

The Linear Quadratic Regulator with Chance Constraints

Georg Schildbach,* Paul Goulart,* and Manfred Morari*

Abstract—This paper is concerned with the design of linear state feedback control laws for linear systems with additive Gaussian disturbances. The objective is to find the feedback gain that minimizes a quadratic cost function in closed-loop operation, while observing chance constraints on the input and/or the state. It is shown that this problem can be cast as a semi-definite program (SDP), in which the chance constraints appear as linear or bilinear matrix inequalities. Both individual chance constraints (ICCs) and joint chance constraints (JCCs) can be considered. In the case of ICCs only, the resulting SDP is linear and can be solved efficiently as a convex optimization program. In the presence of JCCs the SDP becomes bilinear, however it can still be solved efficiently by an iterative algorithm, at least to a local optimum. The application of the method is demonstrated for several numerical examples, underscoring its flexibility and ease of implementation.

I. INTRODUCTION

Chance constraints, i.e. constraints that must be satisfied with at least some probability, are a natural notion for systems that are subject to unbounded disturbances, e.g. following a Gaussian distribution [3], [4]. They also arise in many practical applications, such as process control [1] or building control [2].

The appropriate handling of chance constraints in control systems, however, remains a major challenge. If the design problem is restricted to the class of (stabilizing) linear state feedback controllers and additive Gaussian disturbances, then the stationary distribution of the system's state and input can be computed explicitly. For this stationary operation condition, finding the controller that minimizes the expectation of a quadratic stage cost can be formulated as an Linear Quadratic Regulator (LQR) problem, which has been studied extensively in the past. This paper is concerned with the same problem, under additional chance constraints on the input and/or the state that limit the choice of feasible linear controllers.

Individual chance constraints (ICCs), where the feasible region is a half-space or the intersection of two parallel half-spaces, and joint chance constraints (JCCs), where the feasible region is an arbitrary intersection of half-spaces, are both addressed. The chance constraints, as well as the stationarity condition and the cost function, are reformulated in terms of linear (or bilinear) matrix inequalities (LMIs, BMIs). Hence the synthesis problem can be solved as a semi-definite program (SDP) [5] by means of convex optimization [6].

Chance-constrained control approaches are not new, and have been considered mainly in the field of stochastic model

predictive control (SMPC), e.g. [1], [7]. However, SMPC requires heavy online computations and fails to provide guarantees for the asymptotic behavior of the closed-loop system (at least for unbounded disturbances). The presented method is similar to that of [8], where a chance-constrained linear feedback controller is designed using Chebyshev bounds. However, [8] considers finite-horizon control problems, while the approach of this paper concerns the stationary operation of a controller, so that its properties hold asymptotically in time.

The paper is organized as follows. Section II outlines the control problem, including the assumptions about the linear system, the cost function, and the chance constraints. The stationary distribution induced by a linear controller exists, if the controller is stabilizing, and can be computed explicitly through a Lyapunov equation, which can be posed as a LMI. Section III shows how this distribution can be shaped according to chance constraints on the states/input. Chebyshev-type bounds are used to reformulate ICCs (JCCs) as a system of LMIs (BMIs). Section IV discusses the problem of minimizing the expected quadratic stage cost evaluated on the stationary state distribution, subject to chance constraints. Section V applies the theory to numerical examples, showing some of the advantages and shortcomings of the method. Section VI presents the main conclusions.

II. PROBLEM STATEMENT

A. Control System

Consider a linear discrete-time control system

$$x^+ = Ax + Bu + Ew, \quad (1)$$

where $x^+ \in \mathbb{R}^n$ denotes the *state* subsequent to $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ denotes the *control input*, and $w \in \mathbb{R}^p$ denotes an additive exogenous input. The system matrices A, B, E are of appropriate dimensions.

Assumption 1 (Control System) (a) *The matrix pair (A, B) is stabilizable.* (b) *The matrix pair (A, E) is controllable.* (c) *Full state information is available for feedback design.* (d) *The state disturbances are independent and identically distributed (i.i.d.) random variables.* (e) *Each disturbance w follows a multi-variate normal distribution with mean $\mathbf{E}[w] = 0$ and variance $\mathbf{E}[ww^T] = W$; in short $w \sim \mathcal{N}(0, W)$.*

A basic control objective is to find a *linear feedback gain* $K \in \mathbb{R}^{m \times n}$ such that $u = Kx$ stabilizes the system about the origin. If this holds, then the system trajectory $\{x_k\}_{k \in \mathbb{N}}$ converges to a *stationary distribution* that is normal and does not depend on x_0 (Section II-B).

