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H_∞ output feedback control of commensurate fractional order systems

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***Abstract*— This paper addresses the problem of H_∞ output feedback control of commensurate Fractional Order Systems (FOS) of order $1 < \nu < 2$. Along the lines of recent work on H_∞ norm computation for FOS, an extension of Bounded Real Lemma for FOS is proposed. This lemma is used to derive a method to design H_∞ output feedback control laws. The efficiency of this method is evaluated on a numerical example.**

I. INTRODUCTION

In control systems theory, the fractional operator has firstly been used in CRONE control [16]. The compacity of that operator was used in the synthesis of robust controllers for Integer Order Systems (IOS). CRONE control has then been applied to various systems like car suspension [17] and more recently to the maximization of wind turbine production [10].

Fractional differentiation has aroused a growing interest taking benefit of the fractional operator compacity for modeling various physical phenomena (thermal systems [2], batteries [18], neurons [1]...) with less parameters than IOS. That is why efficient methods have been developed to study their properties. Concerning stability, [14] established a criterion based on the location of the state matrix eigenvalues in the complex plane ; [3] also pursued stability analysis via pole location for fractional delay systems ; [21] and [9] proposed Linear Matrix Inequalities (LMI) based results for commensurate FOS stability analysis. Performances were also considered in [12] where a method to evaluate H_2 norm of a FOS is given.

This paper addresses commensurate FOS control. More precisely, our objective is to extend to FOS the H_∞ control methods developed for IOS. Some analysis results on the computation of FOS H_∞ norm have recently been published in [7]. The pseudo Hamiltonian matrix of a fractional order system was defined in that paper and two methods to compute a FOS H_∞ norm based on this pseudo Hamiltonian matrix were proposed. The first one was a dichotomy algorithm and the second one used LMI formalism. LMIs are efficient ways to express control theory problems [5] and offer better flexibility than analytical methods. Based on these analysis results, a method to design H_∞ state feedback controllers was proposed in [8].

In this paper, the objective is to propose a method to design H_∞ output feedback controllers for FOS. First, commensurate FOS are defined and some results on their stability are given. Next, some H_∞ norm computation methods for commensurate FOS are given and the proof of the extension by [22] to FOS of the Bounded Real Lemma [4] is completed. Then, this lemma is used to derive the H_∞ output feedback controller synthesis method proposed in this paper. At last, that synthesis method is evaluated on a numerical example.

<u>Notations:</u> The transpose of a matrix A is denoted A^T, its conjugate $\bar{A}$ and its conjugate transpose A^*. For Hermitian matrices, $A \succ B$ iff $A - B$ is (semi) positive definite. H_∞ norm of a transfer function $T(s)$ is denoted $\|T(s)\|_\infty$.

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II. FRACTIONAL ORDER SYSTEMS

A. LTI Commensurate Fractional Order Systems

In this paper are considered LTI commensurate FOS admitting a pseudo state space representation of the form

$$P \left\{ \begin{array}{rcl} D^\nu x(t) & = & A\ x(t) + B\ u(t) + B_w\ w(t) \\ y(t) & = & C\ x(t) + D\ u(t) + D_w\ w(t) \\ z(t) & = & C_z\ x(t) + D_z\ u(t) + D_{zw}\ w(t) \end{array} \right. \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the pseudo state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control signals vector, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous input, $y(t) \in \mathbb{R}^{n_y}$ is the sensed outputs vector, $z(t) \in \mathbb{R}^{n_z}$ is the performance outputs vector, $1 < \nu < 2$ is the fractional order of the system and A, B, B_w, C_y, C_z, D_y, D_z, D_{yw} and D_{zw} are constant real matrices. D^ν is the fractional differentiation operator of order ν (presented results are valid whatever definition used: Riemann-Liouville [15], Caputo [6] or others [23]). Transfer matrix between $u(t)$ and $y(t)$ is $G(s) = C\left(s^\nu I - A\right)^{-1} B + D$ and impulse response matrix is $g(t) = \mathcal{L}^{-1}\left(G(s)\right)$.

Remark 1: For a FOS, the knowledge of $x(t_0)$ (t_0 being the initial time) is not sufficient to determine the future behavior of the system [11]. Consequently, vector x does not strictly represent the state of the system and is denoted "pseudo state" in this paper [19] [20].

B. Stability of commensurate fractional order systems

Definition 1 ([14]): A linear FOS of impulse response g is Bounded-Input Bounded-Output (BIBO) stable iff $\forall\ u \in \mathcal{L}^\infty\left(\mathbb{R}^+, \mathbb{R}^{n_u}\right)$, $y = g \star u \in \mathcal{L}^\infty\left(\mathbb{R}^+, \mathbb{R}^{n_y}\right)$. ■

LTI IOS stability can be checked via the location of the state matrix A eigenvalues in the complex plane. This result was extended to LTI commensurate FOS of order $0 < \nu < 1$ by D. Matignon and of order $1 < \nu < 2$ by R. Malti.

