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On Stabilization Over a Gaussian Interference Channel

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***Abstract*— The problem of feedback stabilization of LTI plants over a Gaussian interference channel is considered. Two plants with arbitrary distributed initial states are monitored by two separate sensors which communicate their measurements to two separate controllers over a Gaussian interference channel under average transmit power constraints. The necessary conditions for mean square-stabilization over a memoryless symmetric Gaussian interference channel are derived. These conditions are shown to be tight for some system parameters. Further it is shown that linear memoryless sensing and control schemes are optimal for stabilization in some special cases.**

I. Introduction

In a networked control system there are various agents such as sensors, controllers, actuators, and plants. These agents need to communicate to meet certain control objectives, and this communication should preferably take place over wireless links to reduce cabling cost and to provide flexible and mobile solutions. A major hurdle in implementing wireless networked control systems is the interference which happens due to the cross-talk between various agents while using shared communication resources. There are also external sources of interference such as other radio devices communicating in the neighborhood. In certain situations, interference can be a major factor to limit performance of a networked control system. Therefore it is essential to study and understand the behavior of networked control systems subject to interference. This paper makes an effort in this direction.

In order to study the problems of communication under control constraints, an interaction between stochastic control theory and information theory needs to be addressed. The authors of [1–12, ?–12] have used ideas from information theory to address the problems in control over communication channels. Signal-to-noise ratio requirements for stabilization over various Gaussian channels have been studied in [13–19], where necessary and sufficient conditions for stabilization have been derived. In [19] we studied the problem of mean square stabilization of two linear plants over a symmetric Gaussian interference channel and derived sufficient conditions for stabilization using linear sensing and control schemes.

The main contribution of this paper is that it provides necessary conditions for mean-square stabilization over the symmetric Gaussian interference channel. We evaluate the gap between the necessary conditions derived in this paper and the sufficient conditions derived in [19]. It is demonstrated that linear sensing and control schemes can be close to optimal in some regimes. Moreover in some special cases they are shown to be exactly optimal.

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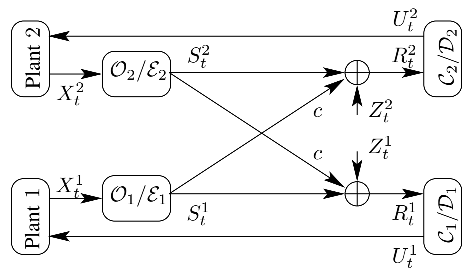

Fig. 1: System Model.

II. Problem Setup

We consider two discrete-time LTI plants whose state equations are given by,

$$X_{t+1}^i = A_i X_t^i + U_t^i, \quad i = 1, 2, \tag{1}$$

where $X_t^i \in \mathbb{R}^n$ and $U_t^i \in \mathbb{R}^n$, are the state and control processes of the i-th plant. We assume that the initial state X_0^i is a random variable with arbitrary probability distribution and a given covariance matrix Λ_0^i with Trace$\{\Lambda_0^i\} < \infty$. Let $\{\lambda_{i,1}, \lambda_{i,2}, \cdots, \lambda_{i,n}\}$ denote the eigenvalues of the system matrix A_i. Without loss of generality we assume that all the eigenvalues of the system matrix A_i are outside the unit disc, i.e., $|\lambda_{i,j}| > 1$ for all i, j. The unstable modes can be decoupled from the stable modes by a similarity transformation. If the system in (1) is one dimensional then A_i is scalar and we use the notation $A_i = \lambda_i$, where $|\lambda_i| > 1$.

The setup for control over symmetric Gaussian interference channel is depicted in Fig. 1. There are two separate observers $\{\mathcal{O}_1, \mathcal{O}_2\}$ and separate controllers $\{\mathcal{C}_1, \mathcal{C}_2\}$ for the two plants. In order to communicate the observed state values to the controllers, an encoder $\mathcal{E}_i$ is lumped with the observer $\mathcal{O}_i$ and a decoder $\mathcal{D}_i$ is lumped with the controller $\mathcal{C}_i$. At any time instant t, the encoders $\mathcal{E}_1$ and $\mathcal{E}_2$ transmit S_t^1 and S_t^2 respectively. The decoders $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively receive

$$R_t^1 = S_t^1 + cS_t^2 + Z_t^1,$$
$$R_t^2 = S_t^2 + cS_t^1 + Z_t^2,$$

where $c \in \mathbb{R}$ is the cross channel gain, and $Z_t^1 \sim \mathcal{N}(0, N)$ and $Z_t^2 \sim \mathcal{N}(0, N)$ are white noise components with a fixed cross-correlation coefficient $\rho_z \triangleq \frac{\mathbb{E}[Z_t^1 Z_t^2]}{N} \in [-1, 1]$. The cross-correlation between the two noise components models a common noise or common interference in the two signals.

