\Xi_{11} := N_A x(k) + N_B u(k)$ and η is a positive scalar. Now we are ready to give the controller design for N-step prediction case.

OP2:

Off-line part : Compute a RPI set as the terminal set $\mathbb{X}_N$. Then using the method introduced in Theorem 3, the set sequence $\{\mathbb{X}_j\}$, $0 \leq j \leq N-1$ are obtained together with the feedback controller.

Online part : At each time k, find the index i such that $x(k) \in \mathbb{X}_{i-1} \setminus \mathbb{X}_i$. If $x(k) \in \mathbb{X}_N$, set i as N. Solve the following constrained optimization problem

$$\min_{u(k),\xi,\lambda,\eta} \xi$$

subject to (19), (27) and $u(k) \in \mathbb{U}$.

Note that for **OP2**, the problem is always feasible when $x(k) \in \mathbb{S}$ ($\mathbb{S} := \mathbb{X}_0 \bigcup \mathbb{X}_1 \bigcup \cdots \bigcup \mathbb{X}_N$), since $u(k) = K_{i-1}x(k)$ is a feasible input when $x(k) \in \mathbb{X}_{i-1} \setminus \mathbb{X}_i$. Though BMIs are involved in the proposed strategy, they are solved in the off-line computation. The computations carried out online are bisection searches as well as solving a convex optimization problem subject to LMIs.

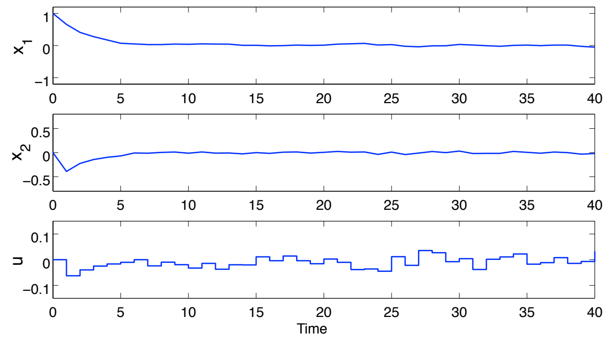

Fig. 1. Closed-loop response of the solenoid system using **OP1**.

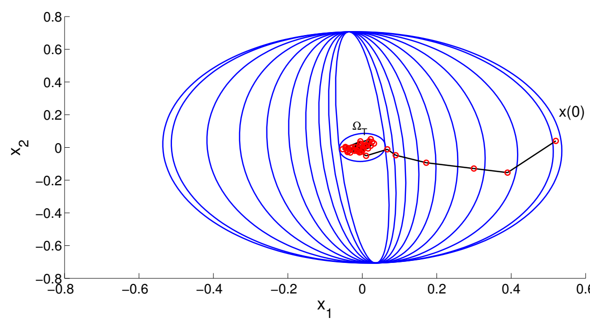

Fig. 2. Control effects of **OP2** method; Blue ellipses are approximate robust one-step sets, red dots are system states.

V. Numerical Example

This section gives simulation results of the proposed controllers. Consider the solenoid system presented in [15]. The state space system model is given by $x(k+1) = (A + MT(k)N_A)x(k) + (B + MT(k)N_B)u(k) + Ed(k)$, where

$$A = \begin{bmatrix} 0.6148 & 0.0315 \\ -0.3155 & -0.0162 \end{bmatrix}, B = E = \begin{bmatrix} 0.0385 \\ 0.0315 \end{bmatrix},$$

and $M = I, N_A = 0.2A, N_B = 0.2B$. The input $u(k)$ satisfies and $|u(k)| \leq 1$. For the disturbance $d(k)$, it holds $d^T(k)d(k) \leq 1$. Q and R in the cost function are chosen as $[2\ 0; 0\ 2]$ and 1 respectively. Fig. 1. shows the control effects of **OP1** strategy. We select the initial system states $x(0)$ as $[1; 0]$. It can be seen that the system states are stabilized neighboring the origin. Since the disturbances always exist, asymptotic stability can not be achieved. For the **OP2** method, the terminal set is a RPI set with appropriate size. It can be computed based on Thoerem 1. In the simulation, the terminal set is obtained as $\{x | x^T X_N^{-1} x \leq 1\}$ with $X_N = [0.0038\ 0.0005; 0.0005;\ 0.0073]$. A sequence of 10 approximate ellipsoidal one-step sets are obtained using Theorem 3. The initial states $x(0) = [0.52; 0.04]$, which lies in $\{\mathbb{X}_{10} \setminus \mathbb{X}_9\}$. Fig. 2 presents the evolution of the system states. It is observed that the system states are steered into the terminal set after 6 steps.

