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Time Convergence Estimation of a Perturbed Double Integrator: Family of Continuous Sliding Mode Based Output Feedback Synthesis

Raúl Santiesteban
Instituo Tecnológico de Culiacán,
Department of Metal-Mecánica.
E-mail:raulcos@mexico.com

Abstract—**In this paper mechanical systems of relative degree two, *i.e.* a perturbed double integrator is under study. A sliding mode based algorithm is under study using strict non smooth Lyapunov functions, such that compensation of growing perturbations together with state variables is shown. Indeed, the well known twisting algorithm and a generalized smooth family of this algorithm are considered. A strict non-smooth Lyapunov function is proposed allowing to design tuning rules for the gains of a family of controllers such that global exact finite time stability of the origin is shown. The proposed methodology estimate an upper bound for convergence time of the closed loop system in spite of growing perturbation with respect to the state. To illustrate performance and robustness properties a numerical experiment is presented, using one-link pendulum as a test bed.**

Keywords: Second-order sliding modes; Lyapunov function; Stability analysis.

I. INTRODUCTION

Second order sliding mode algorithms (SOSM) have become very important for control theory because of their properties such as finite time convergence to the origin in the presence of bounded, persisting external disturbances and parametric uncertainties. One of the first SOSM algorithms, the twisting algorithm (see [1]) , became very popular due its advantage to consider Coulomb friction as part of the controller. However, this kind of algorithm presents a high frequency phenomena known as “chattering” causing an early wearing of actuators.

In this paper, a generalization of the twisting algorithm is studied in order to design continuous finite time- stabilizing feedback controllers (see [2], [3]). The double integrator, one-link pendulum, is used as an illustrative example for this objective. In order to avoid the undesired chattering phenomenon that appears in the closed-loop system (see [4]) a smooth version of ”Twisting algorithm“ is studied. The closed loop double integrator under study is a family of homogeneous continuous controllers, where twisting algorithm is a special case. In [3], global asymptotic stability of the perturbed double integrator is established by applying the invariance principle, using weak Lyapunov function design in spite of external growing perturbations. Since the algorithm is homogeneous, global finite time stability of the origin can be concluded. However, since a weak Lyapunov function is proposed the upper bound for the growing perturbations is with respect to one state variable only.

Based on previews work [5], a strict non smooth Lyapunov function is proposed in order to show the stability of the generalized algorithm. Allowed by this methodology, an estimation of time convergence of the closed loop system to the origin is given. Moreover, the strict Lyapunov function allows a deeply study of the robustness of this algorithm. Indeed, a first contribution of this work is an extension of the gain restrictions given in [3] where the upper bound of the growing perturbation is with respect to both state variables.

Since a family of controllers is under study, the results presented in this paper must support previews ones. For example, the tuning rules of twisting algorithm, a special case of this generalized algorithm, gives a restriction that is commonly known in sliding mode theory. Moreover, since an observer for this generalization of twisting algorithm is needed, tuning rules for closed loop system are given.

For simplicity purposes, in this paper the full vector

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state of the dynamic system is consider to be available for measurement. A lot of information is available if one of the state is not available for measurement. In [6] a non smooth Lyapunov function is proposed, in order to proof finite time stability of super-twisting as a observer. Another example, in [7] the tuning rules of the observer gains, ensuring that the observer error converges to zero before the trajectories of the closed-loop system escape the region specified by the user are under study. Moreover, the proof of the separation principle for the output feedback problem when the super-twisting is applied is well known (see for example [8]).

The stability analysis of the closed loop system is made within strict non-smooth Lyapunov methodology for discontinuous systems (see [9]). Finite time convergence and robustness of the control law design are supported using numerical experiments. A one-link pendulum affected by Coulomb friction and growing perturbations with respect to the state is used as a test bed. In section 2 the problem statement is presented: the stabilization of an uncertain system is under study. In section 3, the stability of the unperturbed closed loop system is showed. In section 3 the closed loop affected by external bounded perturbations is analyzed. In both cases, finite time stability for the closed loop system is concluded. In section 4 to support theoretical results, a numerical example is shown and in section 5 are the conclusions of this work.

