

\mathcal{H}_2 and \mathcal{H}_∞ Error Bounds for Model Order Reduction of Second Order Systems by Krylov Subspace Methods

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Abstract—We present rigorous bounds on the \mathcal{H}_2 and \mathcal{H}_∞ norm of the error resulting from model order reduction of second order systems by KRYLOV subspace methods. To this end, we use a strictly dissipative state space realization of the model and perform a factorization of the error system. The derived error expressions are easy to compute and can therefore be applied to models of very high order, as is demonstrated in numerical examples. In fact, all results hold true for arbitrary state space models in strictly dissipative realization that do not necessarily have to originate from second order systems.

I. INTRODUCTION

KRYLOV subspace methods are a powerful tool for the order reduction of linear time invariant (LTI) systems, as—compared to other model order reduction (MOR) methods—they require little memory and computational effort, which makes them applicable even to systems of very high order. However, KRYLOV subspace methods suffer from a number of drawbacks, among them the possible loss of stability during the reduction and the lack of reliable, efficient error bounds [11].

Although over time various error estimators have been proposed, e. g. [2], [13], none of them can be considered rigorous, which is why error bounds for Rational KRYLOV (RK) are still considered an open problem. Early approaches to the stability issue were implicitly restarted Arnoldi or Lanczos methods with look-ahead-mechanisms [10]. More recently, systems in so-called dissipative (also called contractive) realization have attained considerable attention, as one-sided (GALERKIN-type) reductions preserve their stability [20], [15]. Although in theory any asymptotically stable system can be transformed to a strictly dissipative realization by solving a LYAPUNOV equation or inequality, this is only feasible for original systems of moderate order.

For second order systems fulfilling typical definiteness properties, on the other hand, a strictly dissipative realization can be found with very little numerical effort [17]. It turns out, furthermore, that the resulting state space models can not only be reduced under preservation of stability, but admit rigorous error bounds, as will be proven in this work.

After reviewing preliminaries in Sec. II, we derive \mathcal{H}_2 and \mathcal{H}_∞ bounds on large-scale systems in strictly dissipative realization (Sec. III) and show their use as error bounds in model order reduction by KRYLOV subspace methods in Sec. IV. Numerical demonstrations and conclusions follow.

II. PRELIMINARIES

A. Second order systems and strict dissipativity

Assume a second order system to be given by

$$\mathbf{G}(s) : \begin{cases} \mathbf{M} \ddot{\mathbf{z}}(t) + \mathbf{D} \dot{\mathbf{z}}(t) + \mathbf{K} \mathbf{z}(t) = \mathbf{F} \mathbf{u}(t), \\ \mathbf{y}(t) = \overline{\mathbf{C}} \mathbf{z}(t), \end{cases} \quad (1)$$

$\mathbf{u}(t) \in \mathbb{R}^m$, and $\mathbf{y}(t) \in \mathbb{R}^p$ contain the m inputs and p outputs of the system. $\mathbf{M}, \mathbf{D}, \mathbf{K} \in \mathbb{R}^{n \times n}$ are called mass, damping, and stiffness matrix, respectively. For many finite element models, $\mathbf{M} = \mathbf{M}^T > \mathbf{0}$, $\mathbf{K} = \mathbf{K}^T > \mathbf{0}$ and $\mathbf{D} = \mathbf{D}^T > \mathbf{0}$ holds. Then, a particular state space realization

$$\mathbf{G}(s) = \left[\begin{array}{c|c} \mathbf{E}, \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{0} \end{array} \right] : \begin{cases} \mathbf{E} \dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C} \mathbf{x}(t), \end{cases} \quad (2)$$

whose order is $N = 2n$, is given by the matrices [17]

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} -\alpha \mathbf{K} & \mathbf{K} - \alpha \mathbf{D} \\ -\mathbf{K} & -\mathbf{D} + \alpha \mathbf{M} \end{bmatrix}, & \mathbf{B} &= \begin{bmatrix} \alpha \mathbf{F} \\ \mathbf{F} \end{bmatrix}, \\ \mathbf{E} &= \begin{bmatrix} \mathbf{K} & \alpha \mathbf{M} \\ \alpha \mathbf{M} & \mathbf{M} \end{bmatrix}, & \mathbf{C} &= [\overline{\mathbf{C}} \quad \mathbf{0}]. \end{aligned} \quad (3)$$

It has been shown in [17] that for all $\alpha \in \mathbb{R}$ fulfilling

$$0 < \alpha < \lambda_{\min} \left[\mathbf{D}(\mathbf{M} + \frac{1}{4} \mathbf{D} \mathbf{K}^{-1} \mathbf{D})^{-1} \right],$$

\mathbf{E} is positive definite and admits a CHOLESKY decomposition, while at the same time \mathbf{A} is *strictly* dissipative:

$$\mathbf{E} = \mathbf{E}^T = \mathbf{L}^T \mathbf{L} > \mathbf{0} \quad \text{and} \quad \mathbf{A} + \mathbf{A}^T < \mathbf{0}. \quad (4)$$

The logarithmic 2-norm of \mathbf{A} , defined as the right-most eigenvalue of the symmetric (HERMITE) part of \mathbf{A} ,

$$\mu_2(\mathbf{A}) := \lambda_{\max} \left(\frac{\mathbf{A} + \mathbf{A}^H}{2} \right) < 0,$$

is then strictly negative and can be used to bound the norm of the matrix exponential [5]:

$$\|e^{\mathbf{A}t}\|_2 \leq e^{\mu_2(\mathbf{A})t}.$$

The (generalized) logarithmic \mathbf{E} -norm of \mathbf{A} , derived from the inner product induced by \mathbf{E} , is negative, as well:

$$\mu_{\mathbf{E}}(\mathbf{A}) := \lambda_{\max} \left(\frac{\mathbf{A} + \mathbf{A}^H}{2}, \mathbf{E} \right) = \mu_2(\mathbf{L}^{-T} \mathbf{A} \mathbf{L}^{-1}) < 0,$$

where $\lambda_i(\mathbf{A}, \mathbf{E}) = \lambda_i(\mathbf{A} \mathbf{E}^{-1})$ denotes the generalized eigenvalues solving $\det(\mathbf{A} - \lambda_i \mathbf{E}) = 0$.

In this paper, we will assume the given system $\mathbf{G}(s)$ to be *strictly dissipative*, i. e. to fulfill (4). As was pointed out, this can be achieved for typical second order systems.