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Let f_t^i denote the i-th observer/encoder policy, then we have $S_t^i = f_t^i(X_{[0,t]}^i)$ where $X_{[0,t]}^i := \{X_1^i, X_2^i, \cdots, X_t^i\}$. The sensors must satisfy an average transmit power constraint $\mathbb{E}\left[(S_t^1)^2\right] = \mathbb{E}\left[(S_t^2)^2\right] = P$. We define the correlation between S_t^1 and S_t^2 as $\tilde{\rho}_t := \frac{\mathbb{E}\left[S_t^1 S_t^2\right]}{P} \in [-1, 1]$. Further let π_t^i denote the ith decoder/controller policy, then $U_t^i = \pi_t^i(R_{[0,t]}^i)$. The common objective of the sensors and the controllers is to mean square stabilize respective plants in the mean-square sense, which is defined as follows.

***Definition** 2.1:* A system is said to be *mean square stable* if there exists a constant $M < \infty$ such that $\mathbb{E}[\|X_t\|^2] < M$ for all t.

III. Stability Conditions

In the following theorem we present necessary conditions for mean-square stabilization, which is the main result of this paper.

***Theorem** 3.1:* The two LTI systems in (1) can be mean square stabilized over the given symmetric Gaussian interference channel only if

$$\log\left(|\det(A_1)|\right) \le \frac{1}{2}\log\left(1 + \frac{P(1+c)^2}{N}\right), \tag{2}$$

$$\log\left(|\det(A_2)|\right) \le \frac{1}{2}\log\left(1 + \frac{P(1+c)^2}{N}\right), \tag{3}$$

$$\begin{aligned}&\log\left(|\det(A_1)|\right) + \log\left(|\det(A_2)|\right) \le \\ &\frac{1}{2}\max_{0\le\rho\le 1}\Bigg\{\log\left(1 + \frac{P\left(1 + c^2 + 2c\rho\right)}{N}\right) + \\ &\log\left(\frac{N\left(1-\rho_z^2\right) + P\left(1 + c^2 - 2c\rho_z\right)\left(1-\rho^2\right)}{\left(1-\rho_z^2\right)\left(Pc^2\left(1-\rho^2\right) + N\right)}\right)\Bigg\}.\end{aligned} \tag{4}$$

Proof: In order to prove conditions (2) and (3), we make use of the following Lemma.

Lemma *3.1:* The i-th linear system in (1) can be mean square stabilized over the Gaussian interference channel only if

$$\log\left(|\det(A_i)|\right) \le \liminf_{T\to\infty} \frac{1}{T} I\left(X_{[0,T-1]}^i \to R_{[0,T-1]}^i\right), \tag{5}$$

where $I\left(X_{[0,T\text{-}1]}^i \to R_{[0,T\text{-}1]}^i\right) = \sum_{t=0}^{T\text{-}1} I\left(X_t^i; R_t^i | R_{[0,t\text{-}1]}^i\right)$ is the directed information from the sequence of the i-th plant state variables $X_{[0,T\text{-}1]}^i$ to the sequence of variables $R_{[0,T\text{-}1]}^i$ received by the i-th controller.

Proof: The proof can be found in Appendix I. ■

In the following we obtain an upper bound on the directed information $I\left(X_{[0,T\text{-}1]}^i \to R_{[0,T\text{-}1]}^i\right)$ and then use Lemma 3.1 to derive the necessary conditions given in (2) and (3). The directed information can be bounded as,

$$\begin{aligned}
I\left(X_{[0,T\text{-}1]}^i \to R_{[0,T\text{-}1]}^i\right) &\overset{(a)}{=} \sum_{t=0}^{T-1} I\left(X_t^i; R_t^i | R_{[0,t\text{-}1]}^i\right)\\
&\overset{(b)}{=} \sum_{t=0}^{T-1}\left(h\left(R_t^i | R_{[0,t\text{-}1]}^i\right) - h\left(R_t^i | R_{[0,t\text{-}1]}^i, X_t^i\right)\right)\\
&\overset{(c)}{\le} \sum_{t=0}^{T-1}\left(h\left(R_t^i | R_{[0,t\text{-}1]}^i\right) - h\left(R_t^i | S_{[0,t]}^i, R_{[0,t\text{-}1]}^i, X_t^i\right)\right)\\
&\overset{(d)}{=} \sum_{t=0}^{T-1}\left(h\left(R_t^i | R_{[0,t\text{-}1]}^i\right) - h\left(R_t^i | S_{[0,t]}^i, R_{[0,t\text{-}1]}^i\right)\right)\\
&= \sum_{t=0}^{T-1} I\left(S_{[0,t]}^i; R_t^i | R_{[0,t\text{-}1]}^i\right) \overset{(e)}{\le} I\left(S_{[0,T-1]}^i; R_{[0,T-1]}^i\right)\\
&\overset{(f)}{\le} I\left(S_{[0,T-1]}^1, S_{[0,T-1]}^2; R_{[0,T-1]}^i\right)\\
&= h\left(R_{[0,T-1]}^i\right) - h\left(R_{[0,T-1]}^i | S_{[0,T-1]}^1, S_{[0,T-1]}^2\right)\\
&\overset{(g)}{=} h\left(R_{[0,T-1]}^i\right) - h\left(Z_{[0,T-1]}^i\right)\\
&\overset{(h)}{=} \sum_{t=0}^{T-1}\left[h\left(R_t^i | R_{[0,t-1]}^i\right) - h\left(Z_t^i\right)\right]\\
&\overset{(i)}{\le} \sum_{t=0}^{T-1}\left[h\left(R_t^i\right) - h\left(Z_t^i\right)\right]\\
&\overset{(j)}{\le} \sum_{t=0}^{T-1} \frac{1}{2}\log\left(1 + \frac{P + c^2P + 2c\tilde{\rho}_t P}{N}\right)\\
&\overset{(k)}{\le} \frac{T}{2}\log\left(1 + \frac{P(1+c)^2}{N}\right),
\end{aligned} \tag{6}$$

