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State feedback $\mathcal{H}_\infty$ control of time-delay switched linear systems: a descriptor approach

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Abstract— **In this paper we propose contributions on the control of switched linear systems subject to time-delays. Our methods allow to synthesize state feedback switching laws able to ensure the stabilization and the fulfillment of a prescribed $\mathcal{H}_\infty$ disturbance attenuation property. The proposed criteria are delay-dependent and based on an application of the descriptor approach, previously introduced in the literature to study standard time-delay systems. The core of our procedure is based on the solution of sets of matrix inequalities, which allow simultaneously to assign a suitable switching law and to determine a feasible piecewise quadratic Lyapunov-Krasovskii functional associated to it.**

I. INTRODUCTION

In recent years, the interest in the study of time-delay switched systems has considerably grown. Concepts and results from the theory of time-delay systems (see [1] for a survey) have been extended to this class of systems and the literature witnesses an effort towards the formulation of results concerning some fundamental control theoretical topics, such as stability and stabilization.

One of the first issues to be studied is related to the translation of the concept of common Lyapunov functions, often referred to in the case of standard switched systems, to formulations involving either the Lyapunov-Razumikhin approach or the use of Lyapunov-Krasovskii functionals (see [2], [3], [4], [5] and [6] for instance). More recently, multiple (often piecewise quadratic) Lyapunov-type functionals were also introduced, see for example [5], [7], [8], [9] and [10].

Piecewise quadratic Lyapunov-Krasovskii functionals are a fundamental tool also for the theoretical setup proposed in this paper, which deals with switched linear systems subject to a delay acting on the state. The main focus here is on solving stabilization and $\mathcal{H}_\infty$ disturbance attenuation problems by the synthesis of delay-dependent switching laws based on state feedback. Our work is based on ideas originally proposed in [11] and [12], where the stabilization issue of switched linear systems (with no delays) is dealt with by introducing special Lyapunov-Metzler inequalities. Previous work on this topic includes [13], where we introduced both delay-independent and delay-dependent strategies for the state feedback H_∞ control of switched linear systems with delays. In particular, the delay-dependent strategy proposed therein was based on the use of the Euler formula and a standard completion of squares method, exploited to bound the cross terms appearing in the Lyapunov-Krasovskii functional derivative along the solutions of the system. Other relevant references include [14], in which we discussed state- and output-feedback delay-independent criteria, [15] and [16], containing some control design methods for this class of systems based on a small-gain approach.

A key novelty of the contributions proposed in this paper is that here we exploit a descriptor representation of delay systems, first introduced in [17] out of the context of switched systems. Compared with other approaches found in the literature, one of the advantages of such a representation is that it does not introduce additional dynamics with respect to the original system and requires a less conservative bounding of the cross-terms appearing in the derivative of the Lyapunov-Krasovskii functional along the trajectories of the system. As a result, we are able to derive switching strategies which prove less conservative with respect to other delay-dependent ones. Among them, we will specifically refer to the above mentioned delay-dependent strategy from [13]. In particular, we will provide a numerical comparison between the two approaches, by computing the maximum admissible delay under the two different design methods.

Notably, in our formulations we do not require that the modes of the system are individually stable in order to find a feasible switching law. Furthermore, this is computed as the solution of a set of matrix inequalities associated to the modes of the system and it only depends on the current value of the state. In our view, the latter property has relevant practical implications, because it only requires a finite amount of information for synthesis, compared to the infinite dimensionality of the state of the system.

The paper is organized as follows: in Section II we summarize some definitions and preliminaries; Sections III and IV address the state feedback stabilization and $\mathcal{H}_\infty$ control problems by means of switching, respectively; Section V presents two numerical examples; finally, there are some concluding remarks.