*Automatic Control Laboratory, Swiss Federal Institute of Technology, Zurich, Switzerland.
schildbach|pgoulart|morari@control.ee.ethz.ch

A further objective is to select K such that the stationary state x and input $u = Kx$ observe linear chance constraints (Section II-C). From all K that satisfy this condition, the one that minimizes the expected value of a quadratic cost function should be selected (Section II-D).

Convergence of the state trajectory to a stationary distribution (for any initial condition x_0) indeed means that the chance constraints are satisfied, and the target function minimized, asymptotically in time; see [9, Cha. 17].

B. Stationary Distribution

Lemma 2 (Stationarity Condition) *Assume that K is chosen such that $A + BK$ is strictly stable. Then the state trajectory of the closed-loop system*

$$x^+ = (A + BK)x + Ew \quad (2)$$

converges to a normal stationary distribution $\mathcal{N}(0, S)$, whose (positive definite) covariance $\mathbf{E}[xx^T] = S \succ 0$ satisfies the discrete Lyapunov equation

$$(A + BK)S(A + BK)^T + EWE^T - S = 0. \quad (3)$$

Proof: Convergence to a normal distribution with $\mathbf{E}[x] = 0$ is straightforward from the previous assumptions; a rigorous proof can be found in [9, Thm. 17.6.2].

As the distribution is preserved between two subsequent steps, in particular its second moment is preserved:

$$\mathbf{E}[(x^+)(x^+)^T] = \mathbf{E}[xx^T].$$

Substituting (2) and using the independence of w and x (where the latter depends only on past disturbances) yields

$$\begin{aligned} \mathbf{E}[\left((A + BK)x + Ew\right)\left((A + BK)x + Ew\right)^T] &= \mathbf{E}[xx^T], \\ \mathbf{E}[(A + BK)xx^T(A + BK)^T] + \mathbf{E}[Eww^TE^T] &= \mathbf{E}[xx^T]. \end{aligned}$$

The Lyapunov equation (3) follows immediately from this condition. By Assumption 1(b), S is positive definite ($S \succ 0$), and not just positive semi-definite ($S \succeq 0$). ■

Observe that the stationary state distribution being normal $\mathcal{N}(0, S)$, with $S \succ 0$, implies that any constraints on the state x in \mathbb{R}^n can not be satisfied with probability one. Similarly, constraints on the input $u = Kx$ in \mathbb{R}^m either have to be trivial, i.e. meaning that certain input channels are never used, or they can not be satisfied with probability one. Therefore, the constraints introduced in the next section are interpreted as probabilistic, or *chance constraints*.

C. Chance Constraint

The state trajectory should satisfy the chance constraints

$$\mathbf{P}[Gx \leq h] \geq (1 - \varepsilon_x) \quad (4)$$

asymptotically in time. In other words, in stationary operation of the regulator, the state should fall outside of the *constraint set* $[Gx \leq h] \subset \mathbb{R}^n$ with a *probability level* of at most $\varepsilon_x \in (0, 1)$. If G is a vector and h is a scalar, i.e. the constraint set is a half space, then (4) is called an *individual chance constraint* (ICC). If G is a matrix and h is a vector,

i.e. the constraint set is a polyhedron (' \leq ' is interpreted element-wise), then (4) is a *joint chance constraint* (JCC).

Analogously, chance constraints on the inputs

$$\mathbf{P}[Fu \leq g] \geq (1 - \varepsilon_u), \quad (5)$$

can be considered. This means that in stationary operation of the regulator, the input $u = Kx$ should violate $Fu \leq g$ with a probability of at most $\varepsilon_u \in (0, 1)$. Again, F and g can be vector and scalar (ICC), or matrix and vector (JCC).

Remark 3 (Constraints) (a) *By symmetry of the stationary distribution, two-sided bounds of the type $|Gx| \leq h$ and $|Fu| \leq g$, where F and G are vectors, can also be handled as ICCs.* (b) *Any combination of multiple state constraints and/or input constraints can be handled using the methods presented, but are excluded for notation simplicity.*

D. Target Function

The previous section has introduced the set of *feasible* linear feedback gains K , i.e. those stabilizing the system and satisfying the chance constraints (4), (5). Suppose that multiple choices for K are feasible, then the *optimal* K is defined as the one that minimizes the quadratic stage cost

$$\min \mathbf{E}[x^T Qx + u^T Ru] \quad (6)$$

in stationary operation. As for the chance constraints, the target function (6) is hence minimized asymptotically in time, rather than considering the transient response from an initial condition x_0 (as for an LQR).

