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Symbolic models for stochastic control systems without stability assumptions

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***Abstract*— Symbolic approaches provide a mechanism to construct discrete and possibly finite abstractions of continuous control systems. Discrete abstractions are in turn amenable to automata-theoretic techniques targeted at the construction of controllers satisfying complex specifications that would be difficult to enforce over continuous models with conventional control design methods. Although the construction of discrete abstractions has been extensively studied for non-probabilistic continuous-time control systems, it has received scant attention on their stochastic counterparts. In this paper, we propose an abstraction technique that is applicable to any stochastic continuous-time control system, as long as we are only interested in its behavior over a compact set. The effectiveness of the proposed results is illustrated with the synthesis of a controller for a jet engine model, which is not stable, is affected by noise, and is subject to a schedulability constraint expressed by a finite automaton.**

I. Introduction

Symbolic models are abstract descriptions of physical systems where each state represents a collection, or an aggregate, of states of the continuous system. Symbolic models are as well employed in the description of software and hardware, which are often characterized by discrete, digital components. The composition of continuous and discrete models captures the behavior of physical systems interacting with digital, computational devices, and results in the general framework known as Cyber-Physical Systems (CPS) [17]. The problems of verification and of controller synthesis over models as general as CPS can be algorithmically studied using methodologies and tools developed in the computer science, as long as there exist symbolic models describing the overall behaviors of CPS.

The quest for symbolic abstractions has a rich and recent history with numerous results on non-probabilistic continuous control systems [5], [7], [10], [13], [14], [16], [19]. For stochastic systems the results are less abundant, and deal with discrete-time autonomous systems [1], [3], [4], with discrete-time control systems [2] equipped with a finite number of control actions and investigated over reachability analysis, and finally with continuous-time control systems under some stability assumptions [18]. As an extension of the results in [18], this paper shows that a symbolic model of a continuous-time stochastic control system exists even in the absence of stability assumptions. More specifically, the main contribution of this work is to establish the following claim: *for every continuous-time stochastic control system satisfying some completeness assumption (as per Definition 2.2), one can construct a symbolic model that is alternatingly approximately simulated (as discussed in Definition 3.3) by the stochastic control system and that approximately simulates (as discussed in Definition 3.2) the stochastic control system.*

The mentioned relationships are weaker than that of approximate bisimulation relation established in [18], but they apply to a larger class of continuous-time stochastic control systems since they no longer require any sort of stability assumptions. Moreover, the relationships established in this paper are still sufficient to guarantee that any controller synthesized for the symbolic model enforces the desired specifications on the original stochastic control system. However, they can no longer guarantee, as it was the case in [18], that the existence of a controller for the original stochastic control system leads to the existence of a controller for the symbolic model.

The technical results in this work are illustrated on a Moore-Greitzer jet engine model, which is affected by noise and dwells in a no-stall mode that does not satisfy the stability assumptions required in [18]. The novel abstraction approach presented in this paper is used to synthesize a controller stabilizing the jet engine, despite further schedulability constraints imposed by executing the controller actions on a microprocessor running other tasks.

II. Stochastic Control Systems

A. Notation

The identity map on a set A is denoted by 1_A. If A is a subset of B we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any point $a \in A$ to $\imath(a) = a \in B$. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}_0^+$ denote the set of natural, integer, real, positive real, and nonnegative real numbers, respectively. The symbols I_n, 0_n, and $0_{n\times m}$ denote the identity matrix, the zero vector and zero matrix in $\mathbb{R}^{n\times n}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, and by $\|x\|$ the infinity norm of x, namely, $\|x\| = \max\{|x_1|, |x_2|, ..., |x_n|\}$, where $|x_i|$ denotes the absolute value of x_i. Given a matrix $M = \{m_{ij}\} \in \mathbb{R}^{n\times m}$, we denote by $\|M\|$ the infinity norm of M, namely, $\|M\| = \max_{1\leq i\leq n} \sum_{j=1}^{m} |m_{ij}|$, and by $\|M\|_F$ the Frobenius norm of M, namely, $\|M\|_F = \sqrt{\text{Tr}\,(MM^T)}$ where $\text{Tr}(P) = \sum_{i=1}^{n} p_{ii}$, for any $P = \{p_{ij}\} \in \mathbb{R}^{n\times n}$. The diagonal set $\Delta \subset \mathbb{R}^n \times \mathbb{R}^n$ is defined as: $\Delta = \{(x, x) \mid x \in \mathbb{R}^n\}$.

The closed ball centered at $x \in \mathbb{R}^n$ with radius ε is defined by $\mathcal{B}_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq \varepsilon\}$. A set $B \subseteq \mathbb{R}^n$ is called a *box* if $B = \prod_{i=1}^{n} [c_i, d_i]$, where $c_i, d_i \in \mathbb{R}$ with $c_i < d_i$ for each $i \in \{1, \ldots, n\}$. The *span* of a box B is defined as $span(B) = \min\{|d_i - c_i| \mid i = 1, \ldots, n\}$. By defining $[\mathbb{R}^n]_\eta = \{a \in \mathbb{R}^n \mid a_i = k_i\eta,\ k_i \in \mathbb{Z},\ i = 1, \cdots, n\}$, the set $\bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_\lambda(p)$ is a countable covering of $\mathbb{R}^n$ for any $\eta \in \mathbb{R}^+$ and $\lambda \geq \eta$. Define the η-approximation $[B]_\eta = [\mathbb{R}^n]_\eta \cap B$

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for a box $B \subset \mathbb{R}^n$ and $\eta \leq span(B)$. Note that $[B]_\eta \neq \varnothing$ for any $\eta \leq span(B)$. Geometrically, for any $\eta \in \mathbb{R}^+$ with $\eta \leq span(B)$ and $\lambda \geq \eta$, the collection of sets $\{\mathcal{B}_\lambda(p)\}_{p\in[B]_\eta}$ is a finite covering of B, i.e., $B \subseteq \bigcup_{p\in[B]_\eta} \mathcal{B}_\lambda(p)$. We extend the notions of *span* and of approximation to finite unions of boxes as follows. Let $A = \bigcup_{j=1}^M A_j$, where each A_j is a box. Define $span(A) = \min\{span(A_j) \mid j = 1, \ldots, M\}$, and for any $\eta \leq span(A)$, define $[A]_\eta = \bigcup_{j=1}^M [A_j]_\eta$.

Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum (sup norm) of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty = (\text{ess})\sup\{\|f(t)\|, t \geq 0\}$. A function f is essentially bounded if $\|f\|_\infty < \infty$. For a given time $\tau \in \mathbb{R}^+$, define f_τ so that $f_\tau(t) = f(t)$, for any $t \in [0, \tau)$, and $f_\tau(t) = 0$ elsewhere; f is said to be locally essentially bounded if for any $\tau \in \mathbb{R}^+$, f_τ is essentially bounded. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ iff $(a, b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.

B. Stochastic control systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0}$ satisfying usual conditions of completeness and right continuity [8, p. 48]. Let $(W_s)_{s\geq 0}$ be a p-dimensional $\mathbb{F}$-Brownian motion.

Definition 2.1: A *stochastic control system* is a tuple $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$, where

- $\mathbb{R}^n$ is the state space;
- $\mathsf{U} \subseteq \mathbb{R}^m$ is the input set;
- $\mathcal{U}$ is a subset of the set of all measurable, locally essentially bounded functions of time from $\mathbb{R}_0^+$ to U;
- $f : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ is a continuous function of its arguments satisfying the following Lipschitz assumption: there exist constants $L_x, L_u \in \mathbb{R}^+$ such that: $\|f(x, u) - f(x', u')\| \leq L_x\|x - x'\| + L_u\|u - u'\|$ for all $x, x' \in \mathbb{R}^n$ and all $u, u' \in \mathsf{U}$;
- $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ is a continuous function satisfying the following Lipschitz assumption: there exists a constant $Z \in \mathbb{R}^+$ such that: $\|\sigma(x) - \sigma(x')\| \leq Z\|x - x'\|$ for all $x, x' \in \mathbb{R}^n$.

A stochastic process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ is said to be a *solution process* of Σ if there exists $\upsilon \in \mathcal{U}$ satisfying:

$$\mathrm{d}\,\xi = f(\xi, \upsilon)\,\mathrm{d}\,t + \sigma(\xi)\,\mathrm{d}\,W_t, \tag{II.1}$$

$\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.). We also write $\xi_{a\upsilon}(t)$ to denote the value of the solution process at time $t \in \mathbb{R}_0^+$ under the input υ and from the initial condition $\xi_{a\upsilon}(0) = a$ $\mathbb{P}$-a.s., in which a is a random variable that is measurable in $\mathcal{F}_0$. Note that $\mathcal{F}_0$, in general, is not a trivial sigma-algebra, thus the stochastic control system Σ may start from a random initial condition. Let us emphasize that the solution process is uniquely determined, since the assumptions on f and σ ensure the existence and the uniqueness of solutions [12, Theorem 5.2.1, p. 68].

C. Completeness notion

The main result presented in this paper requires a certain property on Σ that we introduce in this section.

Definition 2.2: A stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$ is incrementally forward complete in the qth moment (δ-FC-M$_q$), where $q \geq 1$, if there exist continuos functions $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $\gamma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that for every $s \in \mathbb{R}^+$, the functions $\beta(\cdot, s)$ and $\gamma(\cdot, s)$ belong to class $\mathcal{K}_\infty$, and for any $\mathbb{R}^n$-valued random variables a and a', which are measurable in $\mathcal{F}_0$, any $t \in \mathbb{R}_0^+$, and any $\upsilon, \upsilon' \in \mathcal{U}$, the following condition is satisfied:

$$\mathbb{E}\left[\|\xi_{a\upsilon}(t) - \xi_{a'\upsilon'}(t)\|^q\right] \leq \beta\left(\mathbb{E}\left[\|a - a'\|^q\right], t\right) + \gamma\left(\|\upsilon - \upsilon'\|_\infty, t\right). \tag{II.2}$$

The notion of δ-FC-M$_q$ can be described in terms of Lyapunov-like functions. We start by introducing the following definition, which is inspired by the notion of incrementally forward complete (δ-FC) Lyapunov function presented in [19] for non-probabilistic control systems.

Definition 2.3: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$ and a continuous function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ that is continuously differentiable on $\{\mathbb{R}^n \times \mathbb{R}^n\}\backslash\Delta$. Function V is called a δ-FC-M$_q$ Lyapunov function for Σ, where $q \geq 1$, if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, ρ, and a constant $\kappa \in \mathbb{R}$, such that

(i) $\underline{\alpha}$ (resp. $\overline{\alpha}$) is a convex (resp. concave) function;
(ii) for any $x, x' \in \mathbb{R}^n$, $\underline{\alpha}\left(\|x - x'\|^q\right) \leq V(x, x') \leq \overline{\alpha}\left(\|x - x'\|^q\right)$;
(iii) for any $x, x' \in \mathbb{R}^n : x \neq x'$, and for any $u, u' \in \mathsf{U}$,

$$\begin{aligned}\mathcal{L}^{u,u'}V(x, x') &:= [\partial_x V \;\; \partial_{x'} V] \begin{bmatrix} f(x, u) \\ f(x', u') \end{bmatrix} \\ &+ \frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix}\sigma(x) \\ \sigma(x')\end{bmatrix}\begin{bmatrix}\sigma^T(x) & \sigma^T(x')\end{bmatrix}\begin{bmatrix}\partial_{x,x}V & \partial_{x,x'}V \\ \partial_{x',x}V & \partial_{x',x'}V\end{bmatrix}\right) \\ &\leq \kappa V(x, x') + \rho(\|u - u'\|),\end{aligned}$$

where $\mathcal{L}^{u,u'}$ is the infinitesimal generator associated to the stochastic control system (II.1) [12, Section 7.3], which in this case depends on two separate control inputs u, u'. The symbols ∂_x and $\partial_{x,x'}$ denote first- and second-order partial derivatives with respect to x and x', respectively.