where (a) follows by the definition of directed information [20]; (b) follows by writing mutual information in terms of differential entropies [21, Theorem 2.4.1]; (c) follows from the fact that conditioning reduces entropy [21, Theorem 2.6.5]; (d) follows from the Markov chain $X_{[0,t]}^i - \{S_{[0,t]}^i, R_{[0,t\text{-}1]}^i\} - R_t^i$; (e) follows from the fact that adding more variables cannot decrease mutual information; (f) follows from the fact that adding side information cannot decrease mutual information; (g) follows from $R_t^i = S_t^i + cS_t^j + Z_t^i$ and mutual independence of $\{Z_t^1, Z_t^2\}$ and $\{S_t^1, S_t^2\}$; (h) follows from the assumption that $Z_{[0,T\text{-}1]}^i$ is a sequence of independent variables; (i) follows from conditioning reduces entropy; (j) follows from

$$h\left(Z_t^i\right) = \frac{1}{2}\log\left(2\pi e N\right),$$
$$h\left(R_t^i\right) \le \frac{1}{2}\log\left(2\pi e\left(P + c^2P + 2c\tilde{\rho}_t P\right)\right),$$

where equality holds if we assume Gaussian distributed variables since the Gaussian distribution maximizes differential entropy for a given variance [21, Theorem 8.6.5]; and (k) follows by the maximization of the R.H.S. of (j) subject to $-1 \le \tilde{\rho}_t \le 1$ for all t. The maximum value is attained by choosing $\tilde{\rho}_t = 1$ if $c \ge 0$ and $\tilde{\rho}_t = -1$ if $c \le 0$. Now by using (6) in Lemma 3.1, we get the necessary conditions for mean square stabilization given in (2) and (3).

In the following we derive the necessary condition (4) using a genie-aided bound approach [22]. Consider a superior system where the controllers have some side (extra) information. We define $Y_t^i := cS_t^i + Z_t^j$ for $i \ne j$ and assume that at any time t, the i-th controller has access to $Y_{[0,t]}^i$ in addition to $R_{[0,t]}^i$. In the following we obtain a necessary condition for mean square stabilization of this superior system. Note that a condition which is necessary for the stabilization of this superior system is also necessary for the stabilization of the actual system. Similar approaches have been used in information theory community to derive outer bounds on capacity regions [22, 23]. We will again use Lemma 3.1 to derive a necessary condition for the superior system. The following lemma reveals some functional relationship between different variables in the superior system, which will be useful in the derivation of the necessary condition.

Lemma *3.2:* For the superior system with side information $Y_t^i := hS_t^i + Z_t^j$ at the i-th controller, we have the following relationships for all $i, j \in \{1, 2\}$ and $i \neq j$:

$$X_t^i = A_i^t X_0^i + \mu_t^i \left(R^i_{[0,t\text{-}1]} \right), \tag{7}$$

$$S_t^i = \nu_t^i \left(X_0^i, Y^j_{[0,t\text{-}1]} \right), \tag{8}$$

$$S_t^i = \upsilon_t^i \left(X_0^i, R^i_{[0,t\text{-}1]} \right), \tag{9}$$

where $\mu_t^i : \mathbb{R}^{t\text{-}1} \to \mathbb{R}$, $\nu_t^i : \mathbb{R}^t \to \mathbb{R}$, and $\upsilon_t^i : \mathbb{R}^t \to \mathbb{R}$.