II. PROBLEM STATEMENT

The general model of second-order mechanical systems, written in the state space form is given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= f(x,y) + g(x,y)\tau + \delta(t,x,y) \end{aligned} \qquad (1)$$

where $g(x,y) \neq 0$ the nominal known part of the system dynamics is represented by the function $f(x,y)$, while the uncertainties are concentrated in $\delta(t,x,y)$ such as growing perturbations with respect to the state variables (see [3], [10]). Since the right hand side of the equation (1) has discontinuous terms, the solutions of system (1) are understood in the Filippov sense (see [11]). For system (1) the following controller design is proposed

$$\tau = \frac{1}{g(x,y)}\left(U - f(x,y)\right) \qquad (2)$$

where U is a new input control. In the next section the unperturbed system will be under study using non smooth strict Lyapunov functions.

A. Description of the unperturbed system

Consider the controlled system (1)-(2) given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= U \end{aligned}$$

where x and $y \in \mathbb{R}$ are scalar state variables and $U \in \mathbb{R}$ is governed by

$$U = -k_1|x|^{\frac{\alpha}{2-\alpha}}\operatorname{sgn}(x) - k_2|y|^{\alpha}\operatorname{sgn}(y) \qquad (3)$$

where k_1, k_2 are positive constants and $0 \leq \alpha < 1$. Let

$$\begin{aligned} V(x,y) &= \frac{2-\alpha}{2}k_1^2|x|^{\frac{4}{2-\alpha}} + k_1|x|^{\frac{2}{2-\alpha}}y^2 \\ &+ |x|^{\frac{3}{2-\alpha}}\operatorname{sgn}(x)y + \frac{1}{2(2-\alpha)}y^4 \end{aligned} \qquad (4)$$

a positive definite Lyapunov function. This function is continuous everywhere but not differentiable for $x = 0$. Equation (4) is a strict Lyapunov function (see [12]) of system (1-3), then finite time stability can be concluded.

Theorem 1: if

$$\begin{aligned} k_1 &> \frac{1}{1+\alpha}\max\{\alpha, k_2\}; \;\; k_2 > \frac{3}{4}(1-\alpha) \\ k_1k_2 &> \left(\frac{3}{2}\right)\frac{1+\alpha}{2-\alpha} \end{aligned} \qquad (5)$$

holds, then system (3) has finite time convergence to the point $(x,y) = (0,0)$ with

$$t_{reach} \leq \frac{1}{\zeta_{min}}\gamma_{max}^{\frac{3+\alpha}{4}}V^{\frac{1-\alpha}{4}}(x(0),y(0)) \qquad (6)$$

as an estimation of the convergence time, with

$$\begin{aligned} \zeta_{min} &= \min\Big\{k_1k_2 - \frac{\alpha}{1+\alpha};\;\; k_1 - \frac{1}{1+\alpha}k_2 \\ k_1k_2 &- \frac{3(1+\alpha)}{2(2-\alpha)};\;\; k_2 - \frac{3}{4}(1-\alpha)\Big\} \end{aligned} \qquad (7)$$

and

$$\eta = \max\{\lambda_{\max}(P), \frac{1}{2(2-\alpha)}\} \qquad (8)$$

.

proof: Let equation (4) a candidate Lyapunov function and it can be written as follows

$$\begin{aligned} V(x,y) &= |x|^{\frac{2}{2-\alpha}}\Big\{\frac{2-\alpha}{2}k_1^2|x|^{\frac{2}{2-\alpha}} + k_1y^2 \\ &+ |x|^{\frac{1}{2-\alpha}}\operatorname{sgn}(x)y\Big\} + \frac{1}{2(2-\alpha)}y^4 \end{aligned} \qquad (9)$$

$$V(x,y) = |x|^{\frac{2}{2-\alpha}}\rho P \rho^T + \frac{1}{2(2-\alpha)}y^4 \qquad (10)$$

where $\rho = [|x|^{\frac{2}{2-\alpha}}\operatorname{sgn}(x)\;\; y]$, and

$$P = \begin{pmatrix} \frac{2-\alpha}{2}k_1^2 & \frac{1}{2} \\ \frac{1}{2} & k_1 \end{pmatrix} \qquad (11)$$

