

# Design of Distributed Controllers and Communication Topologies Considering Link Failures

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**Abstract**—This paper presents results on optimization based design of distributed control laws for linear discrete-time systems interconnected through states, inputs, and a cost function. In the proposed scheme the subsystems may communicate state information, resulting in a trade-off between good closed-loop performance and high communication costs as well as robustness issues if communication links are unreliable. Previous results on optimization based synthesis of a communication topology and associated distributed control laws are extended to the case that (i) some communication links are prone to failure and that (ii) a controller may reconfigure itself, if it is directly affected by a link failure. To include this scenario into the synthesis problem, a link failure is modeled as a change in the network topology and of the affected controllers. The problem is formulated as a mixed-integer semi-definite problem (MISDP) which combines the discrete optimization of the network topology subject to communication constraints and link failures with subproblems for structured controller synthesis.

## I. INTRODUCTION

The subject of this paper is optimization-based synthesis of distributed controllers and network topologies for linear discrete-time systems consisting of subsystems which are interconnected through states, inputs, and a common quadratic cost function. Distributed control of such systems is a challenging problem in many applications, for example in power grids and process control. Within the framework of decentralized control [1], the control of dynamically coupled systems has already been tackled to a considerable extent. There, the structural interconnection constraints are exploited to obtain a decomposition of the overall system. While this allows designing decentralized control laws, the resulting closed loop performance is not considered. For instance, even if a stabilizing decentralized controller exists, performance might be significantly degraded compared to a centralized approach.

Digital communication networks allow for flexible communication between subsystems and for potentially higher performance by exchanging state information between subsystems. The resulting distributed control laws can be used even if no stabilizing decentralized control law exists and typically offer a better performance. On the other hand, this leads to additional communication costs, and the network may induce phenomena such as time-varying delays and packet dropouts. Much of the work in the area of networked control systems (NCS) is focused on these effects for controllers connected to a plant through a communication

network (e.g. [2]). Network topologies are rarely considered in this context, however.

Most of the work on distributed control assumes certain fixed communication topologies, such as communication only between neighbors. In [3], the distributed control of identical and dynamically decoupled systems is covered, and dual decomposition is used for distributed optimization of local controllers in [4]. However, the flexibility of digital communication networks with regard to the topology of the network results in an additional degree of freedom in the controller design. At the same time communication between subsystems also leads to additional costs, such as energy consumption and hardware costs. In addition, communication links may be unreliable or fail. This leads to the problem of finding a topology and associated distributed control laws with good trade-off between communication costs and closed-loop performance, as well as robustness to persistent or intermittent failures of some communication links.

Results on optimal topology design for special cases, such as average-consensus [5], and  $\mathcal{H}_2$ -control of locally controlled systems coupled through outputs [6] have been proposed. In [7], a convex parametrization for optimizing frequency domain controllers subject to structural constraints is proposed. However, for unstable plants a stabilizing controller that satisfies the structural constraints has to be known a priori. Thus, a controller that satisfies all possible structural constraints would have to be a decentralized controller, which often does not exist.

In [8], a method is proposed to simultaneously optimize a distributed control law and its communication topology for linear discrete-time systems which are interconnected through states, inputs, and a performance criterion. Furthermore, the communication topology introduces structural constraints on the controller, resulting in a non-convex controller synthesis problem [9]. To solve this problem, a mixed integer semi-definite programming (MISDP) approach is developed which optimizes distributed controllers and the corresponding communication topology with respect to quadratic infinite horizon cost of the closed-loop system as well as communication costs and constraints. This method does not require the existence of a stabilizing decentralized controller. An iterative LMI approach [10] is used in [11] to design static  $\mathcal{H}_\infty$  controllers with a minimized number of communication links. However, constraints on the communication topology or link failures are not considered.

In this work we, extend our previous method to the case that some communication links are prone to failure. A possible solution to this problem may be to exclude all

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the error-prone links from the set of admissible network topologies a-priori [8]. However, the solution of never using an error-prone link at any time (even if it remains faultless) may clearly not be advantageous, with respect to the overall performance: Regarding the global cost of the closed-loop system, such a control law can perform worse than a control law that makes use of those links as long as they are available, which is our motivation for the presented work.

In the proposed scheme, a subsystem which is directly affected by a link failure can reconfigure its own local controller by switching to different precomputed control laws. Thus the scheme can adapt to a failure of an incoming link in a distributed fashion without requiring global knowledge. The stability properties in the case of intermittent link failures are investigated in a discrete-time average dwell time framework [12]. In Sec. II, we will briefly summarize the results obtained in [8] and present an extension to deal with persistent failure of some communication links. In combination with a Big-M reformulation of the constraints resulting from the network topology, an MISDP is obtained which can be solved by branch and bound techniques.