Proof: The proof is given in Appendix II. ■

The directed information $I\left(X^i_{[0,T\text{-}1]} \to R^i_{[0,T\text{-}1]}\right)$ can be written as,

$$\begin{aligned}
I\left(X^i_{[0,T\text{-}1]} \to R^i_{[0,T\text{-}1]}\right) &= \sum_{t=0}^{T-1} I\left(X_t^i; R_t^i | R^i_{[0,t\text{-}1]}\right) \\
&= \sum_{t=0}^{T-1} h\left(X_t^i | R^i_{[0,t\text{-}1]}\right) - h\left(X_t^i | R^i_{[0,t]}\right) \\
&\overset{(a)}{=} \sum_{t=0}^{T-1} h\left(A_i^t X_0^i + \mu_t^i\left(R^i_{[0,t\text{-}1]}\right) | R^i_{[0,t\text{-}1]}\right) \\
&\quad - h\left(A_i^t X_0^i + \mu_t^i\left(R^i_{[0,t\text{-}1]}\right) | R^i_{[0,t]}\right) \\
&= \sum_{t=0}^{T-1} h\left(A_i^t X_0^i | R^i_{[0,t\text{-}1]}\right) - h\left(A_i^t X_0^i | R^i_{[0,t]}\right) \\
&= I\left(A_i^t X_0^i; R^i_{[0,T\text{-}1]}\right) \overset{(b)}{=} I\left(X_0^i; R^i_{[0,T\text{-}1]}\right),
\end{aligned} \tag{10}$$

where (a) follows by substituting X_t^i from (7); and (b) follows from the fact that A_i is invertible and the mutual information between two variables is invariant with resect to any reversible transformation of one of the variables [24, Page 31]. Now by using (10) in Lemma 3.1, the i-th plant can be mean square stabilized only if

$$\log\left(|A_i|\right) \leq \liminf_{T\to\infty} \frac{1}{T} I\left(X_0^i; R^i_{[0,T-1]}\right). \tag{11}$$

Thus the two plants can be mean square stabilized only if

$$\begin{aligned}
&\log\left(|\det\left(A_1\right)|\right) + \log\left(|\det\left(A_2\right)|\right) \overset{(a)}{\leq} \\
&\liminf_{T\to\infty} \frac{1}{T} I\left(X_0^1; R^1_{[0,T\text{-}1]}\right) + \liminf_{T\to\infty} \frac{1}{T} I\left(X_0^2; R^2_{[0,T\text{-}1]}\right) \\
&\overset{(b)}{\leq} \liminf_{T\to\infty} \frac{1}{T} \left\{ I\left(X_0^1; R^1_{[0,T\text{-}1]}\right) + I\left(X_0^2; R^2_{[0,T\text{-}1]}\right) \right\},
\end{aligned} \tag{12}$$

where (a) follows from (11) and (b) follows from the fact that limit inferior satisfies superaddivity. We can now bound the sum $I\left(X_0^1; R^1_{[0,T\text{-}1]}\right) + I\left(X_0^2; R^2_{[0,T\text{-}1]}\right)$ as

$$\begin{aligned}
&I\left(X_0^1; R^1_{[0,T-1]}\right) + I\left(X_0^2; R^2_{[0,T-1]}\right) \\
&\overset{(a)}{\leq} I\left(X_0^1; R^1_{[0,T-1]}, Y^1_{[0,T-1]}, X_0^2\right) + I\left(X_0^2; R^2_{[0,T-1]}\right) \\
&\overset{(b)}{=} I\left(X_0^1; R^1_{[0,T-1]}, Y^1_{[0,T-1]} | X_0^2\right) + I\left(X_0^2; R^2_{[0,T-1]}\right) \\
&\overset{(c)}{=} h\left(R^1_{[0,T\text{-}1]}, Y^1_{[0,T\text{-}1]} | X_0^2\right) - h\left(R^1_{[0,T\text{-}1]}, Y^1_{[0,T\text{-}1]} | X_0^1, X_0^2\right) \\
&\quad + h\left(R^2_{[0,T\text{-}1]}\right) - h\left(R^2_{[0,T\text{-}1]} | X_0^2\right),
\end{aligned} \tag{13}$$

where (a) follows from the fact that adding side information cannot decrease mutual information; (b) follows from the assumption that X_0^1 and X_0^2 are mutually independent; and (c) follows by writing mutual information in terms of differential entropies. The differential entropy $h\left(R^1_{[0,T\text{-}1]}, Y^1_{[0,T\text{-}1]} | X_0^1, X_0^2\right)$ in (13) can be simplified as

$$\begin{aligned}
&h\left(R^1_{[0,T\text{-}1]}, Y^1_{[0,T\text{-}1]} | X_0^1, X_0^2\right) \\
&= \sum_{t=0}^{T\text{-}1} h\left(R_t^1, Y_t^1 | X_0^1, X_0^2, R^1_{[0,t\text{-}1]}, Y^1_{[0,t\text{-}1]}\right) \\
&\overset{(a)}{=} \sum_{t=0}^{T\text{-}1} h\left(R_t^1, Y_t^1 | X_0^1, X_0^2, R^1_{[0,t\text{-}1]}, Y^1_{[0,t\text{-}1]}, S_t^1, S_t^2\right) \\
&\overset{(b)}{=} \sum_{t=0}^{T\text{-}1} h\left(Z_t^1, Z_t^2 | X_0^1, X_0^2, R^1_{[0,t\text{-}1]}, Y^1_{[0,t\text{-}1]}, S_t^1, S_t^2\right) \\
&\overset{(c)}{=} \sum_{t=0}^{T\text{-}1} h\left(Z_t^1, Z_t^2\right),
\end{aligned} \tag{14}$$