Assumption 4 (Target Function) (a) *The state weight $Q \succeq 0$ can be decomposed into $Q = C^T C$, where the matrix pair (A, C) is observable.* (b) *The input weight is $R \succ 0$, and thus it can be decomposed into $R = D^T D$ with $D \succ 0$.*

III. MINIMUM PROBABILITY LEVELS

From (3) it is clear that $S \succ 0$ must be lower bounded, at least by $S \succeq EWE^T$. Therefore, in general it is not possible to make the probability level ε_x or ε_u of a given chance constraint arbitrarily small.

Consider the case of a single ICC or JCC on the state (or input). By monotonic feasibility for increasing probability levels, there must exist some lower bound $\varepsilon_{lb} \geq 0$ such that a feasible gain K can be found for all $\varepsilon_x > \varepsilon_{lb}$ (or $\varepsilon_u > \varepsilon_{lb}$) and none for $\varepsilon_x < \varepsilon_{lb}$ (or $\varepsilon_u < \varepsilon_{lb}$). The goal of this section is to determine a (good) upper bound $\bar{\varepsilon}$ for ε_{lb} .

First, the cases of an ICC (in III-A) or a JCC (in III-B) are considered separately. Then, the presence of multiple constraints in the design problem is discussed (in III-C).

A. Minimum Probability Level for ICCs

Theorem 5 (Lowest Probability Level for ICC)

(a) *Consider a single ICC on the state. The solution $\bar{\varepsilon}$ to*

the semi-definite program

$$\min_{S, V, \bar{\varepsilon}} \bar{\varepsilon} \quad (7a)$$

$$\text{s.t.} \quad \begin{bmatrix} S - EWE^T & (AS + BV) \\ (AS + BV)^T & S \end{bmatrix} \succeq 0, \quad (7b)$$

$$\frac{1}{6h^2} GSG^T \leq \bar{\varepsilon}, \quad (7c)$$

$$S \succ 0, \quad (7d)$$

represents a feasible probability level for this ICC; it is achieved by $K = VS^{-1}$, where S, V are optimal in (7).

(b) Consider a single ICC on the input. The solution $\bar{\varepsilon}$ to the semi-definite program

$$\min_{S, V, \bar{\varepsilon}} \bar{\varepsilon} \quad (8a)$$

$$\text{s.t.} \quad \begin{bmatrix} S - EWE^T & (AS + BV) \\ (AS + BV)^T & S \end{bmatrix} \succeq 0, \quad (8b)$$

$$\begin{bmatrix} (6g^2)\bar{\varepsilon} & (FV) \\ (FV)^T & S \end{bmatrix} \succeq 0, \quad (8c)$$

$$S \succ 0, \quad (8d)$$

represents a feasible probability level for this ICC; it is achieved by $K = VS^{-1}$, where S, V are optimal in (8).

Proof: The stationarity condition (3) can be reformulated as constraint (7b) and (8b):

$$S - (A + BK)S(A + BK)^T - EWE^T \succeq 0$$

$$\Downarrow$$

$$\begin{bmatrix} S - EWE^T & (A + BK) \\ (A + BK)^T & S^{-1} \end{bmatrix} \succeq 0$$

$$\Downarrow$$

$$\begin{bmatrix} S - EWE^T & AS + BKS \\ (AS + BKS)^T & S \end{bmatrix} \succeq 0.$$

In the first line, (3) has been relaxed to an inequality; minimization of the probability level ensures that S is not chosen larger than required. The second line is based on a Schur complement reformulation [6, Sec. 2.1]. The third line is obtained by left and right multiplication with

$$\begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix} \succ 0,$$

where I is the identity matrix. Constraint (7b), (8b) follows with the variable substitution $V := KS$. This also explains how $K = VS^{-1}$ is retrieved from the solution.

The ICC can be approximated by a tail bound on the uni-variate distribution of Gx or FKx , respectively. As $x \sim \mathcal{N}(0, S)$, these distributions are normal with variance $\mathcal{N}(0, GSG^T)$ and $\mathcal{N}(0, FKS^T F^T)$, respectively. Hence it is possible to apply the Chebyshev-type inequality from [10] for normal distributions.

In case (a) for state constraints this yields

$$\mathbf{P}[|Gx| > h] \leq \frac{GSG^T}{3h^2}, \quad (9)$$

or by exploiting the symmetry of the distribution

$$\mathbf{P}[Gx > h] \leq \frac{GSG^T}{6h^2}. \quad (10)$$

The right-hand side should not be larger than $\bar{\varepsilon}$, giving (7c).