## II. PROBLEM FORMULATION

### A. System Dynamics and Performance Criterion

The considered class of systems is one where the global system consists of a set of  $N$  linear discrete-time subsystems that are coupled through their states, the inputs, and a performance criterion. The dynamics of the global system is defined by the following difference equation:

$$x_{k+1} = Ax_k + Bu_k, \quad (1)$$

where  $x_k \in \mathbb{R}^{n_g}$  and  $u_k \in \mathbb{R}^{m_g}$  are the global states and inputs obtained by collecting the local states  $x_k^{(i)} \in \mathbb{R}^{n_i}$  and inputs  $u_k^{(i)} \in \mathbb{R}^{m_i} \forall i \in \mathcal{N}$ , where  $\mathcal{N} := \{1, \dots, N\}$ . The matrices  $A$  and  $B$  can be partitioned according to the states and inputs of the subsystems:

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \cdots & A_{N,N} \end{bmatrix}, B = \begin{bmatrix} B_{1,1} & \cdots & B_{1,N} \\ \vdots & \ddots & \vdots \\ B_{N,1} & \cdots & B_{N,N} \end{bmatrix}, \quad (2)$$

where  $A_{i,j} \in \mathbb{R}^{n_i \times n_j}$ ,  $B_{i,j} \in \mathbb{R}^{n_i \times m_j} \forall (i,j) \in \mathcal{N} \times \mathcal{N}$ . The aim of this work is to compute a set of control laws with gain matrices  $K^t \in \mathbb{R}^{m_g \times n_g}$  depending on different communication topologies  $t \in \{0, \dots, N_i\}$ :

$$u_k = \underbrace{\begin{bmatrix} K_{1,1}^t & \cdots & K_{1,N}^t \\ \vdots & \ddots & \vdots \\ K_{N,1}^t & \cdots & K_{N,N}^t \end{bmatrix}}_{:=K^t} x_k, \quad (3)$$

where  $K_{i,j}^t$  denotes the feedback from subsystem  $j$  to subsystem  $i$  in topology  $t$ . As performance measure for the closed-loop system, consider an optimal infinite horizon quadratic cost function for the discrete-time dynamics (1), with (3):

$$V^t(x_k) = \min_{K^t} \sum_{i=0}^{\infty} x_{k+i}^T (Q + (K^t)^T R K^t) x_{k+i}. \quad (4)$$

Here,  $Q = Q^T > 0 \in \mathbb{R}^{n_g \times n_g}$  and  $R = R^T > 0 \in \mathbb{R}^{m_g \times m_g}$  are symmetric positive definite weighting matrices for the states and control inputs, which may be used to formulate coupling through the cost function. The optimal infinite horizon cost function is a quadratic function, such that:

$$V^t(x_k) = x_k^T P^t x_k, \quad (5)$$

where  $P^t = (P^t)^T > 0 \in \mathbb{R}^{n_g \times n_g}$ . The  $P^t$  and  $K^t$  which are optimal with respect to (4) can be obtained by the following minimization problem (cf. [8], Prop. 1):

$$\min_{P^t, K^t} \text{trace}(P^t), \quad \text{s.t. } P^t = (P^t)^T > 0, \quad (6)$$

$$P^t - (A + BK^t)^T P^t (A + BK^t) - Q - (K^t)^T R K^t \geq 0. \quad (7)$$

### B. Communication Topology

The structure of the control system consisting of the communication network, the interconnected subsystems, and the distributed controllers is shown in Fig. 1. We consider the case that the network topology can be chosen within constraints and only some links are subject to, possibly persistent, failures. In a system which consists of many systems a link failure can usually only be detected by few subsystems. Hence, the goal of this work is to reconfigure the local controllers in a distributed fashion if a communication link fails, in order to continue operation with only moderately degraded performance. If, for instance, a subsystem no longer receives information from another subsystem because of a link failure, it switches to a different precomputed control law. In contrast, the control laws of unaffected subsystems remain unchanged. A communication topology  $t$  is represented by a matrix  $D^t \in \mathcal{D}$ , where  $\mathcal{D} \subseteq \{0, 1\}^{N \times N}$  is the set of admissible network topologies. The set can be used to define constraints on the network topology, see [8]. The matrix  $D^t$  has the following structure:

$$D^t = \begin{bmatrix} \delta_{1,1}^t & \cdots & \delta_{1,N}^t \\ \vdots & \ddots & \vdots \\ \delta_{N,1}^t & \cdots & \delta_{N,N}^t \end{bmatrix}, \quad (8)$$

where the boolean entries  $\delta_{i,j}^t \in \{0, 1\}$  indicate whether or not information is communicated from subsystem  $j$  to the controller of subsystem  $i$ . The topology without link

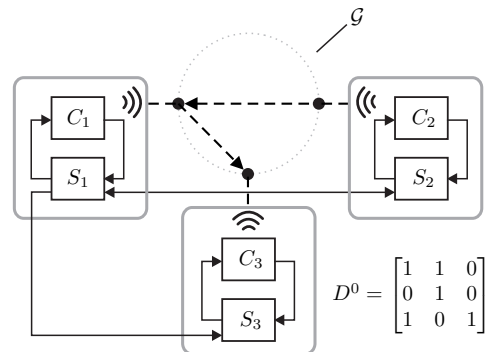


Fig. 1. Example of a graph  $\mathcal{G}$  of a communication network (dashed), interconnected local subsystems  $S_i$  and local controllers  $C_i$ .

failures is denoted by  $D^t$ ,  $t = 0$ . In order to consider link failures, new topologies  $D^t \neq D^0$ ,  $t \in \mathcal{T} \setminus 0$  are introduced, in which some links are disabled due to failure. Herein,  $\mathcal{T} := \{0, \dots, N_t\}$  denotes the index set of different topologies and  $N_t$  is the number of topologies arising from possible combinations of single link failures.