Remark 4: The results of approximate one-step set might be conservative with respect to the actual size, i.e., the state outside the set may also be steered into the inner set by applying the corresponding control law. One reason is that the shape of the sets is restrict to ellipsoid. Another reason is the use of S-procedure, which renders sufficient conditions only in the derivation.

VI. CONCLUSIONS

This paper studies robust model predictive control of a discrete-time linear system. The system is subject to structured parameter uncertainties and also persistent external disturbances. Based on the S-procedure, we present a method to compute the robust positively invariant sets for the considered system, which plays an important role in robust MPC problem. In the controller design, cost function based on nominal predictions is discussed. We have proposed a 0-step RMPC strategy, which minimizes an upper bound of the infinite nominal cost function at each time instant. Though the structure of the 0-step predictive controller is relatively simple, it can not deal with the predictive control problem with a predetermined time-invariant terminal set. To this end, a N-step predictive controller is further proposed. We have shown that a sequence of approximate robust one-step sets can be computed off-line, while the online work is just bisection searches and solving a convex optimization problem subject to LMIs. Thus the online computational burden is relatively affordable, which benefits a lot to practical implementations.

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APPENDIX

A. Proof of Theorem 1

For any $x \in \Omega := \{x | x^T P x \leq \xi\}$, if it holds $x^+ \in \Omega$ though the uncertainties and disturbances, then the set Ω is a RPI set. According to the S-procedure [11], Ω is a RPI set iff one has

$$x^{+T} X^{-1} x^+ - 1 - \lambda_1 (x^T X^{-1} x - 1) \leq 0, \tag{A.28}$$

where λ_1 is a positive scalar. Similar condition can also be found in [12], where a linear system without uncertainties was discussed. Noticing that $d^T d \leq \gamma^2$, the following sufficient condition can be derived for (A.28) to hold

$$x^{+T} X^{-1} x^+ - 1 - \lambda_1 (x^T X^{-1} x - 1) - \lambda_2 (d^T d - \gamma^2) \leq 0, \tag{A.29}$$

where λ_2 is another positive scalar. Substituting $x^+ = \widetilde{A}x + \widetilde{B}Kx + Ed$ into inequality (A.29), one has

$$\begin{bmatrix} x \\ d \\ 1 \end{bmatrix}^T \begin{bmatrix} \Xi_3 & (\widetilde{A} + \widetilde{B}K)^T X^{-1} E & 0 \\ \star & E^T X^{-1} E - \lambda_2 I & 0 \\ \star & \star & -1 + \lambda_1 + \lambda_2 \gamma^2 \end{bmatrix} \begin{bmatrix} x \\ d \\ 1 \end{bmatrix} \leq 0. \tag{A.30}$$

where $\Xi_3 := (\widetilde{A} + \widetilde{B}K)^T X^{-1} (\widetilde{A} + \widetilde{B}K) - \lambda_1 X^{-1}$. Inequality (A.30) holds true if

$$\begin{bmatrix} \Xi_3 & (\widetilde{A} + \widetilde{B}K)^T X^{-1} E & 0 \\ \star & E^T X^{-1} E - \lambda_2 I & 0 \\ \star & \star & -1 + \lambda_1 + \lambda_2 \gamma^2 \end{bmatrix} \leq 0. \tag{A.31}$$

It can be shown that the above inequality is equivalent to

$$\begin{bmatrix} -\lambda_1 X^{-1} & 0 & 0 & (\widetilde{A} + \widetilde{B}K)^T \\ \star & -\lambda_2 I & 0 & E^T \\ \star & \star & -1 + \lambda_1 + \lambda_2 \gamma^2 & 0 \\ \star & \star & \star & -X \end{bmatrix} \leq 0. \tag{A.32}$$

Multiplying $\mathrm{diag}\{X, I, I, I\}$ and its transpose from both sides of (A.32) respectively, and substituting $\widetilde{A} = A +$