Theorem 1 ([13], [14]): System (1), with minimal triplet (A, B, C) and $1 < \nu < 2$, is BIBO stable iff

$$\left|\text{Arg}\left(\text{eig}(A)\right)\right| > \nu\frac{\pi}{2}. \quad (2)$$

■

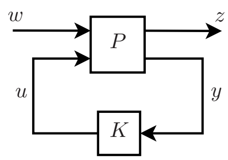

Fig. 1. General control configuration

Stability domain is thus defined as follows:

$$\mathcal{D}_s = \left\{ z \in \mathbb{C} \ : \ |\text{Arg}\,(z)| > \nu \frac{\pi}{2} \right\}. \tag{3}$$

C. H_∞ norm of commensurate FOS

As shown in [7] and [8], H_∞ norm can be interpreted in time domain as the largest energy among output signals resulting from all inputs of unit energy. Consequently, H_∞ norm physical interpretation, in frequency and time domains, is the same for FOS as for IOS.

III. H_∞ PSEUDO-STATE FEEDBACK CONTROL

A. Problem statement

The objective is to design full order H_∞ stabilizing output feedback control laws of the form

$$U(s) = K(s) Y(s), \tag{4}$$

where $K(s)$ is the transfer matrix of a fractional order dynamic output feedback controller with pseudo-state space representation

$$K \left\{ \begin{array}{rcl} D^\nu x_{_K}(t) & = & A_{_K} \ x_{_K}(t) + B_{_K} \ y(t) \\ u(t) & = & C_{_K} \ x_{_K}(t) + D_{_K} \ y(t) \end{array} \right. \tag{5}$$

where $x_{_K} \in \mathbb{R}^n$ is the pseudo state of the controller.

Fig. 1, with P given by relation (1), represents the general control configuration for this problem.

Under the previous conditions, a pseudo-state space representation of the closed-loop T_{zw}^{cl} is

$$T_{zw}^{cl} \left\{ \begin{array}{rcl} D^\nu \tilde{x}(t) & = & \mathcal{A} \ \tilde{x}(t) + \mathcal{B} \ w(t) \\ z(t) & = & \mathcal{C} \ \tilde{x}(t) + \mathcal{D} \ w(t) \end{array} \right. \tag{6}$$

where

$$\tilde{x}(t) = \begin{pmatrix} x^T(t) & x_{_K}^T(t) \end{pmatrix}^T, \tag{7}$$

$$\mathcal{A} = \begin{pmatrix} A + BD_{_K}C & BC_{_K} \\ B_{_K}C & A_{_K} \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} B_w + BD_{_K}D_w \\ B_{_K}D_w \end{pmatrix}, \tag{8}$$

$$\mathcal{C} = \begin{pmatrix} C_z + D_z D_{_K} C & D_z C_{_K} \end{pmatrix}, \ \mathcal{D} = D_{zw} + D_z D_{_K} D_w. \tag{9}$$

Problem: Find K such that $\left\| T_{zw}^{cl} \right\|_\infty < \gamma$, where γ is a positive real number.

B. FOS H_∞ norm computation

Solving the pseudo-state feedback control problem requires to be able to evaluate the H_∞ norm of a stable commensurate FOS. The following Linear Matrix Inequalities based theorems allow to solve this analysis problem.

Theorem 2 ([7]): Let $\gamma > \bar{\sigma}(\mathcal{D})$ be a positive real number. Then $\left\| T_{zw}^{cl}(s) \right\|_\infty < \gamma$ iff pseudo Hamiltonian matrix:

$$H_\gamma = \begin{pmatrix} \mathcal{A} + \mathcal{B}R\mathcal{D}^T\mathcal{C} & \mathcal{B}R\mathcal{B}^T \\ e^{\nu j \pi}\mathcal{C}^T\left(I + \mathcal{D}R\mathcal{D}^T\right)\mathcal{C} & e^{\nu j \pi}\left(\mathcal{A}^T + \mathcal{C}^T\mathcal{D}R\mathcal{B}^T\right) \end{pmatrix} \tag{10}$$

where $R = \left(\gamma^2 I - \mathcal{D}^T\mathcal{D}\right)^{-1}$ has no eigenvalue in set $\mathbb{C}_{\nu 0} = \left\{ (j\omega)^\nu = \omega^\nu e^{\nu j \frac{\pi}{2}}, \omega \in \mathbb{R} \right\}$. ■

A detailed proof of this theorem is given in [7]. That proof is based on the definition of H_∞ norm of a FOS and on the non singularity of system $\phi(s) = \gamma^2 I - T_{zw}^{cl}(-s)^T T_{zw}^{cl}(s)$ whose pseudo-state matrix is H_γ.

Authors of the present paper have derived from theorem 2, the following LMI-based theorem that allows to test if H_∞ norm of T_{zw}^{cl} is lower than a given bound γ.