In case (b) for input constraints, the same argument yields

$$\frac{FKSK^T F^T}{6g^2} \leq \bar{\varepsilon}.$$

With the substitution of $V := KS$ from above,

$$FVS^{-1}V^T F^T \leq (6g^2)\bar{\varepsilon},$$

which is equivalent to (8c) by [6, Sec. 2.1]. \blacksquare

Both SDPs (7) and (8) are convex optimization programs that can be solved efficiently [5] using standard software packages, e.g. [11].

B. Minimum Probability Level for JCCs

Let $G^{(j)}, F^{(j)}$ denote the j -th row of the matrices G, F and $h^{(j)}, g^{(j)}$ denote the j -th element of the vectors h, g .

Theorem 6 (Lowest Probability Level for JCC)

(a) Consider a single JCC on the state. The solution $\bar{\varepsilon}$ to the semi-definite program

$$\min_{S, V, P, q, r, \tau, \bar{\varepsilon}} \bar{\varepsilon}, \quad (11a)$$

$$\text{s.t.} \quad \begin{bmatrix} S - EWE^T & (AS + BV) \\ (AS + BV)^T & S \end{bmatrix} \succeq 0, \quad (11b)$$

$$\begin{bmatrix} P & q \\ q^T & r \end{bmatrix} \succeq 0, \quad (11c)$$

$$\begin{bmatrix} P & q \\ q^T & r - 1 \end{bmatrix} \succeq \tau^{(j)} \begin{bmatrix} 0 & G^{(j)T}/2 \\ G^{(j)}/2 & -h^{(j)} \end{bmatrix}, \quad (11d)$$

$$\tau^{(j)} \geq 0 \quad \forall j, \quad (11e)$$

$$\text{Tr}(PS) + r \leq \bar{\varepsilon} \quad (11f)$$

$$S \succ 0, \quad (11g)$$

represents feasible probability level for this JCC; it is achieved by $K = VS^{-1}$, where S, V are optimal in (11).

(b) Consider a single JCC on the input. The solution $\bar{\varepsilon}$ to the semi-definite program

$$\min_{S, V, Y, P, q, r, \tau, \bar{\varepsilon}} \bar{\varepsilon}, \quad (12a)$$

$$\text{s.t.} \quad \begin{bmatrix} S - EWE^T & (AS + BV) \\ (AS + BV)^T & S \end{bmatrix} \succeq 0, \quad (12b)$$

$$\begin{bmatrix} P & q \\ q^T & r \end{bmatrix} \succeq 0, \quad (12c)$$

$$\begin{bmatrix} P & q \\ q^T & r - 1 \end{bmatrix} \succeq \tau^{(j)} \begin{bmatrix} 0 & F^{(j)T}/2 \\ F^{(j)}/2 & -g^{(j)} \end{bmatrix}, \quad (12d)$$

$$\tau^{(j)} \geq 0 \quad \forall j, \quad (12e)$$

$$\begin{bmatrix} Y & (P^{\frac{1}{2}}V) \\ (P^{\frac{1}{2}}V)^T & S \end{bmatrix} \succeq 0, \quad (12f)$$

$$\text{Tr}(Y) + r \leq \bar{\varepsilon}, \quad (12g)$$

$$S \succ 0, \quad (12h)$$

represents feasible probability level for this JCC; it is achieved by $K = VS^{-1}$, where S, V are optimal in (12).

Proof: The stationarity condition (11b), (12b) has already been derived in the proof of Theorem 5. The major difference to the ICC lies in the Chebyshev-type inequality for polyhedra, which is taken from [12, Sec. 7.4.1].

Define the indicator function $\mathbf{1}_C : \mathbb{R}^n \rightarrow \{0,1\}$ on the complement \mathcal{C} of the constraint set $\{x \in \mathbb{R}^n : Gx \leq h\}$ (or $\{x \in \mathbb{R}^n : FKx \leq g\}$, analogously) as

$$\mathbf{1}_C(x) = \begin{cases} 1 & \text{if } Gx > h \\ 0 & \text{if } Gx \leq h \end{cases} . \quad (13)$$

Then the JCC (4) (or (5), correspondingly) can be equivalently expressed as

$$\mathbf{E}[\mathbf{1}_C(x)] \leq \varepsilon_x , \quad \text{for } x \sim \mathcal{N}(0, S) .$$

Note that by $S \succ 0$ the boundary of the constraint set has probability measure zero, and hence equality $Gx = h$ can be interchanged between the cases of (13).