**Remark 1** *All possible communication topologies which satisfy the structural constraints defined by  $\mathcal{D}$  are considered in the optimization-based synthesis presented in the next section. Since the number of admissible topologies grows exponentially, the problem is hard to solve in general. The set  $\mathcal{D}$  may be used to restrict the synthesis to only certain configurations, for example by excluding topologies which are not structurally controllable.*

Consider a switching function  $\sigma(k) : \mathbb{N}_0 \rightarrow \mathcal{T}$ , which selects a topology  $t$  for each sample time  $k$ . As the switching is induced by faults in the communication network, arbitrary switching sequences and points of switching times are considered. In the following, the topology  $D^0$  and controllers for all topologies  $t$  are computed offline. Constraints on the controllers ensure that controllers can be reconfigured locally if a link failure is detected. Online the reconfiguration is performed as follows: In the fault free case  $D^0$ , a nominal controller  $K^0$  is used. If one or more links fail, the network topology switches to some  $D^{t_1}$  and certain links  $\delta_{i,j}^{t_1}$  become inactive. In this case, only the local controller of a subsystem  $i$  may be able to detect this link failure and it is reconfigured by switching from  $K_{i,j}^0$  to  $K_{i,j}^{t_1}$ . Subsequently, further switching may be induced by additional link failures or by links becoming active again. The scheme proceeds accordingly by locally switching to controller gains which correspond to a new topology  $D^{t_2}$ ,  $t_2 \neq t_1$ . This scheme is repeated at every switching instant. Note, that the topologies are not known a-priori, but they result from the synthesis of the fault free topology  $D^0$ . In Sec. IV, we will analyze stability and performance of such a scheme, in which  $\sigma(k)$  satisfies an average dwell time condition. In order to formally define the conditions for such a decentralized reconfiguration, we require the following assumption and definitions.

**Assumption 1** *It is assumed that a subset of the admissible communication links are subject to failure. This subset can be chosen to include any combination of link failures, what may result in a very large problem, or to include only certain link failures and combinations thereof. Furthermore, it is assumed that each subsystem  $i$  can locally detect failures of incoming links  $\delta_{i,j}^t$  for all  $j \in \mathcal{N} \setminus i$  and can switch to a different precomputed control law  $K_{i,j}^t$ .*

Let  $(i, j) \in \mathcal{F}^t \subseteq (\mathcal{N} \times \mathcal{N}) \setminus (i, i)$  denote the set of links which cannot be used because of structural constraints ( $\mathcal{F}^0$ ) or due to link failure in a topology  $t \neq 0$  ( $\mathcal{F}^t$ ). Then the boolean entries  $\delta_{i,j}^0 \in \{1, 0\}$  of  $D^0$  are given by:

$$\delta_{i,j}^0 := \begin{cases} 0 & \text{if } (i, j) \in \mathcal{F}^0 \text{ or } j \notin \mathcal{N}_i \\ 1 & \text{if otherwise} \end{cases}, \quad (9)$$

where  $\mathcal{N}_i$  is the index set of subsystems  $j \in \mathcal{N} \setminus i$  that transmit information to the  $i$ -th subsystem. In other words  $\delta_{i,j}^0$  is only a decision variable of the subsequent optimization if  $(i, j) \notin \mathcal{F}^0$ . The boolean entries  $\delta_{i,j}^t \in \{1, 0\}$  of the topologies subject to failures are given by:

$$\delta_{i,j}^t := \begin{cases} 0 & \text{if } (i, j) \in \mathcal{F}^t \\ \delta_{i,j}^0 & \text{otherwise} \end{cases}, \quad \forall t \in \mathcal{T} \setminus 0. \quad (10)$$

Based on the boolean variables  $\delta_{i,j}^t$  and the controllers  $K^t$  the following conditions ensure that the controller satisfies the structural constraints and that the controller of the  $i$ -th subsystem is the same for the topologies  $t_1$  and  $t_2$  if subsystem  $i$  is subject to the same faults in  $t_1$  and  $t_2$ .

$$(\delta_{i,j}^t = 0) \Rightarrow (K_{i,j}^t = 0), \quad \forall t \in \mathcal{T} \quad (11)$$

$$(\delta_{i,j}^{t_1} = \delta_{i,j}^{t_2}, \forall j) \Rightarrow (K_{i,j}^{t_1} = K_{i,j}^{t_2}, \forall j), \quad \forall (t_1, t_2) \in \mathcal{T} \times \mathcal{T}. \quad (12)$$

This allows for a reconfiguration of the controller, which only requires local information. Finally, we propose a communication cost function with weights  $c_{i,j} \geq 0$  and  $c_{i,i} = 0$ ,  $\forall i$  for a communication link between the  $j$ -th and  $i$ -th subsystem, such that:

$$J_{com} = \sum_{i=1}^N \sum_{j=1}^N c_{i,j} \delta_{i,j}^0. \quad (13)$$

The costs  $c_{i,j}$  associated with a communication link could, for example, be chosen based on the hardware costs, energy consumption, or the distance between subsystems.