If $det(P) > 0$ then equation (4) is positive definite since $y^4 > 0$ for all $(x,y) \in \mathbb{R}$. Then $k_1^3 > \frac{1}{2(2-\alpha)}$. Using the eigenvalues of matrix P, equation (10) is written as follows in order to find an upper bound

$$V(x,y) \leq \eta \left(|x|^{\frac{1}{2-\alpha}} + |y|\right)^4 \tag{12}$$

where η is given by eqaution (8). In order to show finite time stability, the time derivative of (4) is given by

$$\begin{aligned}
\dot{V}(x,y) &= 2k_1^2|x|^{\frac{2+\alpha}{2-\alpha}}\operatorname{sgn}(x)y \\
&+ \frac{2}{2-\alpha}k_1|x|^{\frac{\alpha}{2-\alpha}}\operatorname{sgn}(x)y^3 \\
&+ 2k_1|x|^{\frac{2}{2-\alpha}}y\left(-k_1|x|^{\frac{\alpha}{2-\alpha}}\operatorname{sgn}(x) - k_2|y|^\alpha\operatorname{sgn}(y)\right) \\
&+ \frac{3}{2-\alpha}|x|^{\frac{1+\alpha}{2-\alpha}}y^2 \\
&+ |x|^{\frac{3}{2-\alpha}}\operatorname{sgn}(x)\left(-k_1|x|^{\frac{\alpha}{2-\alpha}}\operatorname{sgn}(x) - k_2|y|^\alpha\operatorname{sgn}(y)\right) \\
&+ \frac{2}{2-\alpha}y^3\left(-k_1|x|^{\frac{\alpha}{2-\alpha}}\operatorname{sgn}(x) - k_2|y|^\alpha\operatorname{sgn}(y)\right)
\end{aligned} \tag{13}$$

after some algebraic simplifications,

$$\begin{aligned}
\dot{V}(x,y) &= -2k_1k_2|x|^{\frac{2}{2-\alpha}}|y|^{1+\alpha} \\
&+ \frac{3}{2-\alpha}|x|^{\frac{1+\alpha}{2-\alpha}}y^2 - k_1|x|^{\frac{3+\alpha}{2-\alpha}} \\
&- k_2|x|^{\frac{3}{2-\alpha}}|y|^\alpha\operatorname{sgn}(xy) \\
&- \frac{2}{2-\alpha}k_2|y|^{\alpha+3}
\end{aligned} \tag{14}$$

Equation (14) can be simplify as follows

$$\begin{aligned}
\dot{V}(x,y) &\leq -|x|^{\frac{2}{2-\alpha}}\Big(k_1k_2|y|^{1+\alpha} \\
&+ k_1|x|^{\frac{1+\alpha}{2-\alpha}} - k_2|x|^{\frac{1}{2-\alpha}}|y|^\alpha\operatorname{sgn}(xy)\Big) \\
&- |y|^{1+\alpha}\Big(k_1k_2|x|^{\frac{2}{2-\alpha}} - \frac{3}{2-\alpha}|x|^{\frac{1+\alpha}{2-\alpha}}y^{1-\alpha} \\
&+ \frac{2}{2-\alpha}k_2|y|^2\Big)
\end{aligned} \tag{15}$$

In order to show that $\dot{V}(x,y) \leq 0$, consider the following inequalities

$$\begin{aligned}
|x|^{\frac{1}{2-\alpha}}|y|^\alpha &\leq \frac{1}{r_d}\gamma_d^{r_d}|x|^{\frac{1+\alpha}{2-\alpha}} + \frac{1}{s_d}\gamma_d^{-s_d}|y|^{1+\alpha}, \\
&\quad with\ \ r_d = 1+\alpha, s_d = \frac{1+\alpha}{\alpha} \\
|x|^{\frac{1+\alpha}{2-\alpha}}|y|^{1-\alpha} &\leq \frac{1}{r_c}\gamma_c^{r_c}|x|^{\frac{2}{2-\alpha}} + \frac{1}{s_c}\gamma_c^{-s_c}|y|^2, \\
&\quad with\ \ r_c = \frac{2}{1+\alpha}, s_c = \frac{2}{1-\alpha}
\end{aligned} \tag{16}$$

then equation (**??**) can be written as follows

$$\begin{aligned}
\dot{V}(x,y) \leq\ & -|x|^{\frac{2}{2-\alpha}}\left(|y|^{1+\alpha}\left(k_1k_2 - \frac{1}{s_d}\gamma_d^{-s_d}k_2\right)\right. \\
&\left. +|x|^{\frac{1+\alpha}{2-\alpha}}\left(k_1 - \frac{1}{r_d}\gamma_d^{r_d}k_2\right)\right) \\
&-|y|^{1+\alpha}\left(|x|^{\frac{2}{2-\alpha}}\left(k_1k_2 - \frac{1}{r_c}\gamma_c^{r_c}\frac{3}{2-\alpha}\right)\right. \\
&\left. +|y|^2\left(\frac{2}{2-\alpha}k_2 - \frac{1}{s_c}\gamma_c^{-s_c}\frac{3}{2-\alpha}\right)\right)
\end{aligned} \tag{17}$$