Any non-negative function $f : \mathbb{R}^n \rightarrow \mathbb{R}_{0+}$ satisfying $f(x) \geq \mathbf{1}_C(x)$ for all $x \in \mathbb{R}^n$ can be used to establish an upper bound on the violation probability, since

$$\mathbf{E}[f(x)] \geq \mathbf{E}[\mathbf{1}_C(x)] . \quad (14)$$

A particular approach is to pick f as the quadratic function

$$f(x) = x^T P x + 2q^T x + r , \quad P \succ 0 , \quad (15)$$

and to solve for the values for P, q, r in order to make the bound (14) as tight as possible. Clearly, finding the best bound becomes part of problem (11) (or (12), respectively).

Non-negativity of f is guaranteed by constraint (11c), while (11d,e) ensure that $f(x) \geq \mathbf{1}_C(x)$ through the S-procedure [12, Sec. B.2]. The same holds for (12c,d,e).

In the case of (11), the (approximate) violation probability

$$\begin{aligned} \mathbf{E}[f(x)] &= \mathbf{E}[x^T P x + 2q^T x + r] \\ &= \mathbf{E}[\text{Tr}(x^T P x)] + \mathbf{E}[2q^T x] + r \\ &= \text{Tr}(P S) + r , \end{aligned}$$

yields constraint (11f). In the case of (12), the (approximate) violation probability

$$\begin{aligned} \mathbf{E}[f(x)] &= \text{Tr}(P K S K^T) + r \\ &= \text{Tr}(P^{\frac{1}{2}} Y S^{-1} Y^T P^{\frac{1}{2}}) + r , \end{aligned}$$

is reformulated as (12f,g) by introducing the auxiliary variable Y [6, Sec. 2.1]. ■

Both SDPs (11) and (12) are non-convex optimization programs as they are bilinear in the decision variables in (11f) and (12f), respectively. These result from the simultaneous solution for the feedback-related variables S, V , and the parameters of the Chebyshev bound Y, P, q, r, τ .

However, if either of the two sets of variables is fixed, then the SDPs are convex optimization programs and hence can be solved efficiently, e.g. by [11]. A straightforward solution procedure is to iterate between these two, i.e. to solve in an alternating fashion for the feedback variables S, V (while Y, P, q, r, τ are fixed) and the Chebyshev parameters Y, P, q, r, τ (while S, V are fixed). In all practical cases considered for this paper, rapid convergence of this procedure was observed.

C. Combination of Multiple Chance Constraints

Theorems 5 and 6 provide a lower bound $\bar{\varepsilon}$ for the choice of the probability level $\varepsilon_x > \bar{\varepsilon}$ (or $\varepsilon_u > \bar{\varepsilon}$) for a single ICC and JCC, respectively. It is also possible to satisfy a combination of ICCs and JCCs on the state and/or input.

However, the presence of multiple constraints may adversely affect each other. For example, if the input of a system is constrained, this may cause an increase in the lowest probability level that is achievable for a state constraint. Therefore, the minimum probability levels of all inputs and/or states are generally dependent upon each other.

One way to find a feasible combination is to order the constraints by priority. As a first step, one solves for the lower bound $\bar{\varepsilon}$ of the highest priority constraint and selects any probability level greater than this bound. As a second step, one solves for the lower bound $\bar{\varepsilon}$ of the second highest priority constraint, including the highest priority constraint as an *additional LMI* in the problem. Then one proceeds in this manner along a decreasing order of the constraints.

IV. THE CHANCE-CONSTRAINED LQR PROBLEM

A. LQR Optimal Feedback Gain for Single Constraints

Suppose that the desired probability level ε_x (or ε_u) is chosen greater than its lower bound $\bar{\varepsilon}$ from Theorem 5, for an ICC, or Theorem 6, for a JCC. This means that some flexibility in the choice of the stabilizing feedback gain K , satisfying the state (or input) constraint, can be expected.

The following result shows how to find the best feedback gain K , according to the quadratic cost (6) and subject to this chance constraint. It requires but a small reformulation of the previous SDPs (7), (8), (11), and (12), respectively, regarding the target function. The reformulation is identical in all four cases.

Theorem 7 (Optimal Constrained Feedback) *Let $\varepsilon_x > \bar{\varepsilon}$ (or $\varepsilon_u > \bar{\varepsilon}$) be the chosen probability levels, substituted into the constraints (7c) (or (8c)) in case of an ICC, or constraints (11f) (or (12g)) in case of a JCC. Moreover, substitute the reformulation of the target function (6),*

$$\min_{S, V, Z} \quad \text{Tr}(Q S) + \text{Tr}(Z) \quad (16a)$$

$$\text{s.t.} \quad \begin{bmatrix} Z & (D V) \\ (D V)^T & S \end{bmatrix} \succeq 0 , \quad (16b)$$

into the corresponding semi-definite program (7), (8), (11), or (12). Then the LQR optimal feedback gain K (subject to the chance constraint) is obtained from the resulting SDP solution via $K = V S^{-1}$.