### C. Optimization of Performance and Communication Cost

Based on the performance criterion (4), (7) and the communication cost (13), the following optimization problem is obtained:

$$\begin{aligned} \min_{P^t, K^t, D^t} & \sum_{t=0}^{N_t} w^t \text{trace}(P^t) + \sum_{i=1}^N \sum_{j=1}^N c_{i,j} \delta_{i,j}^0 \\ \text{s.t.} & (6), (7), (11), (12), D^t \in \mathcal{D} \quad \forall t \in \mathcal{T}, \end{aligned} \quad (14)$$

with weighting factor  $w^t > 0$  and  $\sum_{t=0}^{N_t} w^t = 1$ , which allows to achieve a trade-off between the nominal performance and the performance in case of link failures. This problem involves the BMI constraint (7) on  $P^t$  and  $K^t$  as well as the logical implication constraints (11) and (12) on  $K^t$  - thus, this is a non-convex problem. The latter constraints can be formulated in a MIP framework. Solutions to MIP problems are obtained by iterative procedures, for example branch and bound algorithms which solve relaxed subproblems to obtain lower bounds on the cost, fix the integer variables which model the topology  $D^0$  and, if an integer feasible solution is found, solve the original problem to obtain an upper bound on the cost. However, BMI solvers are relatively inefficient and provide only locally optimal solutions. Hence, both the integer constraints as well as the BMI constraints have to be relaxed to obtain a lower bound, which is often relatively conservative. If the integer variables are fixed, (14) becomes a structured controller synthesis problem, for which iterative LMI methods have been proposed. However, these methods in general either require a feasible initial solution or do

not guarantee convergence to a feasible solution; also the solutions may not even be locally optimal [10].

Therefore, sufficient and convex conditions were derived in [8] for the case without link failures (i.e.  $t = N_t = 0$ ). Based on an extended LMI parametrization, similar to the well-known approach in [13], the following sufficient LMI conditions for the BMI constraint (7) are used to reformulate (14) into a MISDP.

**Theorem 1** [8] *The non-convex constraint (7) holds with  $P = Y^{-1}$  and  $K = LG^{-1}$  if there exists  $G \in \mathbb{R}^{n_g \times n_g}$ ,  $Y = Y^T > 0 \in \mathbb{R}^{n_g \times n_g}$  and  $L \in \mathbb{R}^{m_g \times n_g}$ , such that the following LMI holds:*

$$\begin{bmatrix} G + G^T - Y & (AG + BL)^T & G^T & L^T \\ AG + BL & Y & 0 & 0 \\ G & 0 & Q^{-1} & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} > 0, \quad (15)$$

where 0 denotes zero matrices of appropriate dimensions.

To ensure that the structural constraints (11) on the controller  $K = LG^{-1}$  are satisfied constraints on  $L$  and  $G$  can be formulated such that  $L = KG$  has a solution  $K$  which complies with  $D \in \mathcal{D}$ . By partitioning  $L$  and  $G$  such that they are compatible with the structure of  $K$  and applying the so called Big-M method [14], the following sufficient conditions for the structural constraint (11) are derived in [8] for a sufficiently large number  $M$  and all  $(i, j) \in \mathcal{N} \times \mathcal{N}$ :

$$-M\delta_{i,j} \leq L_{i,j} \leq M\delta_{i,j} \quad (16)$$

$$-M\delta_{i,j} \leq G_{i,j} \leq M\delta_{i,j} \quad (17)$$

and for all  $(i, j) \in \mathcal{N} \times \mathcal{N}$  and  $z \in \mathcal{N} \setminus \{i, j\}$ :

$$-M(\delta_{i,j} - \delta_{i,z} + 1) \leq G_{z,j} \leq M(\delta_{i,j} - \delta_{i,z} + 1) \quad (18)$$

**Theorem 2** [8] *Suppose the LMI (15) holds for  $L$  and  $G$  subject to (16). Then the controller  $K = LG^{-1}$  satisfies the structural constraint (11) imposed by the communication topology.*

Since the constraints (16) are only sufficient but not necessary, the feasible set is reduced compared to the original problem (14). Thus, the reformulated optimization problem minimizes an upper bound on the original cost in (14).

### III. OPTIMIZATION BASED SYNTHESIS CONSIDERING LINK FAILURE

Based on Theorem 1 and 2, the optimization problem (14), the BMI constraint and structural constraints are reformulated. In order to reformulate the constraint (12), the following variables are introduced:

$$s_{i,j}^{t_1, t_2} := \begin{cases} \delta_{i,j}^{t_2} & \text{if } (i, j) \in \mathcal{F}^{t_1} \\ \delta_{i,j}^{t_1} - \delta_{i,j}^{t_2} & \text{otherwise} \end{cases}. \quad (19)$$

Based on  $s_{i,j}^{t_1, t_2}$  the constraint (12) can be reformulated, resulting in the following for all  $i, j$  and  $(t_1, t_2) \in \mathcal{T} \times \mathcal{T}$ :

$$-M \sum_{r=1}^N s_{i,r}^{t_1, t_2} \leq L_{i,j}^{t_1} - L_{i,j}^{t_2} \leq M \sum_{r=1}^N s_{i,r}^{t_1, t_2}, \quad (20)$$

where  $L_{i,j}^t$  may be interpreted as convexified version of the feedback  $K_{i,j}^t$  in (3), i.e. the feedback from subsystem  $j$  to  $i$  in topology  $t$ . This ensures that  $L_{i,j}^{t_1} = L_{i,j}^{t_2}$  for all  $j$ , if subsystem  $i$  is subject to the same faults in  $t_1$  and  $t_2$  (i.e. if  $\delta_{i,j}^{t_1} = \delta_{i,j}^{t_2}, \forall j$ ). In contrast, if there exists  $j$  such that  $\delta_{i,j}^{t_1} \neq \delta_{i,j}^{t_2}$ , then (20) is relaxed because of  $s_{i,j}^{t_1, t_2} = 1$ . Thus, a different controller for subsystem  $i$  may be obtained for  $t_1$  or  $t_2$  if the subsystem  $i$  is not affected by the same faults in both topologies.