Note that the condition (i) is not required in the context of non-probabilistic control systems [19]. Roughly speaking, condition (ii) implies that the growth rate of functions $\underline{\alpha}$ and $\overline{\alpha}$ are linear, as a concave function is supposed to dominate a convex one. These conditions are not restrictive, provided we are interested in the dynamics of Σ on a compact subset $\mathsf{D} \subset \mathbb{R}^n$, which is often the case in practice. The following theorem describes δ-FC-M$_q$ in terms of the existence of δ-FC-M$_q$ Lyapunov functions.

Theorem 2.4: A stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$ is δ-FC-M$_q$ if it admits a δ-FC-M$_q$ Lyapunov function.

Proof: The proof is similar to the proof of Theorem 3.3 in [18] by enforcing $\kappa \in \mathbb{R}$ rather than $\kappa \in \mathbb{R}^+$ in the proof. ■

Next result shows that any stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$ is indeed δ-FC-M$_1$.

Theorem 2.5: Any stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$ is δ-FC-M$_1$.

Proof: We prove the result by showing that any stochastic control system Σ admits a δ-FC-M$_1$ Lyapunov function as $V(x, x') = \sqrt{(x - x')^T (x - x')}$ and by resorting to the result in Theorem 2.4. It is not difficult to check that the function V satisfies properties (i) and (ii) of Definition 2.3 with functions $\underline{\alpha}(y) := y$ and $\overline{\alpha}(y) := \sqrt{n}y$. It then suffices to verify property (iii). By the definition of V, for any $x, x' \in \mathbb{R}^n$ such that $x \neq x'$, one obtains

$$\partial_x V = -\partial_{x'} V = \frac{(x-x')^T}{V(x,x')},$$
$$\partial_{x,x} V = \partial_{x',x'} V = -\partial_{x,x'} V = \frac{V^2(x,x') I_n - (x-x')(x-x')^T}{V^3(x,x')}.$$

Therefore, following the definition of $\mathcal{L}^{u,u'}$, for any $x, x' \in \mathbb{R}^n$ such that $x \neq x'$, and any $u, u' \in \mathsf{U}$, one obtains

$$\begin{aligned}
\mathcal{L}^{u,u'} V(x,x') &= \frac{(x-x')^T}{V(x,x')} \left(f(x,u) - f(x',u')\right) \\
&\quad + \frac{1}{2} \mathrm{Tr}\left(\begin{bmatrix}\sigma(x)\\ \sigma(x')\end{bmatrix} \begin{bmatrix}\sigma^T(x) & \sigma^T(x')\end{bmatrix} \begin{bmatrix}\partial_{x,x} V & -\partial_{x,x} V\\ -\partial_{x,x} V & \partial_{x,x} V\end{bmatrix}\right) \\
&= \frac{(x-x')^T}{V(x,x')} \left(f(x,u) - f(x',u')\right) \\
&\quad + \frac{1}{2} \mathrm{Tr}\left(\left(\sigma(x) - \sigma(x')\right)\left(\sigma^T(x) - \sigma^T(x')\right) \partial_{x,x} V\right) \\
&= \frac{(x-x')^T}{V(x,x')} \left(f(x,u) - f(x',u')\right) \\
&\quad + \frac{1}{2V^3(x,x')} \Big(\left\|\left(\sigma(x) - \sigma(x')\right)\right\|_F^2 V^2(x,x') \\
&\quad - \left\|(x-x')^T \left(\sigma(x) - \sigma(x')\right)\right\|_F^2 \Big) \\
&\leq \frac{\|x-x'\|}{V(x,x')} \left(L_x \|x-x'\| + L_u \|u-u'\|\right) + \frac{\|\sigma(x) - \sigma(x')\|_F^2}{2V(x,x')} \\
&\leq L_x \|x-x'\| + L_u \|u-u'\| + \frac{\min\{n,q\} n Z^2 \|x-x'\|^2}{2V(x,x')} \\
&\leq \left(L_x + \frac{\min\{n,q\} n Z^2}{2}\right) V(x,x') + L_u \left\|u-u'\right\|,
\end{aligned}$$

where L_x, L_u, and Z are the Lipschitz constants, as introduced in Definition 2.1. Therefore, $V(x,x') = \sqrt{(x-x')^T (x-x')}$ is a δ-FC-M$_1$ Lyapunov function for Σ and as showed in Theorem 2.4, one concludes that Σ is δ-FC-M$_1$. ■

The following result provides a sufficient condition on a particular function V to be a δ-FC-M$_q$ Lyapunov function, when $q \in \{1,2\}$.

Lemma 2.6: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$. Let $q \in \{1,2\}$, $P \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix, and the function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ be defined as follows:

$$V(x,x') := \left(\frac{1}{q} (x-x')^T P (x-x')\right)^{\frac{q}{2}}, \qquad \text{(II.3)}$$

and satisfy

$$(x-x')^T P (f(x,u) - f(x',u)) + \frac{1}{2} \left\|\sqrt{P} \left(\sigma(x) - \sigma(x')\right)\right\|_F^2 \leq \widetilde{\kappa} \left(V(x,x')\right)^{\frac{2}{q}}, \qquad \text{(II.4)}$$

or, if f is differentiable with respect to x, satisfy

$$(x-x')^T P \partial_x f(z,u)(x-x') + \frac{1}{2} \left\|\sqrt{P} \left(\sigma(x) - \sigma(x')\right)\right\|_F^2 \leq \widetilde{\kappa} \left(V(x,x')\right)^{\frac{2}{q}}, \qquad \text{(II.5)}$$

for all $x, x', z \in \mathbb{R}^n$, for all $u \in \mathsf{U}$, and for some constant $\widetilde{\kappa} \in \mathbb{R}$. Then V is a δ-FC-M$_q$ Lyapunov function for Σ.