where (a) follows from (8); (b) follows from $R_t^1 = S_t^1 + cS_t^2 + Z_t^1$ and $Y_t^1 = cS_t^1 + Z_t^2$; and (c) follows from the fact that the channels are memoryless. The entropy term $h\left(R^2_{[0,T\text{-}1]} | X_0^2\right)$ in (13) can be expressed as

$$\begin{aligned}
h\left(R^2_{[0,T\text{-}1]} | X_0^2\right) &= \sum_{t=0}^{T\text{-}1} h\left(R_t^2 | X_0^2, R^2_{[0,t\text{-}1]}\right) \\
&\overset{(a)}{=} \sum_{t=0}^{T\text{-}1} h\left(S_t^2 + Y_t^1 | X_0^2, R^2_{[0,t\text{-}1]}, S^2_{[0,t]}\right) \\
&= \sum_{t=0}^{T\text{-}1} h\left(Y_t^1 | X_0^2, R^2_{[0,t\text{-}1]}, S^2_{[0,t]}\right) \\
&\overset{(b)}{=} \sum_{t=0}^{T\text{-}1} h\left(Y_t^1 | X_0^2, R^2_{[0,t\text{-}1]}, S^2_{[0,t]}, Y^1_{[0,t\text{-}1]}\right) \\
&\overset{(c)}{=} \sum_{t=0}^{T\text{-}1} h\left(Y_t^1 | X_0^2, Y^1_{[0,t\text{-}1]}\right) \\
&= h\left(Y^1_{[0,T\text{-}1]} | X_0^2\right),
\end{aligned} \tag{15}$$

where (a) follows from $R_t^2 = S_t^2 + Y_t^1$ and (9); (b) follows from $Y_t^1 = R_t^2 - S_t^2$; and (c) follows from $R_t^2 = S_t^2 + Y_t^1$

and (8). We can now re-write (13) as,

$$
\begin{aligned}
& I\left(X_0^1; R^1_{[0,T-1]}\right) + I\left(X_0^2; R^2_{[0,T-1]}\right) \\
& \overset{(a)}{\leq} h\left(R^1_{[0,T\text{-}1]}, Y^1_{[0,T\text{-}1]} | X_0^2\right) - \sum_{t=0}^{T\text{-}1} h\left(Z_t^1, Z_t^2\right) \\
& \quad + h\left(R^2_{[0,T\text{-}1]}\right) - h\left(Y^1_{[0,T\text{-}1]} | X_0^2\right) \\
& = h\left(R^1_{[0,T\text{-}1]} | Y^1_{[0,T\text{-}1]}, X_0^2\right) + h\left(R^2_{[0,T\text{-}1]}\right) \\
& \quad - \sum_{t=0}^{T\text{-}1} h\left(Z_t^1, Z_t^2\right) \\
& \overset{(b)}{=} h\left(R^1_{[0,T\text{-}1]} | Y^1_{[0,T\text{-}1]}, X_0^2, S^2_{[0,T\text{-}1]}\right) + h\left(R^2_{[0,T\text{-}1]}\right) \\
& \quad - \sum_{t=0}^{T\text{-}1} h\left(Z_t^1, Z_t^2\right) \\
& \overset{(c)}{\leq} \sum_{t=0}^{T-1} \left[h\left(R_t^1 | Y_t^1, S_t^2\right) + h\left(R_t^2\right) - h\left(Z_t^1, Z_t^2\right)\right] \\
& \overset{(d)}{\leq} \sum_{t=0}^{T-1} \left[\frac{1}{2}\log\left(1 + \frac{P\left(1 + c^2 + 2c\rho_t\right)}{N}\right)\right. \\
& \quad \left. + \log\left(\frac{N\left(1-\rho_z^2\right) + P\left(1 + c^2 - 2c\rho_z\right)\left(1-\rho_t^2\right)}{\left(1-\rho_z^2\right)\left(Pc^2\left(1-\rho_t^2\right) + N\right)}\right)\right] \\
& \overset{(e)}{\leq} \sum_{t=0}^{T-1} \max_{0 \leq \rho_t \leq 1} \left[\frac{1}{2}\log\left(1 + \frac{P\left(1 + c^2 + 2c\rho_t\right)}{N}\right)\right. \\
& \quad \left. + \log\left(\frac{N\left(1-\rho_z^2\right) + P\left(1 + c^2 - 2c\rho_z\right)\left(1-\rho_t^2\right)}{\left(1-\rho_z^2\right)\left(Pc^2\left(1-\rho_t^2\right) + N\right)}\right)\right] \\
& = \frac{T}{2} \max_{0 \leq \rho \leq 1} \left\{\log\left(1 + \frac{P\left(1 + c^2 + 2c\rho\right)}{N}\right)\right. \\
& \quad \left. + \log\left(\frac{N\left(1-\rho_z^2\right) + P\left(1 + c^2 - 2c\rho_z\right)\left(1-\rho^2\right)}{\left(1-\rho_z^2\right)\left(Pc^2\left(1-\rho^2\right) + N\right)}\right)\right\},
\end{aligned} \tag{16}
$$