Lemma 1 ([20]). *A state space model in strictly dissipative realization is asymptotically stable.*

B. Definitions of important system norms

Definition 1. The \mathcal{H}_2 norm of an LTI system is defined as

$$\|\mathbf{G}\|_{\mathcal{H}_2} := \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}(\mathbf{G}^H(i\omega)\mathbf{G}(i\omega)) d\omega}, \quad (5)$$

where $\mathbf{G}(s)$ is the transfer function in the Laplace domain¹

$$\mathbf{G}(s) := \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}.$$

An equivalent formulation is given by [6]

$$\|\mathbf{G}\|_{\mathcal{H}_2} = \sqrt{\text{tr}(\mathbf{B}^T\mathbf{Q}\mathbf{B})}, \quad (6)$$

where \mathbf{Q} solves the generalized LYAPUNOV equation

$$\mathbf{A}^T\mathbf{Q}\mathbf{E} + \mathbf{E}^T\mathbf{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = \mathbf{0}. \quad (7)$$

Definition 2. The \mathcal{H}_∞ norm of an LTI system is defined as

$$\|\mathbf{G}\|_{\mathcal{H}_\infty} := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(\mathbf{G}(i\omega)) = \sup_{\omega \in \mathbb{R}} \|\mathbf{G}(i\omega)\|_2. \quad (8)$$

C. Projective model order reduction of LTI systems

An approximation of $\mathbf{G}(s)$ can be found by model order reduction (MOR) techniques, which provide two projection matrices $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{N \times q}$ and thereby define the reduced order model (ROM) of dimension q :

$$\mathbf{G}_r(s) : \begin{cases} \overbrace{\mathbf{W}^T \mathbf{E} \mathbf{V}}^{\mathbf{E}_r} \dot{\mathbf{x}}_r(t) = \overbrace{\mathbf{W}^T \mathbf{A} \mathbf{V}}^{\mathbf{A}_r} \mathbf{x}_r(t) + \overbrace{\mathbf{W}^T \mathbf{B}}^{\mathbf{B}_r} \mathbf{u}(t), \\ \mathbf{y}_r(t) = \underbrace{\mathbf{C} \mathbf{V}}_{\mathbf{C}_r} \mathbf{x}_r(t). \end{cases} \quad (9)$$

For details, please refer to [1], [22], [8].

Lemma 2 ([15], [20]). *Strict dissipativity—and hence asymptotic stability—of a state space model is preserved by a one-sided (GALERKIN-type) projection, i. e. for $\mathbf{W} := \mathbf{V}$.*

D. The error system in Rational Krylov methods

To measure the approximation quality, we consider the deviation $\mathbf{y}(t) - \mathbf{y}_r(t)$, which is the output of the error system

$$\mathbf{G}_e(s) := \mathbf{G}(s) - \mathbf{G}_r(s). \quad (10)$$

Let \mathbf{V} span a rational (block) input KRYLOV subspace, i. e.

$$\text{span } \mathbf{V} = \text{span} \bigcup_{i=1 \dots K, k=1 \dots k_i} ((\mathbf{A} - \sigma_i \mathbf{E})^{-1} \mathbf{E})^{k-1} (\mathbf{A} - \sigma_i \mathbf{E})^{-1} \mathbf{B} \mathbf{t}_i. \quad (11)$$

K characterizes the number of expansion points σ_i , while k_i and \mathbf{t}_i denote their respective multiplicities and tangential directions [11]. Then \mathbf{V} fulfills the SYLVESTER equation

$$\mathbf{A}\mathbf{V} - \mathbf{E}\mathbf{V}\mathbf{E}_r^{-1}\mathbf{A}_r = \mathbf{B}_\perp \tilde{\mathbf{C}}_r,$$

where

$$\mathbf{B}_\perp := \mathbf{B} - \mathbf{E}\mathbf{V}\mathbf{E}_r^{-1}\mathbf{B}_r, \quad (12)$$

$$\tilde{\mathbf{C}}_r := (\mathbf{B}_\perp^T \mathbf{B}_\perp)^{-1} \mathbf{B}_\perp^T (\mathbf{A}\mathbf{V} - \mathbf{E}\mathbf{V}\mathbf{E}_r^{-1}\mathbf{A}_r), \quad (13)$$

and according to [23], [24], \mathbf{G}_e can be factorized into

$$\mathbf{G}_e(s) = \underbrace{\begin{bmatrix} \mathbf{E}, \mathbf{A} & \mathbf{B}_\perp \\ \mathbf{C} & \mathbf{0} \end{bmatrix}}_{\mathbf{G}_\perp(s)} \cdot \underbrace{\begin{bmatrix} \mathbf{E}_r, \mathbf{A}_r & \mathbf{B}_r \\ \tilde{\mathbf{C}}_r & \mathbf{I} \end{bmatrix}}_{\tilde{\mathbf{G}}_r(s)}. \quad (14)$$

¹ $\mathbf{G}(s)$ will denote the transfer function and the dynamic system itself.

III. BOUNDS ON THE \mathcal{H}_2 AND \mathcal{H}_∞ NORM OF SYSTEMS IN STRICTLY DISSIPATIVE REALIZATION

In the following, we present bounds on the \mathcal{H}_2 and \mathcal{H}_∞ norm of—possibly very large—systems in strictly dissipative realization. Their use as error bounds in KRYLOV-based MOR is then presented in Sec. IV.

A. Bound on the \mathcal{H}_2 norm

We start with a bound on the \mathcal{H}_2 norm. The basic idea is to approximate the solution \mathbf{Q} of the LYAPUNOV equation (7) by a positive semidefinite matrix $\hat{\mathbf{Q}}$ of low rank.