Proof: The only part requiring proof is the reformulation of the target function (6) into (16):

$$\begin{aligned} \mathbf{E}[x^T Q x + u^T R u] &= \mathbf{E}[\text{Tr}(x^T Q x)] + \mathbf{E}[\text{Tr}(x^T K^T D^T D K x)] \\ &= \text{Tr}(Q S) + \text{Tr}(K^T D^T D K S) \\ &= \text{Tr}(Q S) + \text{Tr}(D K S S^{-1} S^T K^T D^T) \\ &= \text{Tr}(Q S) + \text{Tr}(D V S^{-1} V^T D^T) . \end{aligned}$$

Minimizing the second term of the last line is equivalent to

$$\begin{aligned} \min_Z \quad & \text{Tr}(Z) \\ \text{s.t.} \quad & \begin{bmatrix} Z & (DV) \\ (DV)^T & S \end{bmatrix} \succeq 0, \end{aligned}$$

for an auxiliary variable $Z \succeq 0$, see [6, Sec. 2.1]. ■

The SDPs with the target function reformulation of Theorem 7 remain conceptually the same as those solving for the minimum violation probabilities (Section III). Hence the same computation methods apply; i.e. for an ICC the constrained LQR problem can be solved as a convex optimization program (as mentioned in III-A), and for a JCC it can be solved by iterating between the feedback variables S, V, Z and the parameters of the Chebyshev bound Y, P, q, r, τ (as described in III-B).

B. LQR Optimal Feedback Gain for Multiple Constraints

Again, for the constrained LQR-optimal feedback gain K it is possible to satisfy a combination of ICCs and JCCs on the state and/or input simultaneously, by adding further LMIs to the problem of a single constraint (as in Theorem 7).

Indeed, suppose the probability levels of all constraints allow for a feasible choice of K ; e.g. they are selected by the procedure described in III-C. For each ICC on the state (or input), one adds (7c) for the selected ε_x (or (8c) for the selected ε_u) to the problem in Theorem 7. Similarly, for each JCC on the state (or input), one adds (11c-f) for the selected ε_x (or (11c-g) for the selected ε_u) to this problem.

V. EXAMPLES

A. System with Single State Constraint

As a first example, consider the $n = 2$ state system

$$A = \begin{bmatrix} 1.5 & 0.6 \\ 0 & 1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad (17)$$

with the quadratic target function defined by

$$Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad R = 1. \quad (18)$$

The disturbance is normally distributed, with unit variance $W = I$. A single ICC on the first state shall be observed,

$$G = [1 \ 0], \quad h = 1.5, \quad (19)$$

for varying probability levels ε_x . The lower bound on ε_x , obtained through Theorem 5(a), is $\bar{\varepsilon}_x = 4.89\%$. On the other hand, the empirical violation rate of the system under unconstrained ('uc') LQR feedback amounts to $\tilde{\varepsilon}_x^{\text{uc}} = 11.84\%$. Note that all *empirical violation rates* stem from Monte-Carlo simulations, and are indicated by a tilde.

Table I opposes different probability levels ε_x to their corresponding empirical violation rates $\tilde{\varepsilon}_x$. Notice that there is some *conservatism* (i.e. a difference between ε_x and $\tilde{\varepsilon}_x$) in the design, which decreases with increasing probability levels ε_x . It is caused by the Chebyshev inequality (9), giving a bound that is not tight.

TABLE I: Empirical violation rates $\tilde{\varepsilon}_x$ of constrained LQR.

ε_x [in %]	5.00	6.00	7.00	8.00	9.00	10.00	11.00
$\tilde{\varepsilon}_x$ [in %]	3.41	4.76	6.13	7.47	8.65	9.84	10.91

Figure 1 depicts two phase plots for (a) the unconstrained (conventional) LQR and (b) the constrained LQR, designed for $\varepsilon_x = 5\%$. The chance constraint (19) is shown as a dashed line. Notice that the optimal distribution is re-shaped by the presence of the constraint.

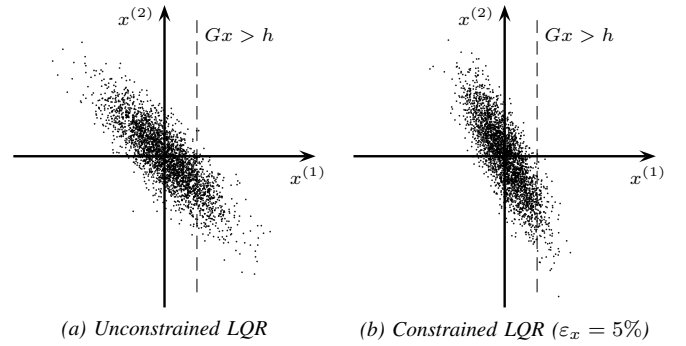


Fig. 1: Phase plot of closed-loop trajectories.