Theorem 3 ([7]): Let $\gamma > \bar{\sigma}(\mathcal{D})$ be a positive real number. Then $\left\| T_{zw}^{cl}(s) \right\|_\infty < \gamma$ iff there exist three positive definite hermitian matrices X_1, X_2 and $X_3 \in \mathbb{C}^{2n \times 2n}$ such that

$$r_1 H_\gamma X_1 + \bar{r_1} X_1 H_\gamma^* + r_2 H_\gamma X_2 + \bar{r_2} X_2 H_\gamma^* - H_\gamma X_3 - X_3 H_\gamma^* \prec 0 \tag{11}$$

and three positive definite hermitian matrices X_4, X_5 and $X_6 \in \mathbb{C}^{2n \times 2n}$ such that

$$\bar{r}_1 H_\gamma X_4 + r_1 X_4 H_\gamma^* + \bar{r}_2 H_\gamma X_5 + r_2 X_5 H_\gamma^* - H_\gamma X_6 - X_6 H_\gamma^* \prec 0 \tag{12}$$

where $r_1 = e^{j(1-\nu)\frac{\pi}{2}}$, $r_2 = e^{-j(1+\nu)\frac{\pi}{2}}$, and matrix H_γ is defined by:

$$H_\gamma = \begin{pmatrix} \mathcal{A} + \mathcal{B}R\mathcal{D}^T\mathcal{C} & \mathcal{B}R\mathcal{B}^T \\ e^{\nu j \pi}\mathcal{C}^T\left(I + \mathcal{D}R\mathcal{D}^T\right)\mathcal{C} & e^{\nu j \pi}\left(\mathcal{A}^T + \mathcal{C}^T\mathcal{D}R\mathcal{B}^T\right) \end{pmatrix} \tag{13}$$

with $R = \left(\gamma^2 I - \mathcal{D}^T\mathcal{D}\right)^{-1}$. ■

Although efficient for analysis purpose, extending theorem 3 to synthesis is complicated since it involves two LMIs and six matrix variables. This is why the following extension of Bounded Real Lemma to FOS from [22] is presented.

Theorem 4 ([22]): Let $\gamma > \bar{\sigma}(\mathcal{D})$ be a positive real number. Then $\left\| T_{zw}^{cl}(s) \right\|_\infty < \gamma$ iff there exists a matrix $X = X^* \in \mathbb{C}^{n \times n}$ such that

$$\begin{pmatrix} \bar{r}\mathcal{A}^T X + X r \mathcal{A} & X\mathcal{B} & \bar{r}\mathcal{C}^T \\ \mathcal{B}^T X & -\gamma^2 I & \mathcal{D}^T \\ r\mathcal{C} & \mathcal{D} & -I \end{pmatrix} < 0, \tag{14}$$

with $r = e^{(1-\nu)j\frac{\pi}{2}}$. ■

Proof of theorem 4 is now completed since nessecity part was incomplete in [22].

Proof: Theorem 2 shows that an upper bound for $\left\| T_{zw}^{cl}(s) \right\|_\infty$ can be found by analyzing matrix H_γ eigenvalues position on $\mathbb{C}_{\nu 0}$. In practice, it is only necessary to test the position of H_γ eigenvalues in relation to segment $\mathbb{C}_{\nu 0}^+ = \left\{ (j\omega)^\nu, \omega \in \mathbb{R}^+ \right\}$. Indeed, we always work with positive frequencies, i.e. $\omega \in \mathbb{R}^+$, and H_γ has been written for positive frequencies. Moreover, for negative frequencies we can write

$$H_\gamma^- = \begin{pmatrix} \mathcal{A}+\mathcal{B}R\mathcal{D}^T\mathcal{C} & \mathcal{B}R\mathcal{B}^T \\ e^{-\nu j\pi}\mathcal{C}^T\big(I+\mathcal{D}R\mathcal{D}^T\big)\mathcal{C} & e^{-\nu j\pi}\big(\mathcal{A}^T+\mathcal{C}^T\mathcal{D}R\mathcal{B}^T\big)\end{pmatrix}. \tag{15}$$

Given that $T_{zw}^{cl}(j\omega)$ gain response is symmetrical with respect to the axis $\omega = 0$ (even function of ω), H_γ and H_γ^- eigenvalues are conjugate, i.e. H_γ has an eigenvalue on $\mathbb{C}_{\nu 0}^+$ if and only if H_γ^- has a conjugate eigenvalue on $\mathbb{C}_{\nu 0}^- = \{(j\omega)^\nu, \omega \in \mathbb{R}^-\}$.

Now, in order to demonstrate theorem 4, we will use the following theorem.

Theorem 5 ([25]): Hamiltonian matrix

$$H = \begin{pmatrix} \tilde{A} & S \\ -Q & -\tilde{A}^* \end{pmatrix}, \tag{16}$$

$\tilde{A} = \tilde{A}^* \in \mathbb{C}^{n\times n}$, $S = S^T \succ 0 \in \mathbb{R}^{n\times n}$, $Q = Q^T \in \mathbb{R}^{n\times n}$, has no eigenvalue on the imaginary axis if and only if Riccati inequality

$$\tilde{A}^*X + X\tilde{A} + XSX + Q < 0 \tag{17}$$

has an hermitian solution $X = X^* \in \mathbb{C}^{n\times n}$. ∎

Theorem 5 allows to test the position of an Hamiltonian matrix eigenvalues with respect to the imaginary axis via the feasibility of a Riccati inequality. Since, we want to test segment $\mathbb{C}_{\nu 0}^+$, H_γ is rotated so that $\mathbb{C}_{\nu 0}^+$ coincides with the half axis $\mathbb{C}_0^+$ of pure imaginary numbers with positive imaginary part, i.e.