`Notation`: the identity matrix of any dimension is denoted by

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I. For real matrices or vectors, the symbol ($'$) indicates transpose. The squared norm of a signal $\xi(t)$ defined for all $t \geq 0$, denoted by $\|\xi\|_2^2$, is equal to $\int_0^\infty \xi(t)'\xi(t)dt$. Given a signal $\xi(t)$ defined for all $t \geq -h$, with $h \geq 0$, we define $\xi_t = \{\xi(t+\tau) : \tau \in [-h, 0]\}$. The set of all signals such that $\|\xi\|_2^2 < \infty$ is denoted by $\mathcal{L}_2$. For a real matrix M, the Hermitian operator $H_e\{\cdot\}$ is defined as $H_e\{M\} = M + M'$.

II. Problem formulation and preliminary results

This section is devoted to the definition of the model of the switched system under analysis and to the formulation of the control problem of interest, as well as to the presentation of some preliminary results.

A. Model and control objective

Define the following continuous-time switched linear system subject to a single time-invariant delay $h \geq 0$ acting on the state

$$\begin{aligned} \dot{x}(t) =& A_\sigma x(t) + A_{d\sigma} x(t-h) + B_\sigma d(t) \\ z(t) =& E_\sigma x(t) + E_{d\sigma} x(t-h) + F_\sigma d(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $d(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^p$ and, defining the index set $\mathcal{P} = \{1, \ldots, N\}$, $\sigma(t) : t \geq 0 \mapsto \mathcal{P}$ is the switching signal. The input $d(t)$ represents a disturbance acting on the system, while $z(t)$ is the output. The initial state is denoted by x_0. In this paper, the objective is the synthesis of state feedback switching strategies able to ensure that, for a fixed value of h,

- the equilibrium solution $x_t = 0$ of system (1) with $d(t) = 0$ is asymptotically stable;
- for any $d(t) \in \mathcal{L}_2$ and $x_0 = 0$, the following $\mathcal{H}_\infty$ disturbance attenuation property is fulfilled
$$J = \int_0^\infty (z(t)'z(t) - \gamma^2 d(t)'d(t))dt < 0 \tag{2}$$
or, equivalently, $\sup_{d(t) \neq 0 \in \mathcal{L}_2} \|z\|_2^2 / \|d\|_2^2 < \gamma^2$, where $\gamma > 0$ is a given constant.

B. Preliminary results

We now introduce some results from [11] addressing the stabilization problem of delay-free switched linear systems via state feedback switching. Notably, they are based on the use of piecewise quadratic Lyapunov functions (see [18] for an introduction). Consider the following continuous-time switched linear system with no time-delays

$$\dot{x}(t) = A_\sigma x(t) \tag{3}$$

In particular, consider the following piecewise quadratic Lyapunov function

$$v(x) = \min_{i \in \mathcal{P}} x'P_i x = \min_{\lambda \in \Lambda} \left(\sum_{i \in \mathcal{P}} \lambda_i x' P_i x \right) \tag{4}$$

where $\Lambda = \left\{ \lambda \in \mathbb{R}^N : \sum_{i \in \mathcal{P}} \lambda_i = 1, \ \lambda_i \geq 0 \right\}$ and $P_i = P_i' > 0, \ \forall i \in \mathcal{P}$. The function $v(x)$, in general, is not uniformly differentiable in $x \in \mathbb{R}^n$ as the cardinality of the set $\mathcal{I}(x) = \{i \in \mathcal{P} : \ v(x) = x'P_i x\}$ may be greater than 1, i.e., the result of minimization (4) is not unique, see [11]. As a further step, introduce the class of Metzler matrices, composed of all square matrices of fixed dimensions with nonnegative off diagonal entries. In particular, we are interested in the class $\mathcal{M}$ of Metzler matrices $\Pi \in \mathbb{R}^{N \times N}$ satisfying the following constraint, $\forall i \in \mathcal{P}$:

$$\sum_{j \in \mathcal{P}} \pi_{ji} = 0 \tag{5}$$

In other words, for any $\Pi \in \mathcal{M}$, it results $\pi_{ii} = -\sum_{j \neq i \in \mathcal{P}} \pi_{ji}, \ \forall i \in \mathcal{P}$. For simplicity, we introduce the notation $P_{Pi} = \sum_{j \in \mathcal{P}} \pi_{ji} P_j$. A fundamental property of such matrices is the following (see e.g. [11], [19]).