B. System with Both Input and State Constraints

Consider again system (17), (18), (19), where now an additional input constraint is to be included. As the feedback gain $K \in \mathbb{R}^2$ features only two decision variables, the problem with two constraints and a target function would be overdetermined. Hence a second input is added, by changing the system data as follows:

$$B = \begin{bmatrix} 0 & 1 \\ 0.2 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (20)$$

The newly introduced input channel shall be kept small,

$$\mathbf{P}[|u^{(2)}| \leq 1.5] \geq (1 - \varepsilon_u). \quad (21)$$

Recall that (21) can be handled as an ICC, according to Remark 3(a). It is intuitive that the lower bound $\bar{\varepsilon}_u$ is 0%, due to controllability by the first input channel. So any probability level works, e.g. $\varepsilon_u = 10\%$.

The second input channel with constraint (21) is now included in the problem of finding the lower bound for ε_x of the state constraint (19) (compare Section III-C). Theorem 5(a) yields $\bar{\varepsilon}_x = 3.68\%$, down from $\bar{\varepsilon}_x = 4.89\%$ in Section V-A. So the previous probability level of $\varepsilon_x = 5\%$ can be maintained.

The empirical violation rates become $\tilde{\varepsilon}_u = 6.81\%$ and $\tilde{\varepsilon}_x = 3.39\%$, as opposed to $\tilde{\varepsilon}_u^{\text{uc}} = 15.08\%$ and $\tilde{\varepsilon}_x^{\text{uc}} = 7.42\%$ for the unconstrained LQR.

C. Attitude Control of a Spinning Satellite

Consider the model of a spinning satellite [13, Sec. 10.2], as sketched in Figure 2. It consists of two rotating masses: the first mass with inertia $J_1 = 1$ represents the satellite body with thrusters, and the second mass with inertia $J_2 = 0.1$

carries the instruments. The two masses are connected by a boom of low stiffness $k = 0.02$ and damping $b = 0.0001$. Disturbing forces act on both masses, while a torque on the satellite body can be applied for control. The objective is to maintain the instruments in a stable position.

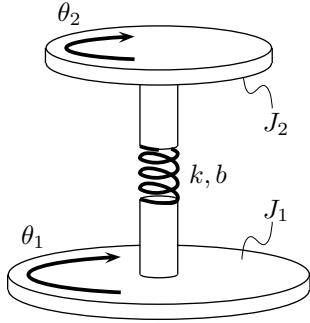


Fig. 2: Model of a spinning satellite.

Defining the state of the satellite as $x = [\theta_2 \ \dot{\theta}_2 \ \theta_1 \ \dot{\theta}_1]$ and using a sampling time of $\Delta t = 0.1$ yields the following system matrices:

$$A = \begin{bmatrix} 0.993 & 0.100 & 0.008 & 0.000 \\ -0.150 & 0.992 & 0.150 & 0.008 \\ 0.002 & 0.000 & 0.999 & 0.100 \\ 0.030 & 0.002 & -0.030 & 0.999 \end{bmatrix}, \quad (22a)$$

$$B = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.001 \\ 0.010 \end{bmatrix}, \quad E = \begin{bmatrix} 0.000 & 0.000 \\ 0.000 & 0.200 \\ 0.000 & 0.000 \\ 0.010 & 0.000 \end{bmatrix}. \quad (22b)$$

Each of the disturbance torques is assumed to be normally distributed with unit variance. The quadratic costs

$$Q = 0.1 \cdot I, \quad R = 1, \quad (23)$$

are supposed to reflect the expensiveness of thruster use in space. Furthermore, the control torque is limited to

$$\mathbf{P}[|u| \leq 0.5] \geq 1 - \varepsilon_u, \quad \varepsilon_u = 10\%. \quad (24)$$

Notice that in this case the lower bound for $\varepsilon_u \geq \bar{\varepsilon} = 0\%$, because the system is stable.