$$H'_\gamma = e^{(1-\nu)j\frac{\pi}{2}} H_\gamma, \tag{18}$$

$$H'_\gamma = \begin{pmatrix} r\left(\mathcal{A}+\mathcal{B}R\mathcal{D}^T\mathcal{C}\right) & r\mathcal{B}R\mathcal{B}^T \\ -\overline{r}\mathcal{C}^T\big(I+\mathcal{D}R\mathcal{D}^T\big)\mathcal{C} & -\overline{r}\big(\mathcal{A}^T+\mathcal{C}^T\mathcal{D}R\mathcal{B}^T\big)\end{pmatrix}, \tag{19}$$

with $r = e^{(1-\nu)j\frac{\pi}{2}}$. This means that $\left\|T_{zw}^{cl}(s)\right\|_\infty < \gamma$ if and only if matrix H'_γ does not have eigenvalues on $\mathbb{C}_0^+$ which is a part of the imaginary axis. But theorem 5 cannot be applied yet to matrix H'_γ as H'_γ is not Hamiltonian. This problem can be solved by using the following similarity transformation:

$$H = UH'_\gamma U^{-1}, \text{ with } U = \begin{pmatrix} I & 0 \\ 0 & e^{(1-\nu)j\frac{\pi}{2}}I \end{pmatrix}. \tag{20}$$

Thus

$$H = \begin{pmatrix} r\left(\mathcal{A}+\mathcal{B}R\mathcal{D}^T\mathcal{C}\right) & \mathcal{B}R\mathcal{B}^T \\ -\mathcal{C}^T\big(I+\mathcal{D}R\mathcal{D}^T\big)\mathcal{C} & -\overline{r}\left(\mathcal{A}^T+\mathcal{C}^T\mathcal{D}R\mathcal{B}^T\right)\end{pmatrix}. \tag{21}$$

Being similar, H'_γ and H have the same eigenvalues and H is an Hamiltonian matrix. Theorem 5 can now be applied with:

$$\tilde{A} = r\left(\mathcal{A}+\mathcal{B}R\mathcal{D}^T\mathcal{C}\right), \tag{22}$$

$$S = \mathcal{B}R\mathcal{B}^T \text{ and } Q = \mathcal{C}^T\big(I+\mathcal{D}R\mathcal{D}^T\big)\mathcal{C}. \tag{23}$$

$\left\|T_{zw}^{cl}(s)\right\|_\infty < \gamma$ if and only if there exists a matrix X such that:

$$\overline{r}\left(\mathcal{A}^T+\mathcal{C}^T\mathcal{D}R\mathcal{B}^T\right)X + Xr\left(\mathcal{A}+\mathcal{B}R\mathcal{D}^T\mathcal{C}\right) + X\mathcal{B}R\mathcal{B}^TX + \mathcal{C}^T\big(I+\mathcal{D}R\mathcal{D}^T\big)\mathcal{C} \prec 0, \tag{24}$$

or

$$\overline{r}\mathcal{A}^TX + Xr\mathcal{A} + \mathcal{C}^T\mathcal{C} + \big(X\mathcal{B}+\overline{r}\mathcal{C}^T\mathcal{D}\big)R\big(\mathcal{B}^TX+r\mathcal{D}^T\mathcal{C}\big) \prec 0. \tag{25}$$

Using Schur complement knowing that $R = \big(\gamma^2 I - \mathcal{D}^T\mathcal{D}\big)^{-1}$, equation (25) can be written:

$$\begin{pmatrix} \overline{r}\mathcal{A}^TX + Xr\mathcal{A} + \mathcal{C}^T\mathcal{C} & \big(X\mathcal{B}+\overline{r}\mathcal{C}^T\mathcal{D}\big) \\ \big(\mathcal{B}^TX + r\mathcal{D}^T\mathcal{C}\big) & -\big(\gamma^2 I - \mathcal{D}^T\mathcal{D}\big)\end{pmatrix} \prec 0, \tag{26}$$

which is equivalent to:

$$\begin{pmatrix} \overline{r}\mathcal{A}^TX + Xr\mathcal{A} & X\mathcal{B} \\ \mathcal{B}^TX & -\gamma^2 I \end{pmatrix} + \begin{pmatrix} \overline{r}\mathcal{C}^T \\ \mathcal{D}^T \end{pmatrix}\begin{pmatrix} r\mathcal{C} & \mathcal{D}\end{pmatrix} \prec 0. \tag{27}$$

Using schur complement again, inequality (27) becomes LMI (14). ∎

Remark 2: Theorem 4 may be considered conservative since the whole imaginary axis is checked for H eigenvalues where only $\mathbb{C}_0^+$ should be checked, but it is not the case. Indeed, let us consider rational fraction:

$$F(x) = \frac{B(x)}{A(x)} = \frac{\prod_{k=0}^{N_z}(x-b_k)}{\prod_{k=0}^{N_p}(x-a_k)}, \tag{28}$$

with $N_z < N_p$, $a_k < 0$ or $a_k = \rho_k e^{j\theta_k}$ where $\rho_k > 0$, $\nu\frac{\pi}{2} < |\theta_k| < \pi$ and $1 < \nu < 2$. Function

$$f : \mathbb{C} \to \mathbb{C}$$
$$x \mapsto F(x) \tag{29}$$

is holomorphic on disc arc $D = \left\{\rho e^{j\theta} | \rho > 0, |\theta| < \nu\frac{\pi}{2}\right\}$ because A has no root in D.