Lemma 1 *For any $\Pi \in \mathcal{M}$ and any $i \in \mathcal{I}(x)$, the inequality $x'P_{Pi}x \geq 0$ holds.*

Indeed, this property follows from (5) and from the fact that whenever $i \in \mathcal{I}(x)$, then $x'P_j x \geq x'P_i x$ for all $j \in \mathcal{P}$. Consider also the following result from [11], whose proof exploits Lemma 1.

Theorem 1 *Assume that there exist matrices $P_i = P_i' > 0$ and $\Pi \in \mathcal{M}$ such that, $\forall i \in \mathcal{P}$, the following Lyapunov-Metzler inequality holds*

$$H_e\{A_i' P_i\} + P_{P_i} < 0 \tag{6}$$

Then, the state feedback switching strategy

$$\sigma(x(t)) = \arg\min_{i \in \mathcal{P}} x(t)'P_i x(t) \tag{7}$$

makes the equilibrium solution $x = 0$ of system (3) *globally asymptotically stable.*

In Theorem 1, a necessary condition for the feasibility of inequality (6) is $A_i + (\pi_{ii}/2)I$ being Hurwitz, $\forall i \in \mathcal{P}$. This does not imply that the set $\{A_1, \ldots, A_N\}$ is composed of Hurwitz matrices, since $\pi_{ii} \leq 0$. On the other side, when all such matrices are Hurwitz the proposed state feedback switching strategy with $\Pi = 0$ preserves stability, since (6) reduces to the condition $A_i'P_i + P_i A_i < 0$. Finally observe that, using (7), no unstable sliding mode may occur, see [11] for more details.

Note that formula (6) is a bilinear matrix inequality (BMI), because of the presence of the Metzler matrix Π in the definition of P_{Pi}. Thus, due to non-convexity, finding a feasible solution is not trivial in general. Anyway, a simplified procedure is illustrated in [11], which consists in searching for solutions to the original problem by means of a line search procedure combined with the solution of a set of LMI's. This method adds a further degree of conservatism, but at the same time strongly reduces the computational effort in searching for feasible solutions. The same observation also holds for the criteria that are proposed in the sequel of this paper.

III. STABILIZATION

In this section we propose criteria for the stabilization of the switched time-delay system (1) by means of state-dependent switching. To this purpose, we rewrite the original system (1) in the following descriptor form, see [17]:

$$\begin{aligned}\dot{x}(t) =&\rho(t)\\ 0\times\dot{\rho}(t) =&-\rho(t)+A_\sigma x(t)+A_{d\sigma}x(t-h)+B_\sigma d(t)\\ z(t) =&E_\sigma x(t)+E_{d\sigma}x(t-h)+F_\sigma d(t)\end{aligned} \quad (8)$$

Here ρ has the same dimension as x. For any switching signal σ, formulations (1) and (8) are equivalent from the viewpoints of both stability and the input/output map from $d(t)$ to $z(t)$, see e.g. [1] and [20].

Having introduced the descriptor representation (8), we are in a position to address the design of stabilizing switching laws for system (1). The following theorem addresses this problem by exploiting the concept of piecewise quadratic Lyapunov-Krasovskii functional.