Proof: The proof is similar to the proof of Lemma 3.4 in [18]. ■

The next result provides an equivalent condition to (II.4) or (II.5) for linear stochastic control systems in the form of a linear matrix inequality (LMI).

Corollary 2.7: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$, where for all $x \in \mathbb{R}^n$, and $u \in \mathsf{U}$, $f(x,u) := Ax + Bu$, for some $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $\sigma(x) := [\sigma_1 x \; \sigma_2 x \; \cdots \; \sigma_p x]$, for some $\sigma_i \in \mathbb{R}^{n \times n}$. Then, function V in (II.3) is a δ-FC-M$_q$ Lyapunov function for Σ, when $q \in \{1,2\}$, if there exists a constant $\widehat{\kappa} \in \mathbb{R}$ satisfying the following LMI:

$$PA + A^T P + \sum_{i=1}^{p} \sigma_i^T P \sigma_i \preceq \widehat{\kappa} P. \qquad \text{(II.6)}$$

Proof: The proof is similar to the proof of Corollary 3.5 in [18]. ■

One can find an appropriate matrix P by solving the LMI (II.6) to have a tighter upper bound in (II.2).

III. Symbolic Models and Approximate Equivalence Notions

A. Systems

We use the notion of systems [17] to describe both stochastic control systems as well as their symbolic models.

Definition 3.1: A system S is a tuple $S = (X, X_0, U, \longrightarrow, Y, H)$ consisting of

- A set of states X;
- A set of initial states $X_0 \subseteq X$;
- A set of inputs U;
- A transition relation $\longrightarrow \subseteq X \times U \times X$;
- An output set Y;
- An output function $H : X \to Y$.

A system S is said to be

- *metric*, if the output set Y is equipped with a metric $\mathbf{d} : Y \times Y \to \mathbb{R}_0^+$;
- *finite*, if X is a finite set.

A transition $(x, u, x') \in \longrightarrow$ is also denoted by $x \xrightarrow{u} x'$. For a transition $x \xrightarrow{u} x'$, state x' is called a u-successor, or simply a successor, of state x. We denote by $\mathbf{Post}_u(x)$ the set of u-successors of a state x and by $U(x)$ the set of inputs $u \in U$ for which $\mathbf{Post}_u(x)$ is nonempty. A system is deterministic if for any state $x \in X$ and any input u, there exists at most one u-successor (there may be none). A system is called nondeterministic if it is not deterministic. Hence, for a nondeterministic system it is possible for a state to have two (or possibly more) distinct u-successors.

B. Relations among systems

First, we recall the notion of approximate simulation relation, introduced in [6], which is useful when analyzing or synthesizing controllers for deterministic systems.

Definition 3.2: Let $S_a = (X_a, X_{a0}, U_a, \underset{a}{\longrightarrow}, Y_a, H_a)$ and $S_b = (X_b, X_{b0}, U_b, \underset{b}{\longrightarrow}, Y_b, H_b)$ be metric systems with the same output sets $Y_a = Y_b$ and metric $\mathbf{d}$, and consider a precision $\varepsilon \in \mathbb{R}^+$. A relation $R \subseteq X_a \times X_b$ is said to be an ε-approximate simulation relation from S_a to S_b, if the following three conditions are satisfied:

(i) $\forall x_{a0} \in X_{a0}, \exists x_{b0} \in X_{b0}$ with $(x_{a0}, x_{b0}) \in R$;
(ii) $\forall (x_a, x_b) \in R$ we have $\mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$;
(iii) $\forall (x_a, x_b) \in R$, $x_a \underset{a}{\xrightarrow{u_a}} x'_a$ in S_a implies the existence of $x_b \underset{b}{\xrightarrow{u_b}} x'_b$ in S_b satisfying $(x'_a, x'_b) \in R$.

System S_a is ε-approximately simulated by S_b or S_b ε-approximately simulates S_a, denoted by $S_a \preceq_{\mathcal{S}}^{\varepsilon} S_b$, if there exists an ε-approximate simulation relation from S_a to S_b.

For nondeterministic systems we need to consider relationships that explicitly capture the adversarial nature of nondeterminism. The notion of alternating approximate simulation relation is shown in [14] to be appropriate for this objective.

Definition 3.3: Let $S_a = (X_a, X_{a0}, U_a, \underset{a}{\longrightarrow}, Y_a, H_a)$ and $S_b = (X_b, X_{b0}, U_b, \underset{b}{\longrightarrow}, Y_b, H_b)$ be metric systems with the same output sets $Y_a = Y_b$ and metric $\mathbf{d}$, and consider a precision $\varepsilon \in \mathbb{R}^+$. A relation $R \subseteq X_a \times X_b$ is said to be an alternating ε-approximate simulation relation from S_a to S_b if conditions (i), (ii) in Definition 3.2 hold, and additionally if the following condition is satisfied:

(iii) for every $(x_a, x_b) \in R$ and for every $u_a \in U_a(x_a)$ there exists $u_b \in U_b(x_b)$ such that for every $x'_b \in \mathbf{Post}_{u_b}(x_b)$ there exists $x'_a \in \mathbf{Post}_{u_a}(x_a)$ satisfying $(x'_b, x'_a) \in R$.