MTN_A, $\widetilde{B} = B + MTN_B$, one has

$$\begin{bmatrix} -\lambda_1 X & 0 & 0 & (AX+BY)^T \\ \star & -\lambda_2 I & 0 & E^T \\ \star & \star & -1+\lambda_1+\lambda_2\gamma^2 & 0 \\ \star & \star & \star & -X \end{bmatrix} + \mathbf{H}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ M \end{bmatrix} T \begin{bmatrix} (N_A X + N_B Y) & 0 & 0 & 0 \end{bmatrix}\right) \leq 0. \quad \text{(A.33)}$$

Form Lemma 1, it is known that (A.33) holds iff there exists a positive scalar such that

$$\begin{bmatrix} -\lambda_1 X & 0 & 0 & (AX+BY)^T \\ \star & -\lambda_2 I & 0 & E^T \\ \star & \star & -1+\lambda_1+\lambda_2\gamma^2 & 0 \\ \star & \star & \star & -X \end{bmatrix} + \eta^{-1}\begin{bmatrix} (N_A X + N_B Y)^T \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} (N_A X + N_B Y) & 0 & 0 & 0 \end{bmatrix} + \eta \begin{bmatrix} 0 \\ 0 \\ 0 \\ M \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & M^T \end{bmatrix} \leq 0. \quad \text{(A.34)}$$

Using the Schur complement, and multiplying $\text{diag}\{I, I, I, I, \eta I\}$ and its transpose on both sides of the resulted matrix respectively, it can be easily shown that (A.34) is equivalent to (5) in the theorem. From the above analysis, it is clear that (5) guarantees the robust invariant property (A.29) for set Ω. LMI (6) guarantees that the feedback control law $u = Kx$ satisfies the input constraint. The derivation can be found in [3]. The proof is thus completed.

B. Proof of Theorem 2

We will first show the recursive feasibility of $\mathbf{OP1}$. Assume that the optimal solution of $\mathbf{OP1}$ at time 0 is denoted as $\{\xi^*, X^*, Y^*, \lambda_1^*, \lambda_2^*, \eta^*\}$, and $P^* := \xi^* X^{*-1}$. At time 1, $x(1)$ is obtained. Inequality (5) leads to the fact that $x^T(1)P^*x(1) \leq \xi^*$, which can also be written as $\bar{x}^T(1)P^*\bar{x}(1) \leq \xi^*$. Thus LMI (10) still holds true at time 1. Since only constraint (10) is state-dependent, the solution of $\mathbf{OP1}$ at time 0 is feasible at time 1. By deduction, it is easily known that the optimization problem $\mathbf{OP1}$ will always be feasible. Then we will show that $V^*(x(k)) := x^T(k)P^*(k)x(k)$ is an ISS-Lyapunov function, where $P^*(k)$ represents the optimal P at time k. It holds

$$\|x(k)\|_2^2 \leq V(x(k)) \leq \rho_{\max}\|x(k)\|_2^2, \quad \text{(A.35)}$$

where $\rho_{\min} := \min\{\underline{\rho}(P^*(k)) | k \geq 0\}$, $\rho_{\max} := \max\{\overline{\rho}(P^*(k)) | k \geq 0\}$, and $\underline{\rho}(\cdot)$, $\overline{\rho}(\cdot)$ represent the minimal and maximal eigenvalue respectively. Since inequality (7) is guaranteed by LMI (11), it holds,

$$\begin{aligned} & V^*(\bar{x}(k+1|k)) - V^*(x(k)) \\ < & -x^T(k)Qx(k) - u^T(k)Ru(k) \\ \leq & -\|x(k)\|_Q^2. \end{aligned} \quad \text{(A.36)}$$