Theorem 2 *Consider system* (1) *and assume that there exist matrices* $P_{1i}=P_{1i}'>0,\ Q=Q'>0,\ T=T'>0,\ P_{2i}$, P_{3i}, $\Pi\in\mathcal{M}$ *and a scalar* $\bar{h}\geq 0$ *such that,* $\forall i\in\mathcal{P}$,

$$\Phi_{1i}(\bar{h})<0 \quad (9)$$

where

$$\Phi_{1i}(\bar{h})=\begin{bmatrix} H_e\{P_{2i}'A_i\}+Q-T+P_{P_{1i}} & \star & \star\\ P_{1i}-P_{2i}+P_{3i}'A_i & -H_e\{P_{3i}\}+\bar{h}^2T & \star\\ T+A_{di}'P_{2i} & A_{di}'P_{3i} & -Q-T\end{bmatrix} \quad (10)$$

and $P_{P_{1i}}=\sum_{j\in\mathcal{P}}\pi_{ji}P_{1j}$. *Then the state feedback switching strategy*

$$\sigma(t)=\arg\min_{i\in\mathcal{P}}x(t)'P_{1i}x(t) \quad (11)$$

makes the equilibrium solution $x_t=0$ *of system* (1) *with* $d(t)=0$ *asymptotically stable for any* $h\in[0,\bar{h}]$.

Proof: First assume $h>0$ and consider the descriptor form (8) associated to system (1) and denote by $\chi(t)=[x(t)'\ \rho(t)']'$ the state vector. Observe that the stability of system (8) with $d(t)=0$ implies the stability of system (1), see [17]. Now associate to each mode $i\in\mathcal{P}$ the Lyapunov-Krasovskii functional candidate

$$\begin{aligned}v_i(\chi_t)=&\chi(t)'EP_i\chi(t)+\int_{t-h}^{t}x(\theta)'Qx(\theta)d\theta\\ &+h\int_{-h}^{0}\int_{t+\theta}^{t}\rho(s)'T\rho(s)dsd\theta\end{aligned} \quad (12)$$

where

$$E=\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix},\ P_i=\begin{bmatrix} P_{1i} & 0\\ P_{2i} & P_{3i}\end{bmatrix},\ P_{1i}=P_{1i}'>0 \quad (13)$$

and define the following piecewise quadratic Lyapunov-Krasovskii functional candidate for system (8)

$$v(\chi_t)=\min_{i\in\mathcal{P}}v_i(\chi_t)$$

which corresponds to the switching law $\sigma(\chi_t)=\arg\min_{i\in\mathcal{P}}v_i(\chi_t)$, equivalent to (11) in view of (12) and (13). Denoting by i the active mode at time t, the upper Dini derivative of $v(\chi_t)$ along the trajectories of system (8) subject to the switching law (11) is given by

$$\begin{aligned}D^+v(\chi_t)=&\limsup_{l\to 0^+}\frac{v(\chi_{t+l})-v(\chi_t)}{l}\\ \leq&\min_{j\in\mathcal{I}(x(t))}2x(t)'P_{1j}\dot{x}(t)\\ &+x(t)'Qx(t)-x(t-h)'Qx(t-h)\\ &+h^2\dot{x}(t)'T\dot{x}(t)-h\int_{t-h}^{t}\dot{x}(\theta)'T\dot{x}(\theta)d\theta\\ =&2x(t)'P_{1i}\dot{x}(t)+x(t)'Qx(t)-x(t-h)'Qx(t-h)\\ &+h^2\dot{x}(t)'T\dot{x}(t)-h\int_{t-h}^{t}\dot{x}(\theta)'T\dot{x}(\theta)d\theta\\ \leq&2x(t)'P_{1i}\dot{x}(t)+x(t)'Qx(t)-x(t-h)'Qx(t-h)\\ &+h^2\dot{x}(t)'T\dot{x}(t)-(x(t)-x(t-h))'T(x(t)-x(t-h))\\ &+2(x(t)'P_{2i}'+\dot{x}(t)'P_{3i}')(A_ix(t)+A_{di}x(t-h)-\dot{x}(t))\end{aligned} \quad (14)$$

where the first inequality follows from Danskin's Theorem (see Theorem 1 in [21], p. 420), the second equality holds since $i\in\mathcal{I}(x(t))$, and the last inequality has been obtained by jointly exploiting the Jensen's inequality, see [17]