System S_a is alternatingly ε-approximately simulated by S_b or S_b alternatingly ε-approximately simulates S_a, denoted by $S_a \preceq_{\mathcal{AS}}^{\varepsilon} S_b$, if there exists an alternating ε-approximate simulation relation from S_a to S_b.

It is readily seen from the above definitions that the notions of approximate simulation and of alternating approximate simulation coincide when the systems involved are deterministic.

IV. Symbolic Models for Stochastic Control Systems

This section contains the main contribution of the paper. We show that any δ-FC-M$_q$ stochastic control system Σ admits a finite symbolic model whenever we are interested in the dynamics of Σ on a compact subset $\mathsf{D} \subset \mathbb{R}^n$. We restrict our attention to stochastic control systems $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$ with $f(0_n, 0_m) = 0_n$, $\sigma(0_n) = 0_{n\times p}$, and input sets U that are assumed to be finite unions of boxes and include $\{0_m\}$. We further restrict our attention to sampled-data stochastic control systems, where input curves belong to set $\mathcal{U}_\tau$ which contains only curves that are constant over intervals of length $\tau \in \mathbb{R}^+$, i.e.

$$\mathcal{U}_\tau = \Big\{ \upsilon : \mathbb{R}_0^+ \to \mathsf{U} \mid \upsilon(t) = \upsilon((k-1)\tau), t \in [(k-1)\tau, k\tau[, k \in \mathbb{N} \Big\}.$$

Given a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$, consider the system

$$S_\tau(\Sigma) = (X_\tau, X_{\tau 0}, U_\tau, \underset{\tau}{\longrightarrow}, Y_\tau, H_\tau),$$

consisting of:

- X_τ is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $X_{\tau 0}$ is the set of random variables that are measurable with respect to trivial sigma-algebra $\mathcal{F}_0$, i.e., the system starts from a deterministic initial condition;
- $U_\tau = \mathcal{U}_\tau$;
- $x_\tau \overset{\upsilon_\tau}{\underset{\tau}{\longrightarrow}} x'_\tau$ if x_τ and x'_τ are measurable, respectively, in $\mathcal{F}_{k\tau}$ and $\mathcal{F}_{(k+1)\tau}$ for some $k \in \mathbb{N} \cup \{0\}$, and there exists a solution process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(k\tau) = x_\tau$ and $\xi_{x_\tau \upsilon_\tau}(\tau) = x'_\tau$ $\mathbb{P}$-a.s.;
- Y_τ is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $H_\tau = 1_{X_\tau}$.

We assume that the output set Y_τ is equipped with the natural metric $\mathbf{d}(y, y') = \left(\mathbb{E}\left[\|y - y'\|^q\right]\right)^{\frac{1}{q}}$, for any $y, y' \in Y$ and some $q \geq 1$.

Before introducing the symbolic model for the stochastic control system, we proceed with the following preliminary lemma.

Lemma 4.1: Consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f, \sigma)$ such that $f(0_n, 0_m) = 0_n$, and $\sigma(0_n) = 0_{n\times p}$. Suppose there exists a δ-FC-M$_q$ Lyapunov function V for Σ such that its Hessian is a positive semidefinite matrix in $\mathbb{R}^{2n\times 2n}$ and $q \geq 2$. Then for any x in a compact set D and any $\upsilon \in \mathcal{U}$, we have

$$\mathbb{E}\left[\left\|\xi_{x\upsilon}(t) - \overline{\xi}_{x\upsilon}(t)\right\|^q\right] \leq h(\sigma, t), \quad \text{(IV.1)}$$

where $\overline{\xi}_{x\upsilon}$ is the solution of the ordinary differential equation (ODE) $\dot{\overline{\xi}}_{x\upsilon} = f\left(\overline{\xi}_{x\upsilon}, \upsilon\right)$ starting from the initial condition x, and the nonnegative valued function h tends to zero as $t \to 0$ or as $Z \to 0$, where Z is the Lipschitz constant, introduced in Definition 2.1.

Proof: The proof is similar to the proof of Lemma 3.7 in [18]. ∎

We refer the interested readers to Lemmas 3.7, 3.9 and Corollary 3.10 in [18] to see how the function h can be similarly computed by enforcing $\kappa \in \mathbb{R}$ in Definition 2.3 rather than $\kappa \in \mathbb{R}_0^+$.

We consider a stochastic control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$, and a tuple $\mathsf{q} = (\tau, \eta, \mu, \theta, l)$ of quantization parameters, where $\tau \in \mathbb{R}^+$ is the sampling time, $\eta \in \mathbb{R}^+$ is the state space quantization, $\mu \in \mathbb{R}^+$ is the input set quantization and $\theta \in \mathbb{R}^+$ and $\ell \in \mathbb{N}$ are design parameters. Given Σ and q, consider the following system:

$$S_{\mathsf{q}}(\Sigma) = (X_{\mathsf{q}}, X_{\mathsf{q}0}, U_{\mathsf{q}}, \underset{\mathsf{q}}{\longrightarrow}, Y_{\mathsf{q}}, H_{\mathsf{q}}), \quad \text{(IV.2)}$$

consisting of: $X_{\mathsf{q}} = [\mathbb{R}^n]_\eta$, $X_{\mathsf{q}0} = [\mathbb{R}^n]_\eta$, $U_{\mathsf{q}} = [\mathsf{U}]_\mu$, and

- $x_{\mathsf{q}} \overset{u_{\mathsf{q}}}{\underset{\mathsf{q}}{\longrightarrow}} x'_{\mathsf{q}}$ if $\left\|\overline{\xi}_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\right\| \leq (\beta(\theta^q, \tau) + \gamma(\mu, \tau))^{\frac{1}{q}} + h(\sigma, \tau)^{\frac{1}{q}} + h(\sigma, \ell\tau)^{\frac{1}{q}} + \eta$, where $\dot{\overline{\xi}}_{x_{\mathsf{q}} u_{\mathsf{q}}} = f\left(\overline{\xi}_{x_{\mathsf{q}} u_{\mathsf{q}}}, u_{\mathsf{q}}\right)$;
- Y_{q} is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $H_{\mathsf{q}} = \imath : X_{\mathsf{q}} \hookrightarrow Y_{\mathsf{q}}$.