Theorem 1. *Let $\mathbf{G}(s)$ be an LTI system in strictly dissipative realization, i. e. with $\mu_{\mathbf{E}}(\mathbf{A}) < 0$. Let furthermore $\hat{\mathbf{Q}}$ be an approximate solution to the LYAPUNOV equation (7) and define the residual*

$$\mathbf{R} := \mathbf{R}(\hat{\mathbf{Q}}) := \mathbf{A}^T \hat{\mathbf{Q}} \mathbf{E} + \mathbf{E}^T \hat{\mathbf{Q}} \mathbf{A} + \mathbf{C}^T \mathbf{C}. \quad (15)$$

Then, an upper bound on the \mathcal{H}_2 norm of $\mathbf{G}(s)$ is given by

$$\|\mathbf{G}\|_{\mathcal{H}_2}^2 \leq \text{tr}(\mathbf{B}^T \hat{\mathbf{Q}} \mathbf{B}) + \|\mathbf{L}^{-T} \mathbf{R} \mathbf{L}^{-1}\|_2 \frac{m \cdot \|\mathbf{B}_\perp^T \mathbf{E}^{-1} \mathbf{B}_\perp\|_2}{-2\mu_{\mathbf{E}}(\mathbf{A})}.$$

Proof. The proof is given in Appendix A. \square

Remark 1. *Although the norm $\|\mathbf{L}^{-T} \mathbf{R} \mathbf{L}^{-1}\|_2$ seems expensive to compute at first sight, please note that neither the dense large matrix \mathbf{R} nor the inverses are required explicitly. Instead, due to*

$$\begin{aligned} \|\mathbf{L}^{-T} \mathbf{R} \mathbf{L}^{-1}\|_2 &= \max_i \sigma_i(\mathbf{L}^{-T} \mathbf{R} \mathbf{L}^{-1}) \\ &= \max_i |\lambda_i(\mathbf{L}^{-T} \mathbf{R} \mathbf{L}^{-1})| \\ &= \max_i |\lambda_i(\mathbf{R}, \mathbf{E})|, \end{aligned} \quad (16)$$

we only need to find the solution λ_i of the generalized, symmetric eigenvalue problem that has largest magnitude, which is easily possible, for instance with a power method [19].

The matrix $\hat{\mathbf{Q}}$ can, for instance, result from techniques like the ADI method for approximate Gramians [14], [18].

Another common approach is the computation of $\hat{\mathbf{Q}}$ with the help of a ROM. After the reduction with suitable \mathbf{V}, \mathbf{W} , one solves the projected (small-scale) LYAPUNOV equation

$$\mathbf{A}_r^T \mathbf{Q}_r \mathbf{E}_r + \mathbf{E}_r^T \mathbf{Q}_r \mathbf{A}_r + \mathbf{C}_r^T \mathbf{C}_r = \mathbf{0} \quad (17)$$

with $\mathbf{A}_r, \mathbf{E}_r$ and \mathbf{C}_r as in (9) and defines $\hat{\mathbf{Q}} := \mathbf{W} \mathbf{Q}_r \mathbf{W}^T$.

Because of Lemma 2, however, it is often reasonable to apply a one-sided projection with only one matrix $\mathbf{Z} \in \mathbb{R}^{N \times \tilde{q}}$ instead of \mathbf{V}, \mathbf{W} . The resulting ROM is then strictly dissipative due to Lemma 2 and solving

$$\hat{\mathbf{A}}_r^T \mathbf{Q}_r \hat{\mathbf{E}}_r + \hat{\mathbf{E}}_r^T \mathbf{Q}_r \hat{\mathbf{A}}_r + \hat{\mathbf{C}}_r^T \hat{\mathbf{C}}_r = \mathbf{0} \quad (18)$$

with the matrices

$$\hat{\mathbf{A}}_r := \mathbf{Z}^T \mathbf{A} \mathbf{Z}, \quad \hat{\mathbf{E}}_r := \mathbf{Z}^T \mathbf{E} \mathbf{Z}, \quad \hat{\mathbf{C}}_r := \mathbf{C} \mathbf{Z}, \quad (19)$$

instead of (17) guarantees existence and uniqueness of \mathbf{Q}_r .

Anyway, $\hat{\mathbf{Q}}$ can always be expressed by

$$\hat{\mathbf{Q}} := \mathbf{Z} \mathbf{Q}_r \mathbf{Z}^T, \quad (20)$$

with $\mathbf{Z} \in \mathbb{R}^{N \times \tilde{q}}$ and $\mathbf{Q}_r \in \mathbb{R}^{\tilde{q} \times \tilde{q}}$.

B. Bound on the \mathcal{H}_∞ norm

Theorem 2. An upper bound on the \mathcal{H}_∞ norm of a system $\mathbf{G}(s)$ in strictly dissipative realization is given by

$$\|\mathbf{G}\|_{\mathcal{H}_\infty} \leq \|\mathbf{C}\mathbf{S}^{-1}\mathbf{B}\|_2 + \sqrt{\|\mathbf{B}^T\mathbf{S}^{-1}\mathbf{B}\|_2 \|\mathbf{C}\mathbf{S}^{-1}\mathbf{C}^T\|_2},$$

where $\mathbf{S} := -(\mathbf{A} + \mathbf{A}^T)$.

Proof. The proof is given in Appendix B. \square

IV. EFFICIENT ERROR BOUNDS IN MODEL ORDER REDUCTION BY KRYLOV SUBSPACE METHODS

The bounds presented in Theorems 1 and 2 can estimate the respective norms of a given large-scale system in strictly dissipative realization. The central idea to utilize them for error bounds in KRYLOV-based MOR is to use the factorized formulation of the error system (14) and apply the bounds to the large-scale factor $\mathbf{G}_\perp(s)$, which shares its matrices \mathbf{A} , \mathbf{E} and \mathbf{C} with the original model $\mathbf{G}(s)$.

A. An \mathcal{H}_2 error bound for KRYLOV-based MOR

We start with an auxiliary lemma.

Lemma 3. The \mathcal{H}_2 norm of the error system in KRYLOV subspace methods is upper bounded by

$$\|\mathbf{G}_e\|_{\mathcal{H}_2} \leq \|\mathbf{G}_\perp\|_{\mathcal{H}_2} \cdot \|\tilde{\mathbf{G}}_r\|_{\mathcal{H}_\infty}. \quad (21)$$

Proof. From the definition (5) of the \mathcal{H}_2 norm it follows

$$\begin{aligned} \|\mathbf{G}_e\|_{\mathcal{H}_2}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathbf{G}_e(i\omega)\|_F^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathbf{G}_\perp(i\omega) \cdot \tilde{\mathbf{G}}_r(i\omega)\|_F^2 d\omega \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathbf{G}_\perp(i\omega)\|_F^2 \cdot \|\tilde{\mathbf{G}}_r(i\omega)\|_F^2 d\omega \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathbf{G}_\perp(i\omega)\|_F^2 \sup_{\omega} \|\tilde{\mathbf{G}}_r(i\omega)\|_F^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left[\mathbf{G}_\perp^H(i\omega) \mathbf{G}_\perp(i\omega) \right] d\omega \cdot \|\tilde{\mathbf{G}}_r\|_{\mathcal{H}_\infty}^2 \square \end{aligned}$$

Together with Theorem 1, this leads to the following result.