if

$$\begin{aligned}
k_1 &> \max\left\{\frac{1}{s_d}\gamma_d^{-s_d},\ \frac{1}{r_d}\gamma_d^{r_d}k_2\right\} \\
k_2 &> \frac{1}{s_c}\gamma_c^{-s_c}\frac{3}{2},\ \ k_1k_2 > \frac{1}{r_c}\gamma_c^{r_c}\frac{3}{2-\alpha}
\end{aligned} \tag{18}$$

hold, the time derivative of the Lyapunov function $V(x,y)$ is negative definite. Since the inequalities (18) are in terms of the positive constants γ_c and γ_d, a solution can be found, always. Let $\gamma_d = 1$ and $\gamma_c = 1$ then inequalities (18) simplifies to (5). In order to show the stability of the system (1-3), let us write the equation (12) as follows

$$V(x,y) \leq \eta \left(|x|^{\frac{1}{2-\alpha}} + |y|\right)^4 \tag{19}$$

$$\left(\frac{V(x,y)}{\eta}\right)^{\frac{1}{4}} \leq \left(|x|^{\frac{1}{2-\alpha}} + |y|\right) \tag{20}$$

and the let us write the equation (17) as follows

$$\dot{V}(x,y) \leq -\zeta_{min}\left(|x|^{\frac{1}{2-\alpha}} + |y|\right)^{3+\alpha} \tag{21}$$

where ζ is given by equation (7). Then the time derivative of $V(x,y)$ can be written as follows

$$\dot{V}(x,y) \leq -\zeta_{min}\left(\frac{V(x,y)}{\eta}\right)^{\frac{3+\alpha}{4}} \tag{22}$$

Consider the following comparison system

$$\dot{\omega} = -a\omega^{\frac{3+\alpha}{4}} \tag{23}$$

The solution of this system is $\omega(t) = (\omega^{\frac{1-\alpha}{4}}(0) - at)^4$, and thus the estimation for reaching time is $t_{reach} = \frac{1}{a}\omega^{\frac{1-\alpha}{4}}(0)$. Summing up, an estimation of an upper bound for the reaching time of the system (3) can be calculated as inequality (6). then, finite time stability for the system (3) can be concluded. ✠ *Example 1.* Considering $\alpha = 0$, the algorithm (3) is known as twisting algorithm, a well-known controller. The function (4) is simplified to (see [5])

$$V(x,y) = k_1|x|^2 + k_1|x|y^2 + |x|^{\frac{3}{2}}\operatorname{sgn}(x)y + \frac{1}{4}y^4 \tag{24}$$

and global finite time stability can be concluded for the twisting algorithm, and inequalities (5) are as follows

$$k_1 > k_2 > 0 \tag{25}$$

In the next section, the stability analysis of the perturbed system will be treated considering external bounded perturbation.

B. Description of the perturbed system

Consider the controlled system (1)-(2) given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= U + \delta(x,y) \end{aligned} \tag{26}$$

where

$$\delta(x,y) \leq \mu_y |y|^{\alpha} + \mu_x |x|^{\frac{\alpha}{2-\alpha}}, \tag{27}$$

using the generalized algorithm (3). Let

$$\begin{aligned} V(x,y) &= \frac{2-\alpha}{2} k_1 \left(k_1 - \mu_x \mathrm{sgn}(x)\right) |x|^{\frac{4}{2-\alpha}} + k_1 |x|^{\frac{2}{2-\alpha}} y^2 \\ &+ |x|^{\frac{3}{2-\alpha}} \mathrm{sgn}(x) y + \frac{1}{2(2-\alpha)} y^4 \end{aligned} \tag{28}$$

a positive definite Lyapunov function. This function is continuous everywhere but not differentiable for $x = 0$. It will be shown that (28) is a strict non smooth Lyapunov function of system (26), then finite time stability will be proved.