where (a) follows from (14) and (15); (b) follows from (8); (c) follows since conditioning from reduces entropy; and (d) from

$$h\left(Z_t^1, Z_t^2\right) = \frac{1}{2}\log\left((2\pi e)^2 N^2 (1-\rho_z^2)\right), \tag{17}$$

$$h(R_t^2) \leq \frac{1}{2}\log\left(2\pi e\left(N + P\left(1 + c^2 + 2c\rho_t\right)\right)\right), \tag{18}$$

$$
\begin{aligned}
& h\left(R_t^1 | Y_t^1, S_t^2\right) \leq \\
& \frac{1}{2}\log\left(2\pi e \frac{N^2\left(1-\rho_z^2\right) + PN\left(1 + c^2 - 2c\rho_z\right)\left(1-\rho_t^2\right)}{Pc^2\left(1-\rho_t^2\right) + N}\right).
\end{aligned} \tag{19}
$$

where the computation of (17) follows from the assumption that noise variables are Gaussian and the inequality (18) follows from the fact that the Gaussian distribution maximizes differential entropy for a given variance. The inequality (19) is obtained as follows. We first bound the conditional variance of R_t^1 given (Y_t^1, S_t^2) as

$$
\begin{aligned}
& \text{Var}\left[R_t^1 | Y_t^1, S_t^2\right] \leq \text{Var}\left[R_t^1\right] - \\
& \text{Cov}\left[R_t^1, (Y_t^1, S_t^2)\right] \text{Var}^{-1}\left[(Y_t^1, S_t^2)\right] \left(\text{Cov}\left[R_t^1, (Y_t^1, S_t^2)\right]\right)^T \\
& = \frac{N\left(1-\rho_z^2\right) + P\left(1 + c^2 - 2c\rho_z\right)\left(1-\rho_t^2\right)}{\left(1-\rho_z^2\right)\left(Pc^2\left(1-\rho_t^2\right) + N\right)},
\end{aligned} \tag{20}
$$

where the equality is achieved with multivariate Gaussian distribution. Further we know that the Gaussian distribution maximizes entropy for a given variance, which gives the inequality (19). This completes the proof. ■

Remark *3.1:* From (4) we observe that for a fixed $|\rho_z|$, negative noise correlation may provide larger stability regions compared to positive correlation. Moreover from (2) and (3) we observe that the presence of interference might actually be useful in some cases. A similar behavior was observed in [19] under a linear sensing and control scheme.

We now state sufficient conditions for mean-square stability over Gaussian interference channel, which were derived in [19] for scalar plants using a linear delay-free sensing and control scheme.

Theorem *3.2:* [19, Theorem 2] The two scalar LTI systems in (1) with $A_i = \lambda_i$ can be mean square stabilized over the given symmetric Gaussian interference channel if

$$\log(|\lambda_i|) < \frac{1}{2}\log\left(\frac{P\left(1 + c^2 + 2c\rho^\star\right) + N}{Pc^2\left(1 - \rho^{\star 2}\right) + N}\right), \tag{21}$$

where $\rho^\star$ is the largest among all the roots in the interval $[0, 1]$ of the following two fourth order polynomials

$$
\begin{aligned}
f_1(\rho) &:= \rho^4 + a_3\rho^3 + a_2\rho^2 + a_1\rho + a_0, \\
f_2(\rho) &:= \rho^4 + b_3\rho^3 + b_2\rho^2 + b_1\rho + b_0,
\end{aligned} \tag{22}
$$

where

$$
\begin{aligned}
a_3 &= \frac{N}{2cP}, \quad a_2 = -2 - \frac{N(4 + c\rho_z)}{2c^2P}, \\
a_1 &= -\frac{N(1 + 2c^2 + 2c\rho_z)}{2c^3P} - \frac{N^2}{c^3P^2}, \\
a_0 &= 1 + \frac{N(2c - \rho_z)}{2c^3P}, \quad b_3 = \frac{2c^2P + 2P + N}{2cP}, \\
b_2 &= \frac{N\rho_z}{2cP}, \quad b_1 = -\frac{(1 + c^2)}{c} - \frac{N(1 + 2\rho_z - 2c^2)}{2c^3P}, \\
b_0 &= -1 - \frac{N(2c - \rho_z)}{2c^3P}.
\end{aligned}
$$

In the following section we analyze the gap between the necessary and the sufficient conditions given in Theorem 3.1 and Theorem 3.2 respectively. This comparison can tell us how tight the necessary conditions are.