$$\int_{t-h}^{t}\dot{x}(\theta)'T\dot{x}(\theta)d\theta\geq\frac{1}{h}\int_{t-h}^{t}\dot{x}(\theta)'d\theta T\int_{t-h}^{t}\dot{x}(\theta)d\theta$$

and adding the term

$$2(x(t)'P_{2i}'+\dot{x}(t)'P_{3i}')(A_ix(t)+A_{di}x(t-h)-\dot{x}(t))=0$$

which is always zero in view of model (8) (with $d(t)=0$). Finally, defining $\xi(t)=[\chi(t)'\ x(t-h)']'$, inequality (14) can be equivalently rewritten as

$$D^+v(x_t)\leq\xi(t)'\Psi_i(h)\xi(t)$$

with

$$\Psi_i(h)=\begin{bmatrix} H_e\{P_{2i}'A_i\}+Q-T & \star & \star\\ P_{1i}-P_{2i}+P_{3i}'A_i & -H_e\{P_{3i}\}+h^2T & \star\\ T+A_{di}'P_{2i} & A_{di}'P_{3i} & -Q-T\end{bmatrix}$$

As a second step, observe that $x(t)'P_{1j}x(t)\geq x(t)'P_{1i}x(t)$ for all $j\in\mathcal{P}$ since $i\in\mathcal{I}(x(t))$; thus, from (9) and Lemma 1 we obtain

$$\xi(t)'\Psi_i(\bar{h})\xi(t)<\xi(t)'\begin{bmatrix} -P_{P_{1i}} & 0\\ 0 & 0\end{bmatrix}\xi(t)\leq 0 \quad (15)$$

Finally, it is easy to verify that inequality (15) holds replacing $\bar{h}$ with h, for any $h\in[0,\bar{h}]$. Therefore, assuming $h>0$, since $v(\chi_t)$ is positive definite, radially unbounded and $D^+v(\chi_t)<0$, we can complete the proof by standard arguments on Lyapunov-Krasovskii functionals (see [1] and related references). Finally, if $h=0$ the Lyapunov-Krasovskii functional (12) reduces to a function of $x(t)$, which can be easily demonstrated to be positive definite and decreasing along the trajectories of the switched system. ■

IV. $\mathcal{H}_\infty$ CONTROL

In this section, the criterion for the stabilization by switching stated in Theorem 2 will be strengthened in order to ensure, beside stabilization, the fulfillment of a disturbance attenuation requirement with respect to any admissible input $d \in \mathcal{L}_2$, as prescribed in Section II.

Theorem 3 *Consider system* (1) *and assume that there exist matrices* $P_{1i} = P'_{1i} > 0$, $Q = Q' > 0$, $T = T' > 0$, P_{2i}, P_{3i}, $\Pi \in \mathcal{M}$ *and scalars* $\bar{h} > 0$, $\gamma > 0$ *such that,* $\forall i \in \mathcal{P}$,

$$\begin{bmatrix} \Phi_{1i}(\bar{h}) & \star \\ \Phi_{2i} & \Phi_{3i}(\gamma) \end{bmatrix} < 0 \tag{16}$$

where $\Phi_{1i}(\bar{h})$ *is defined as in* (10),

$$\Phi_{2i} = \begin{bmatrix} B'_i P_{2i} & B'_i P_{3i} & 0 \\ E_i & 0 & E_{di} \end{bmatrix}$$

and

$$\Phi_{3i}(\gamma) = \begin{bmatrix} -\gamma^2 I & \star \\ F_i & -I \end{bmatrix}$$

Then the state feedback switching law

$$\sigma(t) = \arg\min_{i \in \mathcal{P}} x(t)' P_{1i} x(t)$$

makes the equilibrium $x_t = 0$ *of system* (1) *with* $d(t) = 0$ *asymptotically stable for any* $h \in [0, \bar{h}]$. *Moreover, for any* $d(t) \in \mathcal{L}_2$, $x_0 = 0$ *and any admissible* h, *the* $\mathcal{H}_\infty$ *disturbance attenuation level is upper-bounded by* γ.