Since domain D is bounded, let us define $D_R = \left\{\rho e^{j\theta} | 0 < \rho < R, |\theta| < \nu\frac{\pi}{2}\right\}$. Since f is holomorphic on D_R and continuous on the adherence of D_R, f reaches its maximum on the frontier of D_R. This means that $\forall x \in D_R$,

$$|f(x)| \le \max\Big\{\sup\left(\left|f\left(\rho e^{+j\nu\frac{\pi}{2}}\right)\right|, 0 < \rho < R\right), \sup\left(\left|f\left(\rho e^{-j\nu\frac{\pi}{2}}\right)\right|, 0 < \rho < R\right), \sup\left(\left|f\left(Re^{-j\theta}\right)\right|, -\nu\frac{\pi}{2} < \theta < \nu\frac{\pi}{2}\right)\Big\}. \tag{30}$$

Since $N_z < N_p$, $\sup\left(\left|f\left(Re^{-j\theta}\right)\right|, -\nu\frac{\pi}{2} < \theta < \nu\frac{\pi}{2}\right)$ tends towards 0 when R tends towards $+\infty$. So if we take the limit of inequality (30) when R tends towards $+\infty$, $\forall x \in D$ we can write:

$$|f(x)| \le \max\left\{\sup\left(\left|f\left(\rho e^{+j\nu\frac{\pi}{2}}\right)\right|, 0 < \rho < R\right), \sup\left(\left|f\left(\rho e^{-j\nu\frac{\pi}{2}}\right)\right|, 0 < \rho < R\right)\right\} \tag{31}$$

and since $\left|f\left(\rho e^{+j\nu\frac{\pi}{2}}\right)\right| = \left|f\left(\rho e^{-j\nu\frac{\pi}{2}}\right)\right|$, we get $\forall x \in D$

$$|f(x)| \le \max\left\{\sup\left(\left|f\left(\rho e^{+j\nu\frac{\pi}{2}}\right)\right|, 0 < \rho < R\right)\right\}, \tag{32}$$

which means that

$$\sup\left(f(x), x \in D\right) \le \max\left\{\sup\left(\left|f\left(\rho e^{+j\nu\frac{\pi}{2}}\right)\right|, 0 < \rho < R\right)\right\}. \tag{33}$$

$$\begin{pmatrix} \bar{r}\left(A\boldsymbol{Z}+B\hat{\boldsymbol{C}}\right)+r\left(\boldsymbol{Z}A^T+\hat{\boldsymbol{C}}^TB^T\right) & \bar{r}\left(A+B\hat{\boldsymbol{D}}C\right)+r\hat{\boldsymbol{A}}^T & B_w+B\hat{\boldsymbol{D}}D_w & \bar{r}\left(C_z\boldsymbol{Z}+D_z\hat{\boldsymbol{C}}\right)^T \\ \bar{r}\hat{\boldsymbol{A}}+r\left(A+B\hat{\boldsymbol{D}}C\right)^T & \bar{r}\left(\boldsymbol{Y}A+\hat{\boldsymbol{B}}C\right)+r\left(A^T\boldsymbol{Y}+C^T\hat{\boldsymbol{B}}^T\right) & \boldsymbol{Y}B_w+\hat{\boldsymbol{B}}D_w & \bar{r}\left(C_z+D_z\hat{\boldsymbol{D}}C\right)^T \\ \left(B_w+B\hat{\boldsymbol{D}}D_w\right)^T & \left(\boldsymbol{Y}B_w+\hat{\boldsymbol{B}}D_w\right)^T & -\gamma^2 I & \left(D_{zw}+D_z\hat{\boldsymbol{D}}D_w\right)^T \\ r\left(C_z\boldsymbol{Z}+D_z\hat{\boldsymbol{C}}\right) & r\left(C_z+D_z\hat{\boldsymbol{D}}C\right) & D_{zw}+D_z\hat{\boldsymbol{D}}D_w & -I \end{pmatrix} \prec 0 \quad (35)$$

As the axis $\left\{-xe^{j\nu\frac{\pi}{2}}, x \geq 0\right\}$ is included in D, we can conclude that

$$\left\|f\left(-xe^{j\nu\frac{\pi}{2}}\right)\right\| \leq \left\|f\left(xe^{j\nu\frac{\pi}{2}}\right)\right\|, \forall x \geq 0. \quad (34)$$

This result means that an eigenvalue of H_γ will always appear on the segment $\left\{xe^{j\nu\frac{\pi}{2}}, x > 0\right\}$ corresponding to $\mathbb{C}^+_{\nu 0}$ before it does on segment $\left\{-xe^{j\nu\frac{\pi}{2}}, x > 0\right\}$. Since after the rotation which transforms H_γ into H'_γ, $\mathbb{C}^+_{\nu 0}$ becomes the set of pure imaginary numbers with positive imaginary parts and $\left\{-xe^{j\nu\frac{\pi}{2}}, x > 0\right\}$ becomes the set of pure imaginary numbers with negative imaginary parts, theorem 4 can no longer be considered conservative.

Theorems 3 and 4 are analysis results that allow to determine a commensurate FOS H_∞ norm.

C. Main Results

Based on theorem 4, the following synthesis method is derived.