Applying these reformulations to (14) leads to the following MISDP, which minimizes the sum of the bound on the quadratic infinite horizon cost and the communication cost.

$$\min_{\hat{P}^t, L^t, D^t, Y^t, G} \sum_{t=0}^{N_t} w^t \text{trace}(\hat{P}^t) + \sum_{i=1}^N \sum_{j=1}^N c_{i,j} \delta_{i,j}^0 \quad (21)$$

s.t.

$$\begin{bmatrix} G + G^T - Y^t & (AG + BL^t)^T & G^T & L^{tT} \\ AG + BL^t & Y^t & 0 & 0 \\ G & 0 & Q^{-1} & 0 \\ L^t & 0 & 0 & R^{-1} \end{bmatrix} > 0, \quad \forall t$$

$$\begin{bmatrix} \hat{P}^t & I_{n_g} \\ I_{n_g} & Y^t \end{bmatrix} \geq 0, \quad D^t \in \mathcal{D}, \quad \forall t$$

$$-M\delta_{i,j}^t \leq L_{i,j}^t \leq M\delta_{i,j}^t, \quad \forall i, j, t$$

$$-M\delta_{i,j}^t \leq G_{i,j} \leq M\delta_{i,j}^t, \quad \forall i, j, t$$

$$-M(\delta_{i,j}^t - \delta_{i,z}^t + 1) \leq G_{z,j} \leq M(\delta_{i,j}^t - \delta_{i,z}^t + 1), \quad \forall i, j, z, t$$

$$-M \sum_{r=1}^N s_{i,r}^{t_1, t_2} \leq L_{i,j}^{t_1} - L_{i,j}^{t_2} \leq M \sum_{r=1}^N s_{i,r}^{t_1, t_2}, \quad \forall i, j, t_1, t_2.$$

Note that in general the optimizer of (21) is not unique, i.e. different solutions may achieve the same minimal cost. The first two constraints are used to ensure stability and closed-loop performance. The constraints on  $L_{i,j}^t$  and  $G_{i,j}^t$  are used to formulate that the controller  $K^t = L^t G^{-1}$  has the desired structure, i.e.  $(\delta_{i,j}^t = 0) \implies (K_{i,j}^t = 0)$  (for details see [8]). The last constraint models that the controller can be reconfigured in a decentralized fashion.

**Remark 2** *This formulation includes auxiliary and integer variables for each network topology that may arise due to link failures. However, the variables  $s_{i,j}^{t_1, t_2}$  as well as  $\delta_{i,j}^t$ , for all  $t \neq 0$  can be eliminated by using (10), (19),  $\delta_{i,j}^0$  and the sets  $\mathcal{F}^t$ . This results in an MISDP with  $N^2 - N - |\mathcal{F}^0|$  binary decision variables.*

**Remark 3** *Alternatively, one may optimize over the worst case closed-loop performance by letting  $\hat{P}^t := \hat{P}$  and  $w^t := N_t^{-1}$  for all  $t$ .*

**Theorem 3** *Suppose the constraints in (21) hold. Then, each controller  $K^t = L^t G^{-1}$ :*

- 1) *satisfies the structural constraint (11) imposed by the communication topology,*
- 2) *asymptotically stabilizes the global system (1),*
- 3) *satisfies (12) which ensures that the local controllers can be reconfigured in a distributed fashion.*

*Proof:* Statement 1) is a direct consequence of Theorem 2 applied to each topology  $t$ . To establish 2), note that the cost function (5) can be used as Lyapunov function candidate  $V^{t^*}(x_k)$  according to well known results from LQ theory. Furthermore, condition (7) ensures that  $V^{t^*}(x_k) - V^{t^*}(x_{k+1}) > 0$ , i.e. the cost is decreasing along the closed-loop trajectories for every  $x_k$  and every topology  $t$ . By Theorem 1 the LMI (21) ensures that (7) is satisfied. Thus, each controller  $K^t$  asymptotically stabilizes the system. Finally, if subsystem  $i$  is affected by the same faults in the topologies  $t_1$  and  $t_2$ , it holds that  $\delta_{i,j}^{t_1} = \delta_{i,j}^{t_2} \forall j$ , and (20) enforces  $L_{i,j}^{t_1} = L_{i,j}^{t_2} \forall j$ . By partitioning  $L^t = K^t G$  into blocks corresponding to each subsystem, it follows that:

$$[L_{i,j}^{t_1}, \dots, L_{i,j}^{t_1}] = [K_{i,j}^{t_1}, \dots, K_{i,j}^{t_1}] G, \quad (22)$$

$$[L_{i,j}^{t_2}, \dots, L_{i,j}^{t_2}] = [K_{i,j}^{t_2}, \dots, K_{i,j}^{t_2}] G. \quad (23)$$

Due to the first constraint in (21) the matrix  $G$  always has full rank (c.f. [8]), and it follows that  $K_{i,j}^{t_1} = K_{i,j}^{t_2} \forall j$ , if  $L_{i,j}^{t_1} = L_{i,j}^{t_2} \forall j$ . Hence (12) is satisfied. ■

Controllers for each subsystem and each topology arising from link failures can be computed by solving the MISDP (21). If a subsystem detects a link failure it switches to the corresponding controller which does not require the information transmitted via the failed communication link.