Here β and γ are the functions appearing in (II.2) and h is the function appearing in (IV.1). Note that we have abused notation by identifying $u_{\mathsf{q}} \in [\mathsf{U}]_\mu$ with the constant input curve with domain $[0, \tau[$ and value u_{q}. Notice that the proposed abstraction $S_{\mathsf{q}}(\Sigma)$ is indeed a nondeterministic system governed by an ordinary differential equation. However, in order to establish an (alternating) approximate simulation relation, the output set Y_{q} is defined similarly to our original stochastic system $S_\tau(\Sigma)$. Therefore, in the definition of H_{q}, the inclusion map $\imath$ is meant, with a slight abuse of notation, as a mapping from a deterministic grid point to a random variable with a Dirac probability distribution centered at the grid point.

The transition relation of $S_{\mathsf{q}}(\Sigma)$ is well defined in the sense that for every $x_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ and every $u_{\mathsf{q}} \in [\mathsf{U}]_\mu$ there always exists $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ such that $x_{\mathsf{q}} \overset{u_{\mathsf{q}}}{\underset{\mathsf{q}}{\longrightarrow}} x'_{\mathsf{q}}$. This can be seen since by definition of $[\mathbb{R}^n]_\eta$, for any $\widehat{x} \in \mathbb{R}^n$ there always exists a state $\widehat{x}' \in [\mathbb{R}^n]_\eta$ such that $\|\widehat{x} - \widehat{x}'\| \leq \eta$. Hence, for $\overline{\xi}_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau)$ there always exists a state $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ satisfying $\left\|\overline{\xi}_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\right\| \leq \eta \leq (\beta(\theta^q, \tau) + \gamma(\mu, \tau))^{\frac{1}{q}} + h(\sigma, \tau)^{\frac{1}{q}} + h(\sigma, \ell\tau)^{\frac{1}{q}} + \eta$.

We can now present the main result of the paper, showing that any δ-FC-M$_q$ stochastic control system Σ admits a finite symbolic model.

Theorem 4.2: Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f, \sigma)$ be a stochastic control system, admitting a δ-FC-M$_q$ Lyapunov function V, of the form of (II.3) or the one explained in Lemmas 4.1. For any $\varepsilon \in \mathbb{R}^+$, and any tuple $\mathsf{q} = (\tau, \eta, \mu, \theta, \ell)$ of quantization parameters satisfying $\mu \leq span(\mathsf{U})$ and $h(\sigma, \ell\tau)^{\frac{1}{q}} + \eta \leq \varepsilon \leq \theta$, we have:

$$S_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_{\mathsf{q}}(\Sigma), \qquad \text{(IV.3)}$$

within the time horizon $0, \tau, \cdots, \ell\tau$.

Before providing the proof, it can be readily seen that when we are interested in the dynamics of Σ on a compact subset $\mathsf{D} \subset \mathbb{R}^n$, assumed to be a finite union of boxes, and for a given precision ε, there always exists a sufficiently small choice of τ such that $h(\sigma, \ell\tau)^{\frac{1}{q}} < \varepsilon$. Then by choosing a sufficiently small value of $\eta \leq span(\mathsf{D})$, the condition of Theorem 4.2 is satisfied.

Proof: We start by proving $S_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_\tau(\Sigma)$. Consider the relation $R \subseteq X_\tau \times X_{\mathsf{q}}$ defined by $(x_\tau, x_{\mathsf{q}}) \in R$ if and only if

$$\left(\mathbb{E}\left[\|H_\tau(x_\tau) - H_{\mathsf{q}}(x_{\mathsf{q}})\|^q\right]\right)^{\frac{1}{q}} = \left(\mathbb{E}\left[\|x_\tau - x_{\mathsf{q}}\|^q\right]\right)^{\frac{1}{q}} \leq \varepsilon.$$

For every $x_{\mathsf{q}0} \in X_{\mathsf{q}0}$, by choosing $x_{\tau 0} = x_{\mathsf{q}0}$, we have $(\|x_{\tau 0} - x_{\mathsf{q}0}\|^q)^{\frac{1}{q}} = 0$ and $(x_{\tau 0}, x_{\mathsf{q}0}) \in R$ and condition (i) in Definition 3.3 is satisfied. Now consider any $(x_\tau, x_{\mathsf{q}}) \in R$. Condition (ii) in Definition 3.3 is satisfied by the definition of R. Let us now show that condition (iii) in Definition 3.3 holds. Consider any $u_{\mathsf{q}} \in U_{\mathsf{q}}$. Choose the input $\upsilon_\tau = u_{\mathsf{q}}$ and consider the unique solution process $x'_\tau = \xi_{x_\tau \upsilon_\tau}(\tau) \in \mathbf{Post}_{\upsilon_\tau}(x_\tau)$ in $S_\tau(\Sigma)$. Since Σ is δ-FC-M$_q$, we have:

$$\mathbb{E}\left[\|x'_\tau - \xi_{x_q u_q}(\tau)\|^q\right] \leq \beta\left(\mathbb{E}\left[\|x_\tau - x_q\|^q\right], \tau\right) \leq \beta\left(\varepsilon^q, \tau\right). \qquad \text{(IV.4)}$$