Theorem 6: Fractional order system (1) of order ν is BIBO stabilizable by output feedback control law (5) and $\left\|T^{cl}_{zw}\right\|_\infty < \gamma$ if there exist $\boldsymbol{Z} = \boldsymbol{Z}^T \in \mathbb{R}^{n\times n}$, $\boldsymbol{Y} = \boldsymbol{Y}^T \in \mathbb{R}^{n\times n}$, $\hat{\boldsymbol{A}} \in \mathbb{R}^{n\times n}$, $\hat{\boldsymbol{B}} \in \mathbb{R}^{n\times n_y}$, $\hat{\boldsymbol{C}} \in \mathbb{R}^{n_u\times n}$ and $\hat{\boldsymbol{D}} \in \mathbb{R}^{n_u\times n_y}$ such that LMI (35) is feasable with

$$\begin{pmatrix} \boldsymbol{Z} & I \\ I & \boldsymbol{Y} \end{pmatrix} \succ 0. \quad (36)$$

After solving LMI (35), find non singular matrices M and N such that $MN^T = I - ZY$. The controller pseudo state matrices are then defined by

$$\begin{cases} D_K & := \hat{\boldsymbol{D}} \\ C_K & := \left(\hat{\boldsymbol{C}} - D_K C\boldsymbol{X}\right)M^{-T} \\ B_K & := N^{-1}\left(\hat{\boldsymbol{B}} - \boldsymbol{Y}BD_K\right) \\ A_K & := N^1\left(\hat{\boldsymbol{A}} - NB_K C\boldsymbol{X} - \boldsymbol{Y}BC_K M^T \right. \\ & \quad \left. -\boldsymbol{Y}\left(A+BD_K C\right)\boldsymbol{X}\right)M^{-T} \end{cases} . \quad (37)$$

■

Proof: H_∞ norm of closed-loop system (6) is less than γ if there exist matrices A_K, B_K, C_K, D_K and $X = X^*$ s.t. inequality (14) is feasible with matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ defined by (8-9). The main difficulty of the proof is to find a change of variable that allows to transform this inequelity into an LMI. Such a change of variable is given in [24] for IOS and will be applied here on FOS.

Let $X = X^* \in \mathbb{R}^{2n\times 2n}$ be a solution of analysis LMI (14). Partition X and X^{-1} as

$$X = \begin{pmatrix} \boldsymbol{Y} & N \\ N^T & \star \end{pmatrix} \quad , \quad X^{-1} = \begin{pmatrix} \boldsymbol{Z} & M \\ M^T & \star \end{pmatrix}, \quad (38)$$

where $\boldsymbol{Z}$ and $\boldsymbol{Y}$ are $n \times n$ and symmetric. Given that $XX^{-1} = I$, we pose $X\begin{pmatrix} Z \\ M^T \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}$ which leads to

$$X\Pi_1 = \Pi_2 \quad (39)$$

with

$$\Pi_1 := \begin{pmatrix} \boldsymbol{Z} & I \\ M^T & 0 \end{pmatrix} \quad , \quad \Pi_2 := \begin{pmatrix} I & \boldsymbol{Y} \\ 0 & N^T \end{pmatrix}. \quad (40)$$

The linearizing change of variables is then defined as follows

$$\begin{cases} \hat{\boldsymbol{A}} & := NA_K M^T + NB_K C\boldsymbol{Z} + \boldsymbol{Y}BC_K M^T \\ & \quad +\boldsymbol{Y}\left(A+BD_K C\right)\boldsymbol{Z} \\ \hat{\boldsymbol{B}} & := NB_K + \boldsymbol{Y}BD_K \\ \hat{\boldsymbol{C}} & := C_K M^T + D_K C\boldsymbol{Z} \\ \hat{\boldsymbol{D}} & := D_K \end{cases} \quad (41)$$

where $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{C}}$ are respectively $n\times n$, $n\times n_u$ and $n_y\times n$ matrices. Since M and N can always be chosen with full row rank, the controller matrices A_K, B_K, C_K and D_K can always be computed from $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$, $\hat{\boldsymbol{C}}$, $\hat{\boldsymbol{D}}$, $\boldsymbol{Z}$ and $\boldsymbol{Y}$.

This transformation is motivated by the following identities derived from (39), (40) and (41):

$$r\Pi_1^T X\mathcal{A}\Pi_1 = r\Pi_2^T\mathcal{A}\Pi_1 = r\begin{pmatrix} A\boldsymbol{Z}+B\hat{\boldsymbol{C}} & A+B\hat{\boldsymbol{D}}C \\ \hat{\boldsymbol{A}} & \boldsymbol{Y}A+\hat{\boldsymbol{B}}C \end{pmatrix}, \quad (42)$$

$$\Pi_1^T X\mathcal{B} = \Pi_2^T\mathcal{B} = \begin{pmatrix} B_w+B\hat{\boldsymbol{D}}D_w \\ \boldsymbol{Y}B_w+\hat{\boldsymbol{B}}D_w \end{pmatrix}, \quad (43)$$

$$r\mathcal{C}\Pi_1 = \begin{pmatrix} C_z\boldsymbol{Z}+D_z\hat{\boldsymbol{C}} & C_z+D_z\hat{\boldsymbol{D}}C \end{pmatrix}, \quad (44)$$

$$\Pi_1^T X\Pi_1 = \Pi_1^T\Pi_2 = \begin{pmatrix} \boldsymbol{Z} & I \\ I & \boldsymbol{Y} \end{pmatrix}. \quad (45)$$