Theorem 3. Let $\mathbf{G}(s)$ be an LTI system in strictly dissipative realization and let $\hat{\mathbf{Q}} = \mathbf{Z}\mathbf{Q}_r\mathbf{Z}^T$ be an approximate solution to the LYAPUNOV equation (7) with the residual \mathbf{R} . When the system is reduced by a Rational KRYLOV subspace method and factorized according to (14), then an upper bound on the \mathcal{H}_2 norm of the error system $\mathbf{G}_e(s)$ is given by

$$\|\mathbf{G}_e\|_{\mathcal{H}_2} \leq \sqrt{k_1 + k_2 \cdot \frac{m \cdot \|\mathbf{B}_\perp^T \mathbf{E}^{-1} \mathbf{B}_\perp\|_2}{-2\mu_{\mathbf{E}}(\mathbf{A})}} \cdot k_3,$$

where k_1 , k_2 , and k_3 are defined as

$$\begin{aligned} k_1 &:= \text{tr}(\mathbf{B}_\perp^T \hat{\mathbf{Q}} \mathbf{B}_\perp) = \text{tr}(\mathbf{B}_\perp^T \mathbf{Z} \mathbf{Q}_r \mathbf{Z}^T \mathbf{B}_\perp), \\ k_2 &:= \|\mathbf{L}^{-T} \mathbf{R} \mathbf{L}^{-1}\|_2 = \max_i |\lambda_i(\mathbf{R}, \mathbf{E})|, \\ k_3 &:= \|\tilde{\mathbf{G}}_r\|_{\mathcal{H}_\infty}. \end{aligned}$$

Proof. This follows from Theorem 1 and Lemma 3. \square

Some special cases for the choice of $\hat{\mathbf{Q}}$ follow:

- $\hat{\mathbf{Q}} = \mathbf{0}$, i. e. $\mathbf{Q}_r = \mathbf{0}$ and/or $\mathbf{Z} = \mathbf{0}$.
 $\implies k_1 = 0, k_2 = \|\mathbf{C}\mathbf{E}^{-1}\mathbf{C}^T\|_2.$
- $\text{span } \mathbf{Z} = \text{span } \mathbf{W}$, in particular $\mathbf{V} = \mathbf{W} = \mathbf{Z}$.
 $\implies k_1 = 0.$
- $\hat{\mathbf{Q}} = \mathbf{Q}$ exactly solves the LYAPUNOV equation.
 $\implies k_2 = 0.$
This is the scenario mentioned in [23] for SISO.
- $m = p = 1$ (SISO) and $\mathbf{G}_r(s)$ is \mathcal{H}_2 suboptimal, i. e. it matches at least one moment at the mirror image of each of its poles, cf. [23].
 $\implies k_3 = 1$, as $\tilde{\mathbf{G}}_r$ is all-pass.

Please note that all quantities involved can be computed without significant numerical effort:

- \mathbf{B}_\perp and $\tilde{\mathbf{G}}_r$ for the decomposition of the error system mainly require matrix-vector products.
- The same holds for k_1 .
- The norm $\|\mathbf{B}_\perp^T \mathbf{E}^{-1} \mathbf{B}_\perp\|_2$ requires m solves of large-scale, but sparse systems of equations. Using the CHOLESKY decomposition of $\mathbf{E} = \mathbf{L}^T \mathbf{L}$ is often beneficial: as soon as it has been computed once, the expression simplifies to $\|\mathbf{B}_\perp^T \mathbf{E}^{-1} \mathbf{B}_\perp\|_2 = \|\mathbf{L}^{-T} \mathbf{B}_\perp\|_2^2$.
- $\tilde{\mathbf{G}}_r(s)$ is a small-scale system whose \mathcal{H}_∞ norm k_3 can easily be found.
- The logarithmic norm $\mu_{\mathbf{E}}(\mathbf{A})$ is the solution of a large-scale, but sparse and symmetric generalized eigenvalue problem; see [17] for details on the implementation.
- The same holds for k_2 . In fact, we can exploit the low-rank structure of $\hat{\mathbf{Q}}$ and the sparsity of \mathbf{A} and \mathbf{E} in the power method indicated in Remark 1. A possible MATLAB implementation is

```
Rx = @(x) (x'*A'*Z*Qr*Z'*E + ...
          x'*E'*Z*Qr*Z'*A + x'*C'*C)';
opts = struct('issym',true); N=size(A,1);
k_2 = abs(eigs(Rx,N,E,1,'LM',opts));
```

Another option is to exploit the low rank of \mathbf{R} —which is $2\tilde{q} + p$ or even only $2p$ if \mathbf{Z} spans an outlook KRYLOV subspace (see Sec. V)—and reformulate the norm as a small-scale eigenvalue problem [25].

Note furthermore that \tilde{q} , the column dimension of \mathbf{Z} , can be chosen independently of the reduced order q .

B. An \mathcal{H}_∞ error bound for KRYLOV subspace methods

Theorem 4. Let $\mathbf{G}(s)$ be an LTI system in strictly dissipative realization and define $\mathbf{S} := -\mathbf{A} - \mathbf{A}^T$. When the system is reduced by a Rational KRYLOV subspace method and factorized according to (14), then the \mathcal{H}_∞ norm of the error system $\mathbf{G}_e(s)$ is upper bounded by

$$\|\mathbf{G}_e\|_{\mathcal{H}_\infty} \leq \|\tilde{\mathbf{G}}_r\|_{\mathcal{H}_\infty} \cdot \left(\|\mathbf{C}\mathbf{S}^{-1}\mathbf{B}_\perp\|_2 + \sqrt{\|\mathbf{B}_\perp^T \mathbf{S}^{-1} \mathbf{B}_\perp\|_2 \|\mathbf{C}\mathbf{S}^{-1}\mathbf{C}^T\|_2} \right)$$

Proof. This is a direct consequence from the submultiplicativity of the \mathcal{H}_∞ norm and Theorem 1. \square

Like the \mathcal{H}_2 bound presented above, this expression is easy to evaluate and provides a cheap rigorous error bound.

V. NUMERICAL EXAMPLES

In this section, we demonstrate the effectiveness of the presented bounds with the help of numerical examples.