Theorem 2: if

$$\begin{aligned} k_1 \left(k_2 - \mu_y\right) &> \frac{6}{(1+\alpha)(2-\alpha)} \\ k_2 &> \max\left\{\mu_y;\ \mu_y + \frac{3}{2(1+\alpha)}\right\} \\ k_1 &> \max\left\{4\frac{\alpha}{(3+\alpha)(2-\alpha)};\ \frac{\alpha}{1+\alpha}\right\}\mu_x \\ \frac{1}{2}(k_1 - \mu_x) &> \frac{1}{1+\alpha}(k_2 - \mu_y) > \frac{6}{(3+\alpha)(1+\alpha)}\mu_x \end{aligned} \tag{29}$$

holds, then system (26) has finite time convergence to the point $(x,y) = (0,0)$ with

$$t_{reach} \leq \frac{1}{\zeta_{minp}} \gamma_{max}^{\frac{3+\alpha}{4}} V^{\frac{1-\alpha}{4}}(x(0), y(0)) \tag{30}$$

as an estimation of the convergence time, with

$$\begin{aligned} \zeta_{minp} &= \min\Big\{ k_1 \left(k_2 - \mu_y\right) - \frac{6}{(1+\alpha)(2-\alpha)}; \\ &\quad k_2 - \mu_y;\ k_2 - \mu_y - \frac{3}{2(1+\alpha)}; \\ &\quad k_2 - \mu_y - \frac{6}{3+\alpha}; \\ &\quad \frac{k_1}{2} - 4\frac{\alpha}{(3+\alpha)(2-\alpha)}\mu_x;\ k_1 - \frac{\alpha}{1+\alpha}\mu_x; \\ &\quad \frac{1}{2}(k_1 - \mu_x) - \frac{1}{1+\alpha}(k_2 - \mu_y)\Big\} \end{aligned} \tag{31}$$

$$\gamma_{max} = \max\Big\{\lambda_{mx}(P), \frac{1}{2(2-\alpha)}\Big\}. \tag{32}$$

Example 2. Again, considering $\alpha = 0$, equation (26-27) simplifies to twisting algorithm affected by external bounded perturbations. In order to show finite time stability of this algorithm, the following inequality

$$k_1 - \mu > k_2 > \mu. \tag{33}$$

where $\mu > 0$ must be satisfied at all time. When $\alpha = 0$, inequality (27) simplifies to

$$\delta(x,y) \leq \mu_y + \mu_x, \tag{34}$$

If $\delta(x,y) \leq \mu = \mu_y + \mu_x$ inequality (33) must be satisfied.

In [13] linear terms are added to improve the performance of the algorithm as follows

$$U = -k_1 \mathrm{sgn}(x) - k_2 \mathrm{sgn}(y) + hx + py \tag{35}$$

in [5], a non-smooth Lyapunov function

$$\begin{aligned} W\left(x,y\right) = V_0^2\left(x,y\right) + \left(\gamma + |x|^{\frac{3}{2}}\right)|x|^{\frac{3}{2}}\,\mathrm{sign}\left(x\right)y + \\ + \frac{2}{5}p\left(\gamma + \frac{5}{8}|x|^{\frac{3}{2}}\right)|x|^{\frac{5}{2}} \end{aligned} \tag{36}$$

where

$$V_0\left(x,y\right) = \alpha\left|x\right| + \frac{1}{2}\left(hx^2 + y^2\right) \tag{37}$$

is propose in order to prove finite time stability. Moreover, an upper bound for convergence time is given

$$T_{\mathrm{reach}} \leq \frac{4\eta_{\mathrm{max}}^{\frac{3}{4}}}{\theta_{\mathrm{min}}} W^{\frac{1}{4}}\left(x_0, y_0\right)\ , \tag{38}$$

showing that the system is globally finite-time stable, against external bounded perturbations.

III. NUMERICAL EXPERIMENTS

To illustrate the algorithm performance consider a tracking problem of the one-link pendulum system affected by Coulomb friction and external perturbations bounded by a positive constant. Let us consider that all the state space is available. In other case an observer-controller algorithm can be design such that the control objective is achieved. It is well known that ”supertwisting“ algorithm (sliding mode based observer) has nice properties such as robustness, uniformity and finite time stability.

The state equation of a controlled one-link pendulum (see Fig. 1) is given by

$$(ml^2 + J)\ddot{q} = mglsin(q) - F(\dot{q}) + \tau + \delta(t, q, \dot{q}) \tag{39}$$

where q is the angle made by the pendulum with the vertical,