The existence of a stabilizing controller K such that $\left\|T^{cl}_{zw}\right\|_\infty < \gamma$ is determined by feasibility of LMI (14) with a positive definite solution $X = X^* \succ 0$. If we perform a congruence transformation with $\mathrm{diag}(\Pi_1, I, I)$ on LMI (14) and with Π_1 on matrix X we get

$$\begin{pmatrix} \bar{r}\Pi_1^T\mathcal{A}^T X\Pi_1 + \Pi_1^T Xr\mathcal{A}\Pi_1 & \Pi_1^T X\mathcal{B} & \bar{r}\Pi_1^T\mathcal{C}^T \\ \mathcal{B}^T X\Pi_1 & -\gamma^2 I & \mathcal{D}^T \\ r\mathcal{C}\Pi_1 & \mathcal{D} & -I \end{pmatrix} \prec 0, \quad (46)$$

and equality (45).

We obtain LMI (35) by replacing $r\Pi_1^T X\mathcal{A}\Pi_1$, $\Pi_1^T X\mathcal{B}$ and $r\mathcal{C}\Pi_1$ by their explicit expressions (42), (43) and (44).

Now suppose solutions to LMI (35) verifying (36) have been found. Matrices M and N should be chosen such that

$$MN^T = I - \boldsymbol{ZY}. \tag{47}$$

Since $\boldsymbol{Z}$ and $\boldsymbol{Y}$ satisfy relation (36), square and non singular matrices M and N verifying relation (47) can always be found. The controller pseudo-state matrices D_K, C_K, B_K and A_K are then given by relation (37) which is the inversion of the linearizing change of variable (41). This inversion is possible given that matrices Π_1 and Π_2 which depends on M and N are square and non singular. Hence the constructed controller K leads to $\left\|T_{zw}^{cl}\right\|_\infty < \gamma$. This concludes that proof. ∎

Remark 3: All matrix variables of theorem 6 are real matrices whereas they should be complex since LMI (35) is derived from LMI (14) which comes from complex Riccati inequality (17). That means that theorem 6 normally leads to a controller K with complex pseudo-state matrices which satisfies $\left\|T_{zw}^{cl}\right\|_\infty < \gamma$. In order to derive a controller whose state space matrices are real, we imposed theorem 6 matrix variables to be real. Such a choice would lead to some conservatism in the obtained condition. More precisely, let $\gamma_c = \min_{(\boldsymbol{Y},\boldsymbol{Z},\hat{\boldsymbol{A}},\hat{\boldsymbol{B}},\hat{\boldsymbol{C}},\hat{\boldsymbol{D}})\in\mathbb{C}} \gamma$ s.t. (35-36) and $\gamma_r = \min_{(\boldsymbol{Y},\boldsymbol{Z},\hat{\boldsymbol{A}},\hat{\boldsymbol{B}},\hat{\boldsymbol{C}},\hat{\boldsymbol{D}})\in\mathbb{R}} \gamma$ s.t. (35-36), then $\gamma_c \leq \gamma_r$. Numerical simulations however showed that γ_r is practically equal to γ_c. The authors of the paper are currently working on a proof.

IV. APPLICATION

Theorem 6 has been used to design a stabilizing fractional order controller for an academic Single-Input-Single-Output (SISO) system G. The pseudo-state space representation of $G(s)$ is

$$\left\{\begin{array}{rcl} D^\nu x(t) & = & A_s\, x(t) + B_s\, u(t) \\ y(t) & = & C_s\, x(t) + D_s\, u(t) \end{array}\right. \tag{48}$$

with fractional order $\nu = 1.3$,

$$A_s = \begin{pmatrix} -2 & -1.5 \\ 2 & 0 \end{pmatrix} \quad , \quad B_s = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} \qquad C_s = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad \text{and} \quad D_s = 0. \tag{49}$$

Fig. 2 shows the plant step response. Constraints $W_1(s)$ and $W_2(s)$ have respectively been added on the closed-loop sensitivity function $S(s)$ from w to ϵ and on the complementary sensitivity function $T(s)$ from w to y as shown in fig. 3. The closed-loop system sensitivity functions must verify

$$\|W_1 S\|_\infty < 1 \quad , \quad \|W_2 T\|_\infty < 1 \tag{50}$$

which is equivalent to

$$\|S\|_\infty < \left\|W_1^{-1}\right\|_\infty \quad , \quad \|T\|_\infty < \left\|W_2^{-1}\right\|_\infty . \tag{51}$$

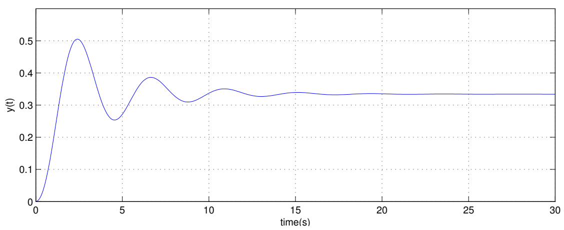

Fig. 2. $G(s)$ step response

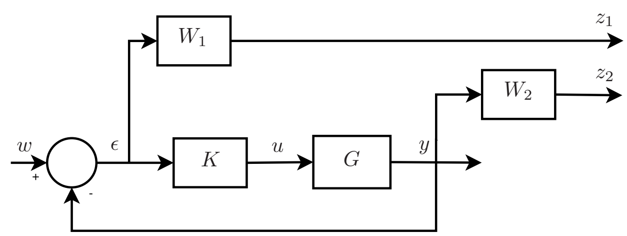

Fig. 3. Closed-loop system

Constraint W_1^{-1} static gain has been chosen low enough to cancel the closed-loop static error. Constraint W_2^{-1} has been chosen in order to attenuate $T(s)$ resonance and thus overshoot and oscillations in time response. Their transfer functions are