We first consider the FEM model of a cantilever beam [16]. Its order is $N = 240$. We follow [17] to find a strictly dissipative state space realization. Then we reduce the system to multiple reduced orders $q_i = 2, 4, \dots, 30$ by one-sided projection, using a KRYLOV subspace (11) with expansion point $\sigma = 0$ and multiplicity q_i . Then we evaluate the \mathcal{H}_∞ bound and several versions of the \mathcal{H}_2 bound for different approximate LYAPUNOV solutions $\hat{\mathbf{Q}}$:

- i) The simplest possible choice: $\hat{\mathbf{Q}} = \mathbf{0}$ (case a).
- ii) Based on our ROM as in (17); $\mathbf{Z} := \mathbf{V}$ (case b).
- iii) \mathbf{Z} is the basis of an output rational KRYLOV (RK) subspace, which is defined dually to the input KRYLOV subspace (11) by replacing $\mathbf{A} \rightarrow \mathbf{A}^T$, $\mathbf{B} \rightarrow \mathbf{C}^T$ in (11) (\mathbf{E} can remain unchanged due to its symmetry, here). Again, we use the expansion point $\sigma = 0$ and multiplicity $\tilde{q}_i = q_i$.
- iv) As iii), but with $\tilde{q}_i = 2q_i$.
- v) We compute the exact LYAPUNOV solution (case c)).

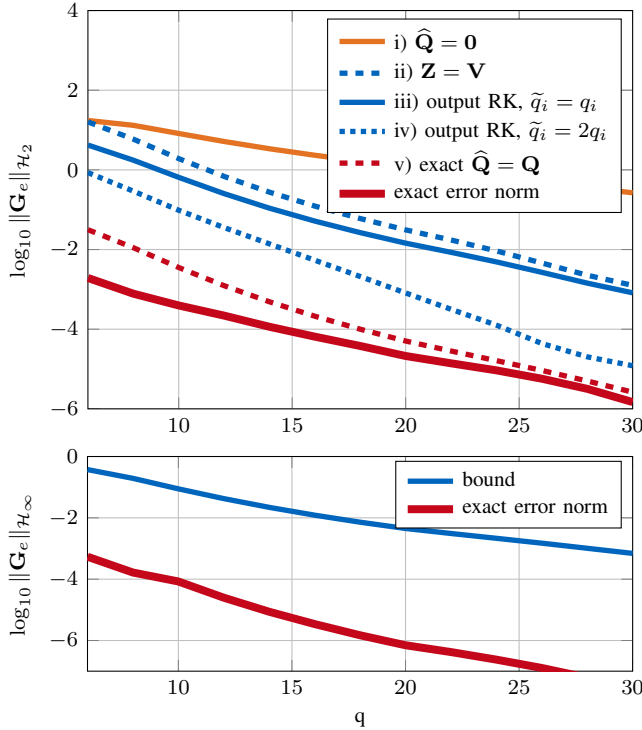


Fig. 1. \mathcal{H}_2 and \mathcal{H}_∞ error bounds for beam benchmark

Fig. 1 shows how all \mathcal{H}_2 bounds decay with increasing q , which is partly due to the quality of the reduced model—the real error decays with increasing order—but also due to the improving approximation of \mathbf{Q} . Please observe how the bounds get tighter the more accurate $\hat{\mathbf{Q}}$ is: $\hat{\mathbf{Q}} = \mathbf{0}$, for instance, gives the worst bound, while for the exact \mathbf{Q} the overestimation is very small.

The \mathcal{H}_∞ bound exhibits notable overestimation, but at least decays almost as fast as the exact error norm.

As a second example, we consider the model of a steel profile from [3]. It is a state space model of order $N = 20209$ which happens to be strictly dissipative, so the bounds apply to the model even though it does not stem from a second order system. Due to space limitations, we only present results for the SISO case given by the transfer behavior from the first input to the first output, here. We proceed as before and evaluate the \mathcal{H}_2 bound for different $\hat{\mathbf{Q}}$, namely the three cases i), iii), and iv) of the above example.

Again, the bounds decay with increasing q . The computation of $\mu_{\mathbf{E}}(\mathbf{A})$ and the CHOLESKY factors of \mathbf{S} and \mathbf{E} lasted 0.7s on an Intel Core2 Duo CPU at 3GHz. The bounds for $q = 150$ then took 0.002s, 9s, and 35s for the three respective cases given in Fig. 2.²

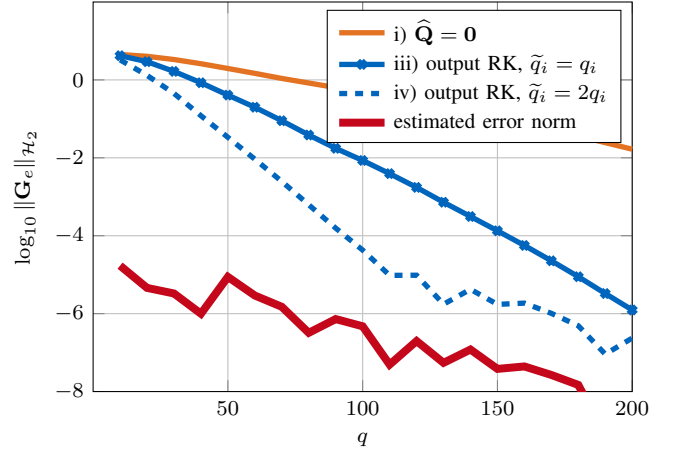


Fig. 2. \mathcal{H}_2 error bounds for steel benchmark

VI. CONCLUSIONS AND FUTURE WORK

We have presented rigorous upper bounds on the \mathcal{H}_2 and \mathcal{H}_∞ error for KRYLOV-based model reduction of LTI systems. They rest upon a decomposition of the error model and hold for state space models in strictly dissipative realization; typical second order systems can easily be transformed into such a formulation. The bounds are easy to evaluate and introduce varying overestimation—from hardly none up to several orders of magnitude, depending on the quality of the approximate LYAPUNOV solution $\hat{\mathbf{Q}}$ (in the \mathcal{H}_2 case).

Future work will concentrate on strategies to diminish the bounds as effectively as possible, in particular for MIMO systems. One possibility to do so is the use of algorithms like K-PIK [21] or ICOP [7] that employ expansions points different from $\sigma = 0$. Besides, the influence of the choice of the parameter α in (3) on the bounds has to be cleared.