As a first case, consider an ICC on the position θ_2 ,

$$\mathbf{P}[|x^{(1)}| \leq 5] \geq 1 - \varepsilon_x, \quad \varepsilon_x = 10\%. \quad (25)$$

The lower bound for ε_x becomes $\bar{\varepsilon}_x = 9.82\%$, accounting for the input constraint (24). If a constrained LQR controller is designed based on $\varepsilon_x = 10\%$, the empirical violations amount to $\tilde{\varepsilon}_u = 6.76\%$ and $\tilde{\varepsilon}_x = 6.79\%$ (unconstrained LQR: $\tilde{\varepsilon}_u^{\text{uc}} = 16.37\%$ and $\tilde{\varepsilon}_x^{\text{uc}} = 5.43\%$).

As a second case, consider a JCC on the position θ_2 and velocity $\dot{\theta}_2$,

$$\mathbf{P}[|x^{(1)}| \leq 5 \wedge |x^{(2)}| \leq 3] \geq 1 - \varepsilon_x, \quad \varepsilon_x = 10\%. \quad (26)$$

The lower bound for $\varepsilon_x \geq \bar{\varepsilon}_x = 43.65\%$ lies far above the desired constraint level of (26). However, if a constrained LQR controller is designed according to Theorem 7, the

empirical violations amount to $\tilde{\varepsilon}_u = 6.71\%$ and $\tilde{\varepsilon}_x = 7.32\% \leq 10\%$ (unconstrained LQR: $\tilde{\varepsilon}_u^{\text{uc}} = 16.35\%$ and $\tilde{\varepsilon}_x^{\text{uc}} = 6.14\%$).

Notice the very large discrepancy between the violation bound based on Chebyshev ($\varepsilon_x = 43.65\%$) and the empirically observed violations ($\tilde{\varepsilon}_x = 7.32\%$) in this case.

VI. CONCLUSION

In this paper, a design approach for a LQR was presented that is optimal with respect to a quadratic target function, while respecting chance constraints on the input and/or state in closed-loop operation. This problem can be reformulated in terms of various LMIs, leading to a linear SDP (in the case of ICCs only) or a bilinear SDP (in the presence of a JCC), which can be solved efficiently by standard tools.

Compared to SMPC, a key advantage is that the computation is offline and finally only a linear controller is implemented. Compared to standard LQR, the same performance can in fact be achieved by a proper choices of Q and R , however the tuning procedure may be tedious in practice and does not bring about an optimality guarantee.

The numerical examples have demonstrated the flexibility of the method. Some conservatism has been observed, due to the Chebyshev bounds, which increases for lower probability levels and is much more significant for JCCs than for ICCs.

ACKNOWLEDGMENT

This research has received funding from the European Union Seventh Framework Programme FP7/2007-2013 under grant agreement number FP7-ICT-2009-4 248940.

REFERENCES

- [1] A. Schwarm and M. Nikolaou, "Chance-constrained model predictive control," *AIChE Journal*, vol. 45(8), pp. 1743–1752, 1999.
- [2] F. Oldewurtel, A. Parisio, C. Jones, D. Gyliastras, M. Gwerder, V. Stauch, B. Lehmann, and M. Morari, "Use of model predictive control and weather forecasts for energy efficient building climate control," *Energy and Buildings*, vol. 45, pp. 15–27, 2012.
- [3] M. Ono, "Joint chance-constrained model predictive control with probabilistic resolvability," in *American Control Conference*, Montreal, Canada, 2012.
- [4] J. Primbs, "A soft constraint approach to stochastic receding horizon control," in *47th IEEE Conference on Decision and Control*, New Orleans (LA), United States, 2007.
- [5] S. Boyd and L. Vandenberghe, "Semidefinite programming," *SIAM Review*, vol. 38-1, pp. 49–95, 1996.
- [6] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia (PA): Society for Industrial and Applied Mathematics, 1994.
- [7] P. Li, M. Wendt, and G. Wozny, "A probabilistically constrained model predictive controller," *Automatica*, vol. 38, pp. 1171–1176, 2002.
- [8] Z. Zhou and R. Cogill, "An algorithm for state constrained stochastic linear-quadratic control," in *American Control Conference*, San Francisco (CA), United States, 2011.
- [9] S. Meyn and R. Tweedie, *Markov Chains and Stochastic Stability*, 2nd ed. Cambridge: Cambridge University Press, 2009.
- [10] A. DasGupta, "Best constants in chebyshev inequalities with various applications," *Metrika*, vol. 51, pp. 185–200, 2000.
- [11] M. Grant and S. Boyd, *CVX Users' Guide, version 1.22*, Palo Alto (CA), 2012. [Online]. Available: http://cvxr.com/cvx/cvx_usrguide.pdf
- [12] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge: Cambridge University Press, 2004.
- [13] G. Franklin, J. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*, 5th ed. Upper Saddle River (NJ): Prentice Hall, 2009.