Note that $\bar{x}(k+1|k) = A\bar{x}(k) + Bu(k)$ and $x(k+1) = (A + MT(k)N_A)x(k) + (B + MT(k)N_B)u(k) + Ed(k)$. Thus $x(k+1) = \bar{x}(k+1|k) + w(k) + Ed(k)$. From Lemma 3, it is known that there exists a Lipschitz constant $\mathcal{L}_P(k)$ for $V^*(x(k)$ such that $|V^*(x_1) - V^*(x_2)| \leq \mathcal{L}_P(k)\|x_1 - x_2\|$. Also, Lemma 4 implies that $\|x(k+1) - \bar{x}(k+1|k)\| \leq \sigma_w(\|w(k)\|) + \sigma_d(\|d(k)\|)$, where σ_w and σ_d are $\mathcal{K}_\infty$-functions. Define $\bar{V}(x(k+1)) := x(k+1)^T P^*(k)x(k+1)$. It holds

$$\begin{aligned} & |\bar{V}(x(k+1)) - V^*(\bar{x}(k+1|k))| \\ \leq & \mathcal{L}_P(k)\sigma_w(\|w(k)\|) + \mathcal{L}_P(k)\sigma_d(\|d(k)\|). \end{aligned} \quad \text{(A.37)}$$

Noting that $\bar{x}(k) = x(k)$, inequality (A.36) and (A.37) imply

$$\begin{aligned} & \bar{V}(x(k+1)) - V^*(k) \\ < & -\|x(k)\|_Q^2 + \mathcal{L}_P(k)\sigma_w(\|w(k)\|) \\ & +\mathcal{L}_P(k)\sigma_d(\|d(k)\|). \end{aligned} \quad \text{(A.38)}$$

For the optimal solution at time $k+1$, it holds

$$V^*(x(k+1)) \leq x^T(k+1)P^*(k)x(k+1). \quad \text{(A.39)}$$

From (A.38) and (A.39), it is easily known that

$$\begin{aligned} & V^*(x(k+1)) - V^*(x(k)) \\ \leq & -\|x(k)\|_Q^2 + \overline{\mathcal{L}}_P\sigma_w(\|w(k)\|) \\ & +\overline{\mathcal{L}}_P\sigma_d(\|d(k)\|), \end{aligned} \quad \text{(A.40)}$$

where $\overline{\mathcal{L}}_P := \max\{\mathcal{L}_P(k) | k \geq 0\}$. Inequalities (A.35) and (A.40) imply that $V^*(x(k))$ is an ISS-Lyapunov function. From Lemma 2, it is known that system (2) is input-to-state stable w.r.t. $w(k)$ and $d(k)$. The proof is thus completed.

C. Proof of Theorem 3

The proof is straightforward referring to that of Theorem 1. It can be proven that inequality (13) implies

$$\begin{bmatrix} -\lambda_1 X_{i-1}^{-1} + \Xi_6 & (\widetilde{A} + \widetilde{B}K_{i-1})^T X_i^{-1} E & 0 \\ \star & -\lambda_2 + E^T X_i^{-1} E & 0 \\ \star & \star & -\sigma \end{bmatrix} \leq 0, \quad \text{(A.41)}$$

where $\Xi_6 := (\widetilde{A} + \widetilde{B}K_{i-1})^T X_i^{-1}(\widetilde{A} + \widetilde{B}K_{i-1})$, and $\sigma := 1 - \lambda_1 - \lambda_2\gamma^2$. Multiplying $[x^T \; d^T \; 1]$ and its transpose from both sides of (A.41) respectively, it can be proven that the following inequality holds,

$$x^{+T} X_i^{-1} x^+ - 1 - \lambda_1(x^T X_{i-1}^{-1} x - 1) - \lambda_2(d^T d - \gamma^2) \leq 0. \quad \text{(A.42)}$$

According to the S-procedure, it is known that for any x satisfying $x^T X_{i-1}^{-1} x - 1 \leq 0$, it holds $x^{+T} X_i^{-1} x^+ - 1 \leq 0$, which implies $x^+ \in \mathbb{X}_i$. Thus the "robust one-step set" property is guaranteed. LMI (14) ensures the satisfaction of input constraints. It should be noted that LMI (15) is an additional constraint added into the computation of set sequence. It guarantees $\bar{x}^+ X_{i-1}^{-1} \bar{x}^+ - \bar{x}^T X_{i-1}^{-1} \bar{x} < -\bar{x}^T Q \bar{x} - u^T R u$ when $u = K_{i-1}\bar{x}$ is applied. Finally, in order to maximize the size of the approximate robust one-step set, determinant maximization is considered as the objective function in the optimization problem [11]. The proof is thus completed.