IV. Optimality of Linear Scheme

The problem at hand has a non-classical information structure, where there are four decision makers (two sensors and two controllers). It is well-known that for such problems linear policies are not optimal in general. A popular example is the Witsenhausen problem [25] where there are two decision makers and linear policies are shown to be sub-optimal.

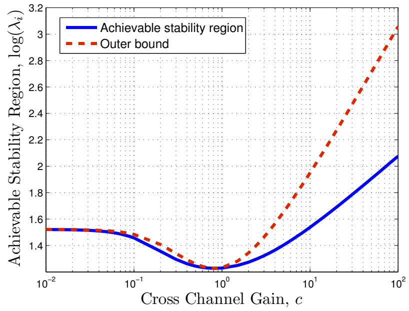

Fig. 2: $\rho_z = 0, P = 20, N = 1$.

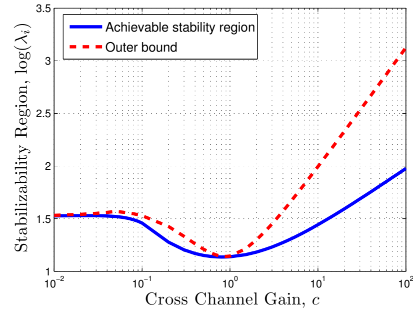

Fig. 3: $\rho_z = 0.5, P = 20, N = 1$.

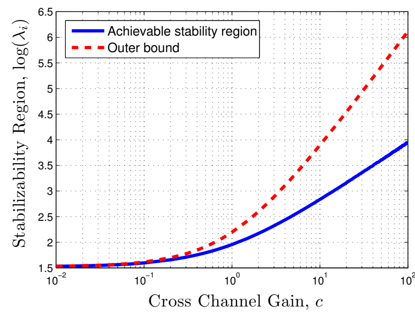

Fig. 4: $\rho_z = -1, P = 20, N = 1$.

In the following we give two special cases where linear policies are actually optimal for mean-square stabilization over the Gaussian interference channel.

Case 1: If the two noise components are fully correlated i.e., $\rho_z = 1$, and further $2c(1 + \frac{c^2 P}{N}) < 1$, then according to [19, Remark 2] the sufficient conditions for mean-square stability are given by

$$\log(|\lambda_i|) < \frac{1}{2}\log\left(1 + \frac{P(1+c)^2}{N}\right), \quad i = 1, 2,$$

which coincide with the necessary conditions given in (2) and (3). Thus the linear scheme proposed in [19] is optimal for this special case.

Case 2: If the two noise components are fully correlated or anti-correlated i.e., $\rho_z = \pm 1$ and further the initial states are fully correlated, then according to [19, Remark 1] the sufficient conditions for mean-square stability are given by

$$\log(|\lambda_i|) < \frac{1}{2}\log\left(1 + \frac{P(1+c)^2}{N}\right), \quad i = 1, 2,$$

which coincide with the necessary conditions given in (2) and (3). Thus the linear scheme proposed in [19] is also optimal for this special case.

In order to investigate further, we numerically evaluate the achievable stability region using Theorem 3.2 and the corresponding outer bound using Theorem 3.1. In Fig. 2-4 we fix $P = 20$, $N = 1$ and plot the achievable stability regions and outer bounds as functions of increasing cross-channel interference c for $\rho_z = 0, 0.5$, and -1. These curves are basically illustrating lower and upper bounds on stability region. We can observe that linear schemes can be quite close to optimal in low to moderate interference regimes, however the gap becomes large in the presence of very strong interference. Whether this increasing gap shows inefficiency of linear schemes or looseness of the outer bound in a high interference region will be a subject of our future research.

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Appendix I
Necessary Condition

The proof essentially follows from the same steps as in Theorem 4.1 of [**?**], however, with some differences. Consider the following series of equalities:

$$\begin{aligned}
& I\left(X^i_{[0,T\text{-}1]} \to R^i_{[0,T\text{-}1]}\right) \overset{(a)}{=} \sum_{t=0}^{T-1} I\left(X^i_t; R^i_t | R^i_{[0,t\text{-}1]}\right) \\
& = I\left(X^i_0; R^i_0\right) + \sum_{t=1}^{T-1} I\left(X^i_t; R^i_t | R^i_{[0,t\text{-}1]}\right) \\
& \overset{(b)}{=} \; I\left(X^i_0; R^i_0\right) + \sum_{t=1}^{T-1}\left(h\left(X^i_t | R^i_{[0,t\text{-}1]}\right) - h\left(X^i_t | R^i_{[0,t]}\right)\right) \\
& \overset{(c)}{=} \sum_{t=1}^{T-1}\left(h\left(A_i X^i_{t\text{-}1} + U^i_{t\text{-}1} | R^i_{[0,t\text{-}1]}\right) - h\left(X^i_t | R^i_{[0,t]}\right)\right) \\
& \quad + I\left(X^i_0; R^i_0\right) \\
& \overset{(d)}{=} \sum_{t=1}^{T-1}\left(\log\left(|\det\left(A_i\right)|\right) + h\left(X^i_{t\text{-}1} | R^i_{[0,t\text{-}1]}\right) - h\left(X^i_t | R^i_{[0,t]}\right)\right) \\
& \quad + I\left(X^i_0; R^i_0\right) \\
& = T\log\left(|\det\left(A_i\right)|\right) + h\left(X^i_0 | R^i_0\right) - h\left(X^i_{T\text{-}1} | R^i_{[0,T\text{-}1]}\right) \\
& \quad + I\left(X^i_0; R^i_0\right) \\
& = h\left(X^i_0\right) + (T\text{-}1)\log\left(|\det\left(A_i\right)|\right) - h\left(X^i_{T\text{-}1} | R^i_{[0,T\text{-}1]}\right),
\end{aligned} \tag{23}$$