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²As we could no more compute the exact error norm, we used $\mathbf{G}_e(s) \approx \mathbf{G}_{ref}(s) - \mathbf{G}_r(s)$, where the reference model $\mathbf{G}_{ref}(s)$ of order 500 is an excellent approximation of $\mathbf{G}(s)$. However, please note that this is not a reliable error information.

APPENDIX A: PROOF OF THEOREM 1

The \mathcal{H}_2 bound presented in Theorem 1 is based on an observation due to HODEL from 1996 [12], which we will firstly generalize to non-standard state space realizations.

Lemma 4. Let $\mathbf{E} = \mathbf{E}^T > \mathbf{0}$ be positive definite and admit the CHOLESKY decomposition $\mathbf{E} = \mathbf{L}^T \mathbf{L}$. Let furthermore $\mathbf{Q} \in \mathbb{R}^{N \times N}$ solve the generalized LYAPUNOV equation (7) and let $\hat{\mathbf{Q}} \in \mathbb{R}^{N \times N}$ be an estimate of \mathbf{Q} . Define the residual

$$\mathbf{R} := \mathbf{R}(\hat{\mathbf{Q}}) := \mathbf{A}^T \hat{\mathbf{Q}} \mathbf{E} + \mathbf{E}^T \hat{\mathbf{Q}} \mathbf{A} + \mathbf{C}^T \mathbf{C}.$$

If the logarithmic norm $\mu_{\mathbf{E}}(\mathbf{A}) = \lambda_{\max} \left(\frac{\mathbf{A} + \mathbf{A}^T}{2}, \mathbf{E} \right) < 0$, then

$$\|\mathbf{L}(\mathbf{Q} - \hat{\mathbf{Q}})\mathbf{L}^T\|_2 \leq \frac{1}{-2\mu_{\mathbf{E}}(\mathbf{A})} \|\mathbf{L}^{-T} \mathbf{R} \mathbf{L}^{-1}\|_2.$$

Proof. Subtracting (15) from (7) yields

$$\mathbf{A}^T (\mathbf{Q} - \hat{\mathbf{Q}}) \mathbf{E} + \mathbf{E} (\mathbf{Q} - \hat{\mathbf{Q}}) \mathbf{A} + \mathbf{R} = \mathbf{0}. \quad (22)$$

Using the CHOLESKY decomposition $\mathbf{E} = \mathbf{L}^T \mathbf{L}$ and defining $\mathbf{X} := \mathbf{L}(\mathbf{Q} - \hat{\mathbf{Q}})\mathbf{L}^T$, we can equivalently rewrite (22) as

$$\mathbf{L}^{-T} \mathbf{A}^T \mathbf{L}^{-1} \mathbf{X} + \mathbf{X} \mathbf{L}^{-T} \mathbf{A} \mathbf{L}^{-1} + \mathbf{L}^{-T} \mathbf{R} \mathbf{L}^{-1} = \mathbf{0}. \quad (23)$$

The (unique) solution of this LYAPUNOV equation is

$$\begin{aligned} \mathbf{X} &= \int_0^{\infty} e^{\mathbf{L}^{-T} \mathbf{A}^T \mathbf{L}^{-1} t} \mathbf{L}^{-T} \mathbf{R} \mathbf{L}^{-1} e^{\mathbf{L}^{-T} \mathbf{A} \mathbf{L}^{-1} t} dt \\ \Rightarrow \|\mathbf{X}\|_2 &\leq \int_0^{\infty} \left\| e^{\mathbf{L}^{-T} \mathbf{A} \mathbf{L}^{-1} t} \right\|_2^2 dt \cdot \|\mathbf{L}^{-T} \mathbf{R} \mathbf{L}^{-1}\|_2 \\ &\leq \int_0^{\infty} e^{2\mu_2(\mathbf{L}^{-T} \mathbf{A} \mathbf{L}^{-1}) t} dt \cdot \|\mathbf{L}^{-T} \mathbf{R} \mathbf{L}^{-1}\|_2 \\ &= \int_0^{\infty} e^{2\mu_{\mathbf{E}}(\mathbf{A}) t} dt \cdot \|\mathbf{L}^{-T} \mathbf{R} \mathbf{L}^{-1}\|_2 \quad \square \end{aligned}$$

This lemma basically allows us to replace the norm of $(\mathbf{Q} - \hat{\mathbf{Q}})$ —which is unknown because we cannot compute \mathbf{Q} —by the norm of the residual. The crucial step is now to split the formulation (6) of the \mathcal{H}_2 norm into two parts. The first summand uses $\hat{\mathbf{Q}}$ instead of \mathbf{Q} for the computation of the norm; the second term provides for the fact that $\hat{\mathbf{Q}}$ is only an approximation, the resulting error of which is upper bounded with the help of the residual \mathbf{R} .

Proof of Theorem 1. Starting from definition (6) of the \mathcal{H}_2 norm, we obtain

$$\begin{aligned} \|\mathbf{G}\|_{\mathcal{H}_2}^2 &= \text{tr}(\mathbf{B}^T \mathbf{Q} \mathbf{B}) \\ &= \text{tr}(\mathbf{B}^T \hat{\mathbf{Q}} \mathbf{B}) + \text{tr}(\mathbf{B}^T (\mathbf{Q} - \hat{\mathbf{Q}}) \mathbf{B}). \end{aligned}$$

The second summand fulfills

$$\begin{aligned} \text{tr}(\mathbf{B}^T (\mathbf{Q} - \hat{\mathbf{Q}}) \mathbf{B}) &\leq m \cdot \|\mathbf{B}^T (\mathbf{Q} - \hat{\mathbf{Q}}) \mathbf{B}\|_2 \\ &\leq m \cdot \|\mathbf{B}^T \mathbf{L}^{-1} \mathbf{L} (\mathbf{Q} - \hat{\mathbf{Q}}) \mathbf{L}^T \mathbf{L}^{-T} \mathbf{B}\|_2 \\ &\leq m \cdot \left\| \mathbf{L}^{-T} \mathbf{B} \mathbf{B}^T \mathbf{L}^{-1} \right\|_2 \cdot \left\| \mathbf{L} (\mathbf{Q} - \hat{\mathbf{Q}}) \mathbf{L}^T \right\|_2 \end{aligned}$$

and the claim follows with Lemma 4. \square

APPENDIX B: PROOF OF THEOREM 2

We start by recalling the following result on the structured complex stability radius.