#### IV. STABILITY AND PERFORMANCE ANALYSIS

In case of persistent link failures, only a finite number of switchings can occur and a stabilizing controller will be used afterwards, thus the system is stabilized. To gain more insight into the problem and to consider intermittent link failures the stability properties of switched discrete-time systems can be analyzed in the average dwell time framework [15]. Let  $N_{\sigma(k)}(k_s, k_e)$  denote the number of discontinuities  $\sigma(k)$  over the interval  $[k_s, k_e]$ . Then  $\sigma(k)$  is said to have average dwell time  $\tau_a$  if:

$$N_{\sigma(k)}(k_s, k_e) \leq N_0 + \frac{k_e - k_s}{\tau_a}, \quad (24)$$

where  $N_0 > 0$  is the number of switches which may occur on any interval shorter than  $\tau_a$ . Here we will consider the case  $N_0 = 0$ . This means that the average time between consecutive switches is at least  $\tau_a$ . This directly leads to specifications for the underlying communication network in the sense of an average loss rate or average time between failures. These specifications have to be guaranteed by the underlying communication network.

To compute a  $\tau_a$  which ensures asymptotic stability, we require the following Theorem for discrete-time switched systems from [12]:

**Theorem 4** [12] *Consider the discrete-time switched system  $x_{k+1} = f_{\sigma(k)}(x_k)$ ,  $\sigma(k) \in \mathcal{T}$  and let  $0 < \alpha < 1$ ,  $\mu > 1$  be given constants. Suppose there exists  $C^1$  functions  $V_{\sigma(k)} : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\sigma(k) \in \mathcal{T}$  and two class  $\mathcal{K}_\infty$  functions  $\beta_1$  and  $\beta_2$  such that  $\forall \sigma(k) = i \in \mathcal{T}$ ,*

$$\beta_1(\|x\|) \geq V_i(x) \geq \beta_2(\|x\|) \quad (25)$$

$$\Delta V_i(x) \leq -\alpha V_i(x) \quad (26)$$

and  $\forall (\sigma(k_l) = i, \sigma(k_l - 1) = j) \in \mathcal{T} \times \mathcal{T}, i \neq j$ ,

$$V_i(x_{k_l}) \leq \mu V_j(x_{k_l}), \quad (27)$$

then the system is globally asymptotically stable for any switching signals with the average dwell time

$$\tau_a \geq \tau_a^* = -\frac{\ln(\mu)}{\ln(1-\alpha)}. \quad (28)$$

Based on this theorem, we can compute how often (on average) the controller may switch between arbitrary topologies. Because the optimal cost-functions are quadratic, (25) is trivially satisfied. The constants  $\mu > 1$  and  $0 < \alpha < 1$  can be computed by solving

$$\min \mu^{t_1, t_2}, \text{ s.t. } V^{t_1}(x_k) \leq \mu^{t_1, t_2} V^{t_2}(x_k), \quad (29)$$

$$\max \alpha^t, \text{ s.t. } V^t(x_k) - V^t(x_{k+1}) \geq \alpha^t V^t(x_k), \quad (30)$$

for all  $(t_1, t_2) \in \mathcal{T} \times \mathcal{T}$  and all  $t \in \mathcal{T}$  after the synthesis problem (21) has been solved.

**Remark 4** *Some specifications for the design of the communication network may be obtained by checking the cost increase  $\mu^{t_1, t_2}$  when switching from  $t_1$  to  $t_2$  as well as the decay rate  $\alpha^{t_2}$ . For instance if  $\ln(\mu^{t_1, t_2})$  is large for some pair  $(t_1, t_2)$  compared to all other cases, then the links which are available in  $t_1$  but not in  $t_2$  should be made more reliable.*

Based on  $\mu = \max \mu^{t_1, t_2}$ ,  $\alpha = \min \alpha^t$ , a lower bound on the average time between consecutive switching can be computed by (28) which ensures asymptotic stability of the switched systems. However, this does not guarantee good performance. In particular the performance of the switched controller for small  $\tau_a \geq \tau_a^*$  may be worse than that of a time-invariant controller which never uses links that may fail (i.e. the controller for the worst-case topology  $t_w$ ). To analyze the performance of the switched control law, we require various bounds on the cost functions, which are shown in Fig. 2. Based on (30) the following upper bound on the worst-case cost is obtained:

$$V^{t_w}(x_k) \leq (1 - \alpha^{t_w})^k V^{t_w}(x_0), \quad (31)$$

where  $\alpha^{t_w} = \alpha$ . This is the best performance that can be guaranteed by average dwell time for the considered setting, since the system may be in the topology  $t_w$  at  $k = 0$  and never switches if  $\tau_a \rightarrow \infty$ . In order to bound the possible performance loss due to switching, a desired upper bound  $\hat{V}(x_k) := (1 - \alpha^d)^k V^{t_w}(x_0)$ , with  $\alpha^d \leq \alpha^{t_w}$  can be chosen and a corresponding  $\tau_a$  can be computed.