Since $\mathbb{R}^n \subseteq \bigcup_{p \in [\mathbb{R}^n]_\eta} \mathcal{B}_\eta(p)$, there exists $x'_{\mathsf{q}} \in X_{\mathsf{q}}$ such that

$$\left(\mathbb{E}\left[\|x'_\tau - x'_{\mathsf{q}}\|^q\right]\right)^{\frac{1}{q}} \leq h(\sigma, \ell\tau)^{\frac{1}{q}} + \eta. \qquad \text{(IV.5)}$$

Using the inequalities $\varepsilon \leq \theta$, (IV.4), and (IV.5), and triangle inequality, we obtain:

$$\begin{aligned}
&\left\|\overline{\xi}_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\right\| = \left(\mathbb{E}\left[\left\|\overline{\xi}_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\right\|^q\right]\right)^{\frac{1}{q}} \\
&\leq \left(\mathbb{E}\left[\left\|\overline{\xi}_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) - \xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau)\right\|^q\right]\right)^{\frac{1}{q}} + \left(\mathbb{E}\left[\|\xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) - \xi_{x_\tau \upsilon_\tau}(\tau)\|^q\right]\right)^{\frac{1}{q}} \\
&\quad + \left(\mathbb{E}\left[\|\xi_{x_\tau \upsilon_\tau}(\tau) - x'_{\mathsf{q}}\|^q\right]\right)^{\frac{1}{q}} \\
&\leq h(\sigma, \tau)^{\frac{1}{q}} + (\beta(\varepsilon^q, \tau))^{\frac{1}{q}} + h(\sigma, \ell\tau)^{\frac{1}{q}} + \eta \\
&\leq (\beta(\theta^q, \tau) + \gamma(\mu, \tau))^{\frac{1}{q}} + h(\sigma, \tau)^{\frac{1}{q}} + h(\sigma, \ell\tau)^{\frac{1}{q}} + \eta,
\end{aligned}$$

which, by the definition of $S_{\mathsf{q}}(\Sigma)$, implies the existence of $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$ in $S_{\mathsf{q}}(\Sigma)$. Therefore, from inequality (IV.5) and since $h(\sigma, \ell\tau)^{\frac{1}{q}} + \eta \leq \varepsilon$, we conclude that $(x'_\tau, x'_{\mathsf{q}}) \in R$ and condition (iii) in Definition 3.3 holds.

In a similar way, we can prove that $S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_{\mathsf{q}}(\Sigma)$. ∎

The following remark readily extends the assertion of Theorem 4.2 to be valid over an infinite time horizon, under an assumption on the observation of the diffusion.

Remark 4.3: Suppose the symbolic model is allowed to periodically observe the system $S_\tau(\Sigma)$ after each period $T := \ell\tau$, for some $\ell \in \mathbb{N}$. Then, the assertion of Theorem 4.2 holds over an infinite horizon, since one can update the initial state of the symbolic model up to precision η with respect to the realization of $S_\tau(\Sigma)$ at time $\ell\tau$, and replicate the same strategy periodically. In particular, if the observation period is the same as the sampling time, then the lower bound of ε reduces to $h(\sigma, \tau)^{\frac{1}{q}} + \eta$ by setting $\ell = 1$.

Let us highlight that the assumption in Remark 4.3 implicitly requires enlarging the class of admissible inputs to the set of stochastic ones. That is, the input signal synthesized in the symbolic model is deterministic within the time horizon $\ell\tau$, but according to the diffusion observation may change from one realization to another.

We note that the results in [19, Theorem 4.1] are fully recovered by the results in Theorem 4.2 if the stochastic control system Σ is not affected by any noise, implying that $h(\sigma, t)$ is identically zero and the δ-FC-M$_q$ property reduces to the δ-FC property. Correspondingly, the definitions of $S_\tau(\Sigma)$ and $S_{\mathsf{q}}(\Sigma)$ need slight modifications.

Remark 4.4: Although we assume that the set U is infinite, Theorem 4.2 still holds when the set U is finite, with the following modifications: first, the system Σ is required to satisfy the property (II.2) for $\upsilon = \upsilon'$; second, assume $U_{\mathsf{q}} = \mathsf{U}$ and set $\gamma(\mu, \tau) = 0$ in the definition of $S_{\mathsf{q}}(\Sigma)$.

V. Symbolic Control Design for a Jet Engine

We illustrate the results of this paper over the Moore-Greitzer jet engine model in no-stall mode, which is affected by noise and unstable [9]. In this model, the unstable equilibrium (in the absence of noise) is transferred to the origin ($\phi = 0$ and $\psi = 0$) using the following change of coordinates: $\phi = \Phi - 1$, $\psi = \Psi - \Psi_{c0} - 2$, where Φ is the mass flow, Ψ is the pressure rise and Ψ_{c0} is a constant. The resulting model Σ is:

$$\begin{bmatrix} \mathrm{d}\,\phi \\ \mathrm{d}\,\psi \end{bmatrix} = \begin{bmatrix} -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \\ \frac{1}{\omega^2}(\phi - \upsilon) \end{bmatrix} \mathrm{d}\,t + \begin{bmatrix} 0.1\phi\,\mathrm{d}\,W_t^1 \\ 0.1\psi\,\mathrm{d}\,W_t^1 \end{bmatrix}, \qquad \text{(V.1)}$$

where ω is a positive constant parameter set to be equal to 1, $\upsilon(t) = \Phi_T(t) - 1$ is the control input and $\Phi_T(t)$ is the mass flow through the throttle. We work on the subset $\mathsf{D} = [-2, 2] \times [-2, 2]$ of the state space of Σ. One can readily verify that Σ satisfies the conditions in Definition 2.1 with $L_u = 1$, $Z = 0.1$, and $L_x = 13$, when we are interested in the behaviors of Σ in D.

We consider a parameter $q = 1$ and show that Σ satisfies (II.5) by finding a suitable matrix P using SOS programming as described in [15]. The constant $\widetilde{\kappa}$ in (II.5) takes the value 1.5 and the resulting matrix is $P = I_2$.