Lemma 5 ([4]). If $\mathbf{G}(s) \not\equiv 0$, then $\|\mathbf{G}\|_{\mathcal{H}_{\infty}} = \frac{1}{r_{\mathbf{C}}}$, where $r_{\mathbf{C}} = r_{\mathbf{C}}(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is the structured complex stability radius, i. e. the smallest number $\epsilon \in \mathbb{R}^+$ that admits a matrix $\Delta \in \mathbb{C}^{m \times p}$ with $\|\Delta\|_2 < \epsilon$ such that the perturbed system

$$\mathbf{G}_{\Delta}(s) := \mathbf{C} \left(s\mathbf{E} - (\mathbf{A} + \mathbf{B}\Delta\mathbf{C}) \right)^{-1} \mathbf{B}$$

is no longer asymptotically stable, but has imaginary poles.

In order to give an upper bound on $\|\mathbf{G}(s)\|_{\mathcal{H}_{\infty}}$, it is therefore sufficient to find a lower bound $\epsilon^* \in \mathbb{R}^+$ on $r_{\mathbf{C}}$, i. e. a real number $\epsilon^* \leq r_{\mathbf{C}}$ for which all perturbed systems $\mathbf{G}_{\Delta}(s)$ with $\|\Delta\|_2 < \epsilon^*$ are still asymptotically stable. The higher this number ϵ^* , the tighter the bound becomes.

In the following, we will use the concept of dissipativity to derive such an ϵ^* . Three supplementary lemmata prepare the main result of Theorem 2.

Lemma 6. If $\mathbf{E} > \mathbf{0}$ and $\mu_2(\mathbf{A} + \mathbf{B}\Delta\mathbf{C}) < 0$ for all Δ with $\|\Delta\|_2 < \epsilon^*$, then $\|\mathbf{G}\|_{\mathcal{H}_{\infty}} \leq \frac{1}{\epsilon^*}$.

Proof. This is a direct consequence of Lemma 1: strict dissipativity is sufficient for asymptotic stability. \square

Lemma 7. An upper bound on $\|\mathbf{G}\|_{\mathcal{H}_{\infty}}$ for systems in strictly dissipative realization is given by

$$\|\mathbf{G}\|_{\mathcal{H}_{\infty}} \leq \max_{\substack{i=1 \dots N \\ \|\tilde{\Delta}\|=1}} \lambda_i \left[(\mathbf{B}\tilde{\Delta}\mathbf{C} + \mathbf{C}^T \tilde{\Delta}^H \mathbf{B}^T) \mathbf{S}^{-1} \right],$$

where

$$\mathbf{S} := -(\mathbf{A} + \mathbf{A}^T) > \mathbf{0}. \quad (25)$$

Proof. Our goal is to find a preferably large ϵ^* such that Lemma 6 still applies. To this end, we define

$$\delta := \|\Delta\|_2 < \epsilon^* \quad \text{and} \quad \tilde{\Delta} := \frac{\Delta}{\|\Delta\|_2}$$

such that $\Delta = \delta \cdot \tilde{\Delta}$ and $\|\tilde{\Delta}\|_2 = 1$. Then,

$$\begin{aligned} \mu_2(\mathbf{A} + \mathbf{B}\Delta\mathbf{C}) &< 0 \\ \Leftrightarrow \mathbf{x}^H (\mathbf{A} + \mathbf{A}^T + \mathbf{B}\Delta\mathbf{C} + (\mathbf{B}\Delta\mathbf{C})^H) \mathbf{x} &< 0 \quad \forall \mathbf{x} \in \mathbb{C}^N \\ \Leftrightarrow \mathbf{x}^H (\mathbf{A} + \mathbf{A}^T) \mathbf{x} + \delta \mathbf{x}^H (\mathbf{B}\tilde{\Delta}\mathbf{C} + (\mathbf{B}\tilde{\Delta}\mathbf{C})^H) \mathbf{x} &< 0 \\ \Leftrightarrow \frac{\mathbf{x}^H (\mathbf{B}\tilde{\Delta}\mathbf{C} + \mathbf{C}^T \tilde{\Delta}^H \mathbf{B}^T) \mathbf{x}}{\mathbf{x}^H (-\mathbf{A} - \mathbf{A}^T) \mathbf{x}} &< \frac{1}{\delta} \end{aligned} \quad (26)$$

must hold for any $\tilde{\Delta}$. Due to $\|\Delta\|_2 = \delta < \epsilon^*$, we know that $\frac{1}{\epsilon^*} < \frac{1}{\delta}$ holds. Consequently, in order to guarantee that condition (26) is met for all $\tilde{\Delta}$ and for all \mathbf{x} , we set

$$\begin{aligned} \frac{1}{\epsilon^*} &:= \max_{\mathbf{x}, \tilde{\Delta}} \frac{\mathbf{x}^H (\mathbf{B}\tilde{\Delta}\mathbf{C} + \mathbf{C}^T \tilde{\Delta}^H \mathbf{B}^T) \mathbf{x}}{\mathbf{x}^H (-\mathbf{A} - \mathbf{A}^T) \mathbf{x}} \\ \Rightarrow \frac{\mathbf{x}^H (\mathbf{B}\tilde{\Delta}\mathbf{C} + \mathbf{C}^T \tilde{\Delta}^H \mathbf{B}^T) \mathbf{x}}{\mathbf{x}^H (-\mathbf{A} - \mathbf{A}^T) \mathbf{x}} &\leq \frac{1}{\epsilon^*} < \frac{1}{\delta}. \end{aligned} \quad (27)$$

The remaining step from (27) is a property of the generalized RAYLEIGH quotient [9]. \square

It remains to find out, for which $\tilde{\Delta}$ the maximum is reached. To this end, we give a final auxiliary lemma.

Lemma 8. For any given $\mathbf{x} \in \mathbb{C}^N$, one solution to the optimization problem

$$\arg \max_{\substack{\tilde{\Delta} \in \mathbb{R}^{m \times p} \\ \|\tilde{\Delta}\|_2 = 1}} \mathbf{x}^H (\mathbf{B}\tilde{\Delta}\mathbf{C} + \mathbf{C}^T \tilde{\Delta}^H \mathbf{B}^T) \mathbf{x}, \quad (28)$$

is given by the rank-1 matrix $\tilde{\Delta}^* = \frac{\mathbf{B}^T \mathbf{x} \mathbf{x}^H \mathbf{C}^T}{\|\mathbf{B}^T \mathbf{x}\|_2 \|\mathbf{C} \mathbf{x}\|_2}$.