Proof: The asymptotic stability property with $d(t) = 0$ is a consequence of Theorem 2, since the feasibility of (16) implies that (9) holds. As a second step, by computing the Schur complement of the bottom right element $-I$ of matrix (16), we obtain the condition

$$\Xi_i(\bar{h}) < -\begin{bmatrix} E'_i \\ 0 \\ E'_{di} \\ F'_i \end{bmatrix} \begin{bmatrix} E'_i \\ 0 \\ E'_{di} \\ F'_i \end{bmatrix}' - \begin{bmatrix} P_{P_{1i}} & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$

where

$$\Xi_i(\bar{h}) = \begin{bmatrix} H_e\{P'_{2i}A_i\} + Q - T & \star & \star & \star \\ P_{1i} - P_{2i} + P'_{3i}A_i & -H_e\{P_{3i}\} + \bar{h}^2 T & \star & \star \\ T + A'_{di}P_{2i} & A'_{di}P_{3i} & -Q - T & \star \\ B'_i P_{2i} & B'_i P_{3i} & 0 & 0 \end{bmatrix} \tag{17}$$

As a consequence, by multiplying (17) to the right by $\eta(t) = [x(t)' \ \rho(t)' \ x(t-h)' \ d(t)']'$ and to the left by its transpose, and finally computing the upper Dini derivative of the Lyapunov-Krasovskii functional (12) along the trajectories of system (8), it results

$$\begin{aligned} D^+ v(\eta_t) &\leq \eta(t)' \Xi_i(\bar{h}) \eta(t) \\ &< -z(t)'z(t) + \gamma^2 d(t)'d(t) \end{aligned}$$

Now observe that $v(x_0) = 0$ since $x_0 = 0$ by assumption and that $\lim_{t\to\infty} v(x_t) = 0$ thanks to asymptotic stability. Then, integrating the latter expression over time from 0 to ∞, we conclude that property (2) holds. ■

V. NUMERICAL EXAMPLES

In this section we discuss two numerical examples, which are applications of Theorems 2 and 3, respectively. In order to introduce them, consider system (1) with $\mathcal{P} = \{1, 2\}$, where the matrices defining the state space realization of each mode are

$$A_i = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix} \quad A_{d1} = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} \quad A_{d2} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$E_i = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad E_{di} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad F_i = 0$$

with $i = 1, 2$. Observe that the eigenvalues associated to $\bar{A}_i = A_i + A_{di}$, $i = 1, 2$, are

$$\lambda_{1,2}(\bar{A}_1) = \left\{ \sqrt{\frac{7}{4}} j, \ -\sqrt{\frac{7}{4}} j \right\}$$
$$\lambda_{1,2}(\bar{A}_2) = \{0.5, \ -2.5\}$$

which implies that the individual systems are not asymptotically stable for $h = 0$. The next examples address the stabilization and $\mathcal{H}_\infty$ disturbance attenuation problems for this system, respectively.

Example 1 ***(Stabilization)***

In order to test our result about the state feedback stabilization of system (1) *by means of switching, we compared the control strategy proposed in Theorem 2 (hereafter referred to as strategy 1) with those presented in Theorems 2 and 5 of [13] (hereafter numbered as 2 and 3). In particular, strategy 2 deals with the synthesis of delay-independent switching laws, while strategy 3 is delay-dependent, similarly to strategy 1.*