**Theorem 5** *Given  $0 \leq \alpha^d \leq \alpha^{t_w}$ , the cost function of the switched control law is bounded by*

$$V^0(x_k) \leq V_{\sigma(k)}(x_k) \leq (1 - \alpha^d)^k V_{\sigma(0)}(x_0) \quad (32)$$

if (24) holds for  $N_0 = 0$  and

$$\tau_a \geq \tau_a^* = \frac{\ln(\mu)}{\ln(1 - \alpha^d) - \ln(1 - \alpha)}. \quad (33)$$

*Proof:* The lower bound trivially follows if all links are available for all  $k \geq 0$ , i.e. no switching occurs and the

nominal controller is used. To establish the upper bound, note that  $V^{\sigma(k)}(x_k)$  can be bounded as follows based on (24) and (30):

$$V^{\sigma(k)}(x_k) \leq \mu^{N_{\sigma(k)}(k,0)} (1 - \alpha)^k V^{\sigma(0)}(x_0). \quad (34)$$

Since we are interested in the case where the average time between consecutive switches is at least  $\tau_a$  the case  $N_0 = 0$  is considered here, this results in:

$$V^{\sigma(k)}(x_k) \leq \left( \mu^{\frac{1}{\tau_a}} (1 - \alpha) \right)^k V^{\sigma(0)}(x_0). \quad (35)$$

Comparing (35) and (32) to guarantee the decay rate  $\alpha^d$  it is required that:

$$\mu^{\frac{1}{\tau_a}} (1 - \alpha) = (1 - \alpha^d). \quad (36)$$

Solving (36) for  $\tau_a$  leads to the condition (33). ■

This result shows that in a worst-case scenario, the performance of the switched control law  $K^{\sigma(k)}$  may be inferior to the performance of a time-invariant controller which only uses reliable links. However, by switching the control law based on the available links, performance might be improved up to the bound  $V^0(x_k) \leq V^{\sigma(k)}(x_k)$  if more links are available. Based on Theorem 5 an average dwell-time  $\tau_a$  can be computed, such that the performance loss in the worst-case is acceptable, while allowing for improved performance by using failure prone links when they are available. For example consider the following scenarios in which the worst-case topology is active at  $k = 0$ :

- All links may become available again after a short time without further failures, in this case  $K^{\sigma(k)}$  recovers the nominal performance, while using  $K^{t_w}$  always yields the same, possibly low, performance.
- The topology may switch (with average dwell time according to (33)) between topologies with relatively low control performance. In this case  $K^{\sigma(k)}$  results in performance close to the bound  $\hat{V}(x_k)$ , while  $K^{t_w}$

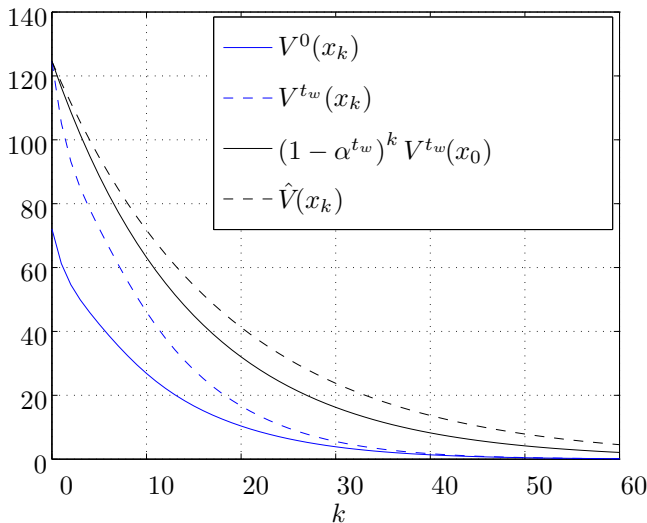


Fig. 2. Cost  $V^0(x_k)$  in the fault free topology, worst case topology  $V^{t_w}(x_k)$  and bounds based on  $\alpha^{t_w}$  and  $\alpha^d$ .

may offer better performance. Theorem 5 quantifies the performance loss for this worst-case scenario in terms of the average dwell time.

## V. SIMULATION RESULTS

To illustrate the proposed method, consider the following system [8]:

$$\begin{bmatrix} x_{k+1}^{(1)} \\ x_{k+1}^{(2)} \\ x_{k+1}^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0.1 & 0.1 & 1 & 0.1 & 0 & 0.1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 1 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 & 0 \\ 0.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} u_k. \quad (37)$$

Here,  $x_k^{(i)} \in \mathbb{R}^2$ ,  $u_k^{(i)} \in \mathbb{R}^1 \forall i = \{1, \dots, 3\}$  are the local states and inputs, and the second subsystem is interconnected with the first and third subsystem. The weighting matrices for the infinite horizon cost function are chosen as follows and introduce further coupling between the first and third subsystem:

$$Q = \begin{bmatrix} 1 & 0.1 & 0 & 0 & 0.2 & 0 \\ 0.1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 1 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}. \quad (38)$$