Using the results of Theorem 2.4, one obtains the following δ-FC-M$_1$ bound for the jet engine model:

$$\mathbb{E}\left[\|\xi_{a\upsilon}(t) - \xi_{a'\upsilon'}(t)\|\right] \leq \sqrt{2}e^{1.5t}\mathbb{E}\left[\|a - a'\|\right] + t\mathsf{e}^{1.5t}|\upsilon - \upsilon'|_\infty.$$

For a given sampling time $\tau = 0.1$ and δ-FC-M$_1$ Lyapunov function $V(x, x') = (x - x')^T(x - x')$, one can compute an $h(\sigma, 0.1) = 0.07$.

We assume that $\mathsf{U} = [-2, 2]$ and that the control input can take only three different values from the set $\{-2, 0, 2\}$. In order to synthesize a controller under this constraint on the input, we select $\mu = 2$. The objective is to design a controller forcing the trajectories of the system to reach and stay (in the 1st moment metric) within the target set $W = [-0.25, 0.25] \times [-0.25, 0.25]$, which can be expressed in the LTL formula $\Diamond\Box W$.

Furthermore, we assume that the controller is implemented on a microprocessor, executing other tasks in addition to the

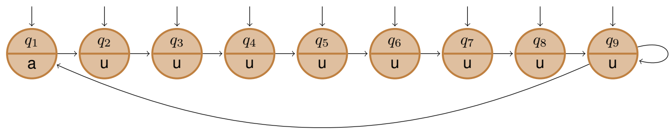

Fig. 1. Finite system describing the schedulability constraint over the controller. The lower part of the states are labeled with the outputs (a and u) denoting the availability and unavailability of the microprocessor, respectively.

control scheme. Let us consider a schedule with epochs of nine time slots, in which at most one slot is allocated to the control task and the rest of them to other tasks. A time slot is an interval of the form $[k\tau, (k+1)\tau[$, with $k \in \mathbb{N}$ and where τ is the sampling time. Therefore, some of the possible microprocessor schedules are given by:

$$|\mathsf{auuuuuuuu}|\mathsf{auuuuuuuu}|\mathsf{auuuuuuuu}|\mathsf{auuuuuuuu}|\cdots,$$

$$|\mathsf{uuuuuuuua}|\mathsf{uuuuuuuuu}|\mathsf{auuuuuuuu}|\mathsf{uuuuuuuuu}|\cdots,$$

where a denotes a slot available for the control task and u denotes a slot allotted to other tasks. The schedulability constraint on the microprocessor can be represented by the finite system (labeled automaton) in Figure 1, where the allowed initial states are distinguished as targets of a sourceless arrow. We embed the schedulability constraint by composing the finite system in Figure 1 by the constructed finite system $S_{\mathsf{q}}(\Sigma)$.

For a precision $\varepsilon = 0.08$, we construct a symbolic model $S_{\mathsf{q}}(\Sigma)$ by choosing $\theta = 0.08$, $\ell = 1$, and $\eta = 0.01$. The computation of the abstraction $S_{\mathsf{q}}(\Sigma)$ is performed in the tool Pessoa [11] on a laptop with CPU 2 GHz Intel Core i7. The resulting number of states after composition is equal to 1447209. The consumed CPU times for computing the abstraction and synthesizing the controller have been 21554 and 542 seconds, respectively. Here we have assumed that symbolic model can observe the diffusion process at each sampling time, as discussed in Remark 4.3.

Figure 2 displays several realizations of the closed-loop trajectory stemming from the initial condition in $(-0.75, -1.75)$ and the evolution of the input signal, where the finite system is initialized at state q_2.

Furthermore, in Figure 3, we show the average value over 200 experiments of the distance in time of the solution process $\xi_{x_0\upsilon}$ to the set W, namely $\|\xi_{x_0\upsilon}(t)\|_W$, where the point-to-set distance is defined as $\|x\|_W = \inf_{w\in W}\|x-w\|$. Notice that the average distance is significantly lower than the precision $\varepsilon = 0.08$, as expected since the conditions based on Lyapunov-like functions can lead to conservative bounds.

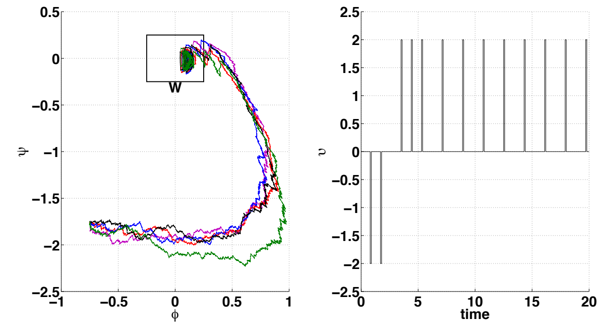

Fig. 2. Several realizations of the closed-loop trajectory with initial condition $(-0.75, -1.75)$ (left panel) and evolution of the input signal (right panel), where the finite system is initialized at state q_2.

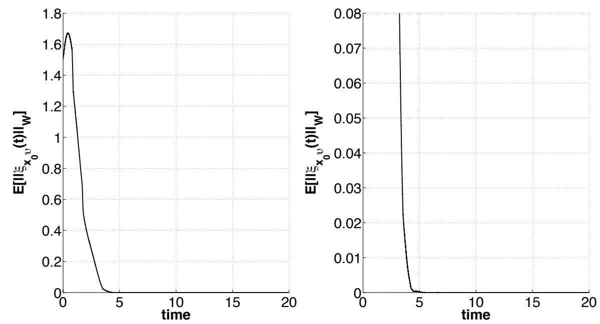

Fig. 3. Average values (over 200 experiments) of the distance of the solution process $\xi_{x_0\upsilon}$ to the set W, in different vertical scales.

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