By applying these methods, it can be observed that no stabilizing switching law according to strategy 2 can be found for the system under analysis. On the other side, strategy 3 provides a solution for $h \in [0, 0.1120]$. *Finally, applying strategy 1, a stabilizing switching law was found for a considerably larger value of the delay, i.e., for* $h \in [0, 0.2646]$, *by choosing the following solution parameters:*

$$P_{11} = \begin{bmatrix} 20.868 & -10.271 \\ -10.271 & 14.063 \end{bmatrix} \quad P_{12} = \begin{bmatrix} 20.100 & -9.109 \\ -9.109 & 12.997 \end{bmatrix}$$
$$P_{21} = \begin{bmatrix} -1.406 & 35.096 \\ -2.431 & -19.434 \end{bmatrix} \quad P_{22} = \begin{bmatrix} 27.802 & -25.597 \\ -7.894 & 10.398 \end{bmatrix}$$
$$P_{31} = \begin{bmatrix} 21.386 & -6.028 \\ -18.320 & 23.291 \end{bmatrix} \quad P_{32} = \begin{bmatrix} 19.731 & -1.554 \\ 1.503 & 6.774 \end{bmatrix}$$
$$Q = \begin{bmatrix} 10.515 & 2.607 \\ 2.607 & 3.749 \end{bmatrix} \quad T = \begin{bmatrix} 16.689 & -20.321 \\ -20.321 & 97.076 \end{bmatrix}$$

with $\pi_{ji} = -\pi_{ii} = 33$, $j \neq i$. *The associated switching lines are described by* $x^{(2)} = 1.776x^{(1)}$ *and* $x^{(2)} = 0.406x^{(1)}$, *using the notation* $x = [x^{(1)} \ x^{(2)}]'$. *The state trajectories of the system for various initial conditions and* $h = \bar{h} = 0.2646$ *are reported in Figure 1. Observe that the presence of a sliding mode does not spoil the stability of the switched* *system. Finally, Figure 2 represents the time evolution of the Lyapunov-Krasovskii functional associated to the solution starting from* $x_0 = [1\ 4]'$.

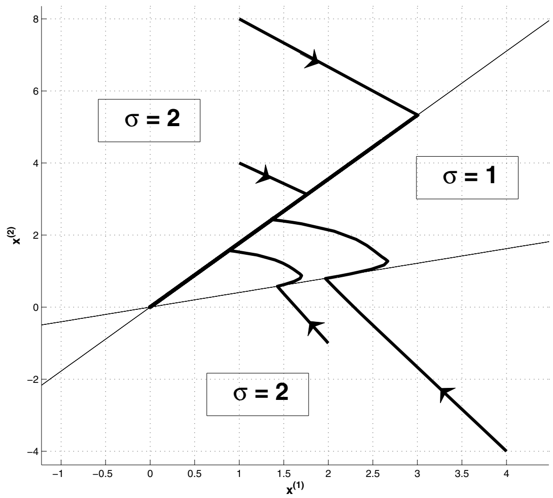

Fig. 1. Example 1: state trajectories (for various initial conditions and $h = \bar{h} = 0.2646$) and switching lines.

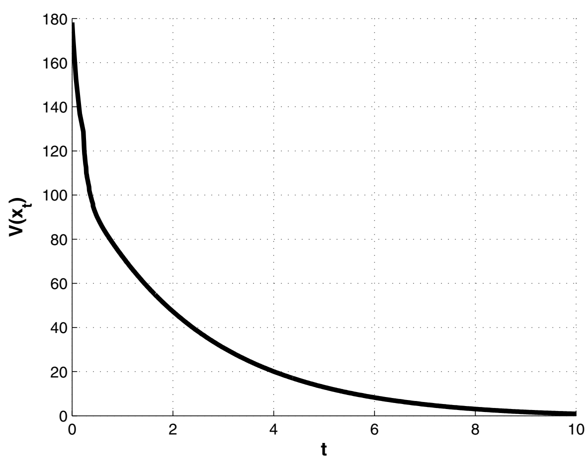

Fig. 2. Example 1: Lyapunov-Krasovskii functional with $x_0 = [1\ 4]'$ and $h = \bar{h} = 0.2646$.