Here, we consider the case  $N_f = 2$ , i.e. at most two links may fail. This results in  $L = 3$  and in the following index sets  $\mathcal{F}^1 = \{(1, 2)\}$ ,  $\mathcal{F}^2 = \{(3, 1)\}$ ,  $\mathcal{F}^3 = \{(1, 2), (3, 1)\}$ . Thus we obtain four different topologies with weights  $w^0 = 0.9$  and  $w^1 = w^2 = w^3 = \frac{1}{30}$  chosen to prefer nominal performance. The communication links which are subject to failure are less expensive:

$$c = \begin{bmatrix} 0 & 1 & 5 \\ 5 & 0 & 5 \\ 1 & 5 & 0 \end{bmatrix}. \quad (39)$$

The MISDP was solved using the Matlab toolbox YALMIP [16] using branch and bound, and the solver SeDuMi for the semi-definite subproblems. The computation time for this example is around 28 sec on an Intel Core i7-2620M. With respect to the computational complexity of the problem, two main factors should be considered. The number of integer variables encoding the nominal topology  $D^0$  grows with the number of subsystems but does not depend on the number of link failures (cf. Remark 2). The branch and bound solver used here is rather inefficient with respect to the number of nodes of the search tree which are explored before convergence (cf. [17] for more efficient techniques). Another aspect is the complexity of problem (21) when the binary variables are either fixed or relaxed to the interval  $[0, 1]$ . If the overall system is large, a large number of structural inequality constraints on  $L_{i,j}^t$ ,  $G_{i,j}$  may arise. Since the LMI is in dual SDP form, this results in a large number of slack variables in the resulting SDP. In particular,

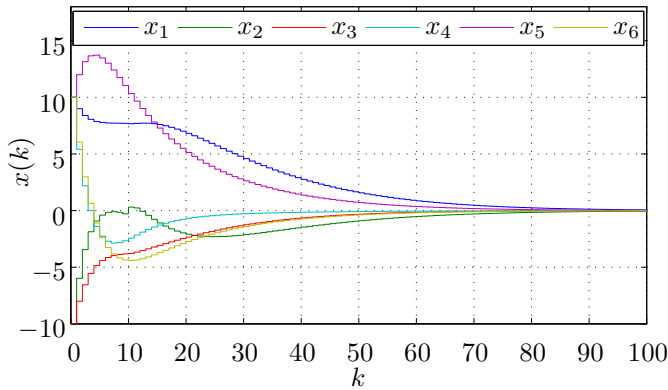


Fig. 3. Simulation results for the distributed controller  $K^0$  and  $K^1$ . At time  $k = 10$  the system switches from  $K^0$  to  $K^1$ .

the resulting memory requirements may be problematic for larger systems. A method to resolve this issue by exploiting the interconnection structure of the system and solving a set of smaller problems instead of (21) has been proposed in [18].

In this example the link  $\delta_{3,1}$  is never used due to the communication costs, i.e.  $\mathcal{F}^2$  does not affect the controller, while  $\mathcal{F}^1$  and  $\mathcal{F}^3$  have the same effect. Thus for  $t = \{0, 2\}$  the following distributed controller is obtained:

$$K^t = \begin{bmatrix} -4.24 & -5.48 & -1.76 & -0.06 & 1.76 & -0.68 \\ 0 & 0 & -2.51 & -4.25 & -2.01 & -0.84 \\ 0 & 0 & 0 & 0 & -1.15 & -2.80 \end{bmatrix}.$$

And for  $t = \{1, 3\}$  we obtain:

$$K^t = \begin{bmatrix} -6.13 & -7.38 & 0 & 0 & 3.62 & -2.52 \\ 0 & 0 & -2.51 & -4.25 & -2.01 & -0.84 \\ 0 & 0 & 0 & 0 & -1.15 & -2.80 \end{bmatrix}.$$

In other words, if  $\delta_{1,2}$  fails the controller is switched from  $K^0$  to  $K^1$ . Simulation results for these controllers are shown in Fig. 3. At time  $k = 10$ , the communication from subsystem 2 to subsystem 1 fails and system 1 switches to a different controller, i.e. the overall system switches from  $K^0$  to  $K^1$ . It can be seen that the link failure only has a small effect on the system and good performance is achieved using the adapted controller  $K^1$ . In other words, the synthesis method included reliable links to achieve good performance if one of the unreliable links fails, and it excluded another unreliable link because it is not critical for performance. Computing  $\tau_a$  based on Theorem (28) yields  $\tau_a \geq 3$  to ensure asymptotic stability. In contrast, bounding the worst-case performance loss by  $\alpha^d = 0.95\alpha^{t_w}$  and computing the average dwell time based on (33) yields  $\tau_a \geq 50$ .

## VI. CONCLUSION

An approach to optimal distributed control considering closed-loop performance, communication costs, as well as failure of communication links was presented. Based on a quadratic performance criterion, sufficient convex LMI conditions for structured controller synthesis are obtained. In combination with a model of the costs of communication

topologies, an MISDP is obtained which allows joint optimization of the network topology and distributed controllers. If a communication link fails, the affected controller locally switches to a different control law, while stability of the global system is preserved, i.e. robustness with regard to link failures is obtained. Stability as well as performance bounds of such a switched control law are obtained based on average dwell time conditions. Improving the efficiency of the proposed method for large systems is seen as interesting area for future research.

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