$$W_1^{-1}(s) = 3.16 \cdot \frac{s^{1.3} + 5.6 \times 10^{-4}}{s^{1.3} + 1} \tag{52}$$

$$W_2^{-1}(s) = 1.99 \times 10^{-5} \cdot \frac{s^{1.3} + 1.6 \times 10^5}{s^{1.3} + 1}. \tag{53}$$

The general control configuration for this H_∞ problem is given by relation (1). Matrices of this representation are not given due to paper size limitation.

Theorem 6 is used to obtain an output feedback controller K such that $\left\|T_{zw}^{cl}\right\|_\infty < 1$, i.e.

$$\left\| \begin{array}{c} W_1 S \\ W_2 T \end{array} \right\|_\infty < 1. \tag{54}$$

Please note that if a feasible solution to LMI (35-36) is found, then relation (54) is true which also implies that relation (50) is fulfilled. However, no conclusion can be drawn if LMI (35-36) is not feasible as (54) $\rightleftarrows$ (50).

Solver SDPT3 [26] is used to solve LMIs associated to theorem 6 and thus to obtain controller K given by relation (5) with

$$A_K = \begin{pmatrix} -1.9 \times 10^5 & 4.7 \times 10^4 & 969.9 & -7.4 \times 10^6 \\ 1.9 \times 10^4 & -4.7 \times 10^3 & -96.9 & 7.4 \times 10^5 \\ -344.3 & 414.2 & -3.5 & 2.2 \times 10^4 \\ 1.1 \times 10^3 & -152.5 & 7.1 & -1.8 \times 10^5 \end{pmatrix} \tag{55}$$

$$B_K = \begin{pmatrix} -8.5 \times 10^6 \\ 8.5 \times 10^5 \\ 2.2 \times 10^5 \\ -2.8 \times 10^4 \end{pmatrix} \quad , \quad D_K = 3.4 \times 10^4, \tag{56}$$

$$C_K = \begin{pmatrix} 764.5 & -188 & -3.8 & 12.9 \times 10^4 \end{pmatrix}. \tag{57}$$

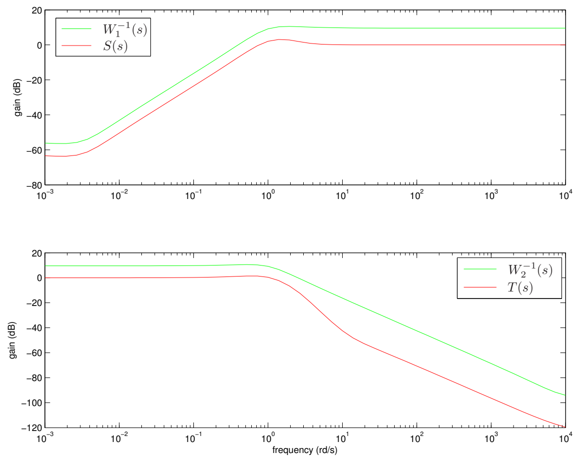

Fig. 4. Constraints and sensitivity functions

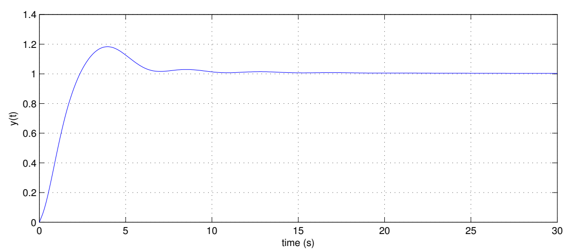

Fig. 5. Closed-loop step response

Computation of pseudo state matrix $\mathcal{A}$ eigenvalues confirms that closed loop system is stable. Indeed they verify theorem 1. Relation (50) on the sensitivity functions is respected as shown in fig. 4. Moreover, as shown in fig. 5, the closed-loop system exhibits less oscillations and is faster since time response has been divided by 2 (with $t_r^{5\%} = 10.8s$ for the plant and $t_r^{5\%} = 6s$ for the closed-loop system). Static error is also negligible.

Remark 4: Concerning the conservatism mentioned in remark 3, minimizing γ in LMIs (35-36) with complex and real matrix variables leads to $\gamma_c = 0.44$ and $\gamma_r = 0.47$ respectively. Besides leading to a controller which cannot be physically implemented, the closed loop system performances would not have been significantly increased by the use of complex matrix variables.

V. CONCLUSION

In this paper was developped a new output feedback synthesis method for commensurate FOS. The particularity of the proposed method is that no integer approximation of the plant transfer function is made before the synthesis and a commensurate fractional controller with the same order as the plant is designed. The efficiency of the method was checked via the output feedback control of an academic SISO fractional system. A stabilizing controller improving the system performance has been found and the shaping constraints on the sensitivity functions has been respected.

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