where (a) follows from the definition of directed information [20]; (b) follows by writing mutual information in terms of differential entropies; (c) follows from (1); (d) follows from the fact that for a matrix A and a random variable X, we have $h\left(AX\right) = h(X) + \log\left(\left|\det\left(A\right)\right|\right)$ [21, Theorem 8.6.4, (8.71)]; Using (23) the directed information rate is given by

$$\begin{aligned}
&\liminf_{T\to\infty}\frac{1}{T}I\left(X^i_{[0,T\text{-}1]}\to R^i_{[0,T\text{-}1]}\right)\\
&= \log\left(\left|\det\left(A_i\right)\right|\right)\\
&\quad+\liminf_{T\to\infty}\frac{1}{T}\left(h\left(X^i_0\right)-h\left(X^i_{T\text{-}1}|R^i_{[0,T\text{-}1]}\right)\right)\\
&\overset{(a)}{=} \log\left(\left|\det\left(A_i\right)\right|\right)-\limsup_{T\to\infty}\frac{1}{T}h\left(X^i_{T\text{-}1}|R^i_{[0,T\text{-}1]}\right)\\
&\overset{(b)}{\geq} \log\left(\left|\det\left(A_i\right)\right|\right)-\limsup_{T\to\infty}\frac{1}{T}\log\left((2\pi e)^n|K|\right)\\
&= \log\left(\left|\det\left(A_i\right)\right|\right),
\end{aligned}$$

where the inequality (a) follows from $h\left(X^i_0\right)<\infty$; and (b) follows from the fact that for a mean square stable system there exists a matrix $K\succ 0$ with $\Lambda_t\preceq K$ for all t. Further we know that for a given covariance matrix K the differential entropy is maximized by the Gaussian distribution.

Appendix II
Proof of Lemma 3.2

Consider the following series of equalities

$$\begin{aligned}
X^i_{t+1} &\overset{(a)}{=} A_iX^i_t+U^i_t \overset{(b)}{=} A_i^{t+1}X^i_0+\sum_{k=0}^{t}A^{t-k}U^i_k\\
&= A_i^{t+1}X^i_0+\sum_{k=0}^{t}A^k\pi^i_k\left(R^i_{[0,k]}\right) &(24)\\
&= A_i^{t+1}X^i_0+\sum_{k=0}^{t}A^k\pi^i_k\left(S^i_{[0,k]}+Y^i_{[0,k]}\right), &(25)
\end{aligned}$$

where (a) follows from (1); (b) follows by recursively applying (a); (24) follows from $U^i_t=\pi^i_t\left(R^i_{[0,t]}\right)$; and (25) follows from $R^i_t=S^i_t+hS^j_t+Z^i_t$ and $Y^i_t=hS^j_t+Z^j_t$ for $i\neq j$. From (24) we see that $X^i_t=A^t_iX^i_0+\mu^i_t\left(R^i_{[0,t\text{-}1]}\right)$, where $\mu^i_t:\mathbb{R}^{t\text{-}1}\to\mathbb{R}$. Since $S^i_t=f^i_t\left(X^i_{[0,t]}\right)$ and $X^i_t=A^t_iX^i_0+\mu^i_t\left(R^i_{[0,t\text{-}1]}\right)$, we have $S^i_t=\upsilon^i_t\left(X^i_0,R^i_{[0,t\text{-}1]}\right)$ where $\upsilon^i_t:\mathbb{R}^t\to\mathbb{R}$. Moreover S^i_t can also be written as

$$\begin{aligned}
S^i_t = f^i_t\left(X^i_{[0,t]}\right) &\overset{(a)}{=} g^i_t\left(X^i_0,S^i_{[0,t-1]},Y^i_{[0,t-1]}\right)\\
&\overset{(b)}{=} \nu^i_t\left(X^i_0,Y^i_{[0,t-1]}\right), &(26)
\end{aligned}$$

where (a) follows from (25) and by defining $g^i_t:\mathbb{R}^{2t\text{-}1}\to\mathbb{R}$; and (b) follows from recursively applying (a) and by defining $\nu^i_t:\mathbb{R}^t\to\mathbb{R}$.

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