Proof. The maximal value is bounded by

$$\begin{aligned} \mathbf{x}^H (\mathbf{B}\tilde{\Delta}\mathbf{C} + \mathbf{C}^T \tilde{\Delta}^H \mathbf{B}^T) \mathbf{x} &\leq 2 \|\mathbf{x}^H \mathbf{B}\tilde{\Delta}\mathbf{C} \mathbf{x}\|_2 \\ &\leq 2 \|\mathbf{x}^H \mathbf{B}\|_2 \cdot \underbrace{\|\tilde{\Delta}\|_2}_1 \cdot \|\mathbf{C} \mathbf{x}\|_2. \end{aligned}$$

This value is, however, actually reached when inserting the matrix $\tilde{\Delta}^*$ in (28):

$$\begin{aligned} \mathbf{x}^H (\mathbf{B}\tilde{\Delta}^* \mathbf{C} + \mathbf{C}^T \tilde{\Delta}^{*H} \mathbf{B}^T) \mathbf{x} &= 2 \frac{\mathbf{x}^H \mathbf{B} \mathbf{B}^T \mathbf{x} \mathbf{x}^H \mathbf{C}^T \mathbf{C} \mathbf{x}}{\|\mathbf{B}^T \mathbf{x}\|_2 \cdot \|\mathbf{C} \mathbf{x}\|_2} \\ &= 2 \frac{\|\mathbf{B}^T \mathbf{x}\|_2^2 \cdot \|\mathbf{C} \mathbf{x}\|_2^2}{\|\mathbf{B}^T \mathbf{x}\|_2 \cdot \|\mathbf{C} \mathbf{x}\|_2} = 2 \|\mathbf{B}^T \mathbf{x}\|_2 \cdot \|\mathbf{C} \mathbf{x}\|_2 \end{aligned}$$

□

Now we are ready to prove the main theorem.

Proof of Theorem 2. Due to Lemma 8, we can assume $\tilde{\Delta}$ to be given by two complex vectors \mathbf{u} , \mathbf{v} fulfilling $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1$:

$$\tilde{\Delta} = \mathbf{u} \mathbf{v}^H \quad (29)$$

The eigenvalue problem in (24) can therefore be reformulated to a 2×2 generalized (HERMITE) eigenvalue problem whose maximal (real) solution is easy to obtain.

$$\begin{aligned} \lambda_{\max} \left[(\mathbf{B}\tilde{\Delta}\mathbf{C} + \mathbf{C}^T \tilde{\Delta}^H \mathbf{B}^T) \mathbf{S}^{-1} \right] &= \\ = \lambda_{\max} \left[(\mathbf{B} \mathbf{u} \mathbf{v}^H \mathbf{C} + \mathbf{C}^T \mathbf{v} \mathbf{u}^H \mathbf{B}^T) \mathbf{S}^{-1} \right] &= \\ = \lambda_{\max} \left[\begin{bmatrix} \mathbf{B} \mathbf{u} & \mathbf{C}^T \mathbf{v} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^H \mathbf{B}^T \\ \mathbf{v}^H \mathbf{C} \end{bmatrix} \mathbf{S}^{-1} \right] &= \\ = \lambda_{\max} \left[\underbrace{\begin{bmatrix} \mathbf{u}^H \mathbf{B}^T \\ \mathbf{v}^H \mathbf{C} \end{bmatrix} \mathbf{S}^{-1} \begin{bmatrix} \mathbf{B} \mathbf{u} & \mathbf{C}^T \mathbf{v} \end{bmatrix}}_{\in \mathbb{R}^{2 \times 2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] &= \\ = \lambda_{\max} \left[\begin{bmatrix} \mathbf{u}^H \mathbf{B}^T \mathbf{S}^{-1} \mathbf{B} \mathbf{u} & \mathbf{u}^H \mathbf{B}^T \mathbf{S}^{-1} \mathbf{C}^T \mathbf{v} \\ \mathbf{v}^H \mathbf{C} \mathbf{S}^{-1} \mathbf{B} \mathbf{u} & \mathbf{v}^H \mathbf{C} \mathbf{S}^{-1} \mathbf{C}^T \mathbf{v} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] &= \\ = \mathbf{v}^H \mathbf{C} \mathbf{S}^{-1} \mathbf{B} \mathbf{u} + \sqrt{\mathbf{u}^H \mathbf{B}^T \mathbf{S}^{-1} \mathbf{B} \mathbf{u} \cdot \mathbf{v}^H \mathbf{C} \mathbf{S}^{-1} \mathbf{C}^T \mathbf{v}}. \end{aligned}$$

Accordingly, we can conclude:

$$\begin{aligned} \|\mathbf{G}\|_{\mathcal{H}_\infty} &\leq \max_{i, \tilde{\Delta}} \lambda_i \left[(\mathbf{B}\tilde{\Delta}\mathbf{C} + \mathbf{C}^T \tilde{\Delta}^H \mathbf{B}^T) \mathbf{S}^{-1} \right] \\ &\leq \max_{\mathbf{u}, \mathbf{v}} \left[\mathbf{v}^H \mathbf{C} \mathbf{S}^{-1} \mathbf{B} \mathbf{u} + \sqrt{\mathbf{u}^H \mathbf{B}^T \mathbf{S}^{-1} \mathbf{B} \mathbf{u} \cdot \mathbf{v}^H \mathbf{C} \mathbf{S}^{-1} \mathbf{C}^T \mathbf{v}} \right] \end{aligned} \quad (30)$$

A worst case estimation of the scalar terms in (30) is obtained using the submultiplicativity of the Euclidian norm, e.g.

$$\begin{aligned} \mathbf{v}^H \mathbf{C} \mathbf{S}^{-1} \mathbf{B} \mathbf{u} &\leq \|\mathbf{v}^H \mathbf{C} \mathbf{S}^{-1} \mathbf{B} \mathbf{u}\|_2 \leq \underbrace{\|\mathbf{v}^H\|_2}_1 \|\mathbf{C} \mathbf{S}^{-1} \mathbf{B}\|_2 \underbrace{\|\mathbf{u}\|_2}_1 \end{aligned}$$

and the claim readily follows. □

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