Example 2 ***($\mathcal{H}_\infty$ control)***

In this example, we compare the $\mathcal{H}_\infty$ disturbance attenuation properties of the delay-dependent strategies achieved with Theorem 3 (strategy 1) and Theorem 5 in [13] (strategy 2). In particular, assuming $\bar{h} = 0.08$ and applying strategy 2, a feasible solution can be found for $\gamma \geq \gamma_{min} = 8.18$. On the other side, by applying strategy 1 under the same assumptions we obtain a solution ensuring considerably better attenuation properties, i.e., $\gamma_{min} = 3.70$. The associated set of solution parameters is reported next:

$$P_{11}=\begin{bmatrix} 14.225 & -5.146 \\ -5.146 & 5.540 \end{bmatrix} \quad P_{12}=\begin{bmatrix} 14.018 & -4.902 \\ -4.902 & 5.378 \end{bmatrix}$$

$$P_{21}=\begin{bmatrix} -0.681 & 19.674 \\ -10.314 & 0.027 \end{bmatrix} \quad P_{22}=\begin{bmatrix} 114.618 & -176.195 \\ -17.813 & 24.876 \end{bmatrix}$$

$$P_{31}=\begin{bmatrix} 11.219 & -5.128 \\ -5.169 & 15.353 \end{bmatrix} \quad P_{32}=\begin{bmatrix} 156.229 & -16.237 \\ -16.739 & 5.023 \end{bmatrix}$$

$$Q =\begin{bmatrix} 1.777 & 2.000 \\ 2.000 & 2.269 \end{bmatrix} \quad T =\begin{bmatrix} 29.376 & -68.929 \\ -68.929 & 244.616 \end{bmatrix}$$

with $\pi_{ji} = -\pi_{ii} = 116,\ j \neq i$, while the corresponding switching lines are $x^{(2)} = 2.522x^{(1)}$ and $x^{(2)} = 0.509x^{(1)}$. The disturbance attenuation property of the resulting switched system with $x_0 = [0\ 0]'$ was tested by applying the $\mathcal{L}_2$-class disturbance input $d(t) = \sin(t)e^{-0.1t}$. Figure 3 represents the quantity $J_T = \int_0^T (z(t)'z(t) - \gamma^2 d(t)'d(t))dt$, confirming that the expected $\mathcal{H}_\infty$ disturbance attenuation property is achieved in correspondence of the specific disturbance input applied to the system.

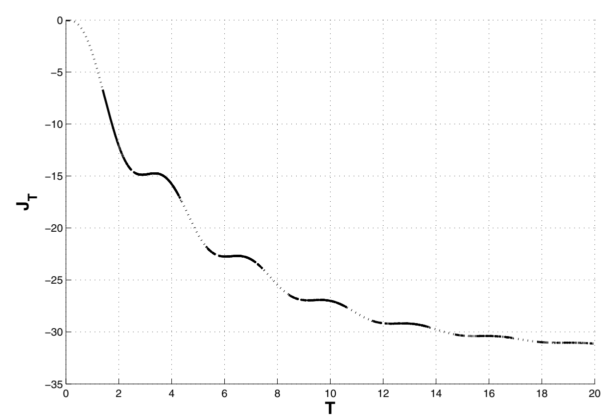

Fig. 3. Example 2: J_T (solid line: mode 1; dashed line: mode 2).

CONCLUSIONS

In this paper we have proposed an application of the descriptor approach to the stabilization and $\mathcal{H}_\infty$ disturbance attenuation of switched linear systems with a time-delay acting on the state. Our criteria are delay-dependent and based on the use of piecewise quadratic Lyapunov-Krasovskii functionals. The computation of feasible switching laws is done by solving sets of matrix inequalities. We also showed that this method is able to outperform other ones previously proposed in the literature, both in terms of maximum admissible time delay in the synthesis of stabilizing switching laws and of $\mathcal{H}_\infty$ disturbance attenuation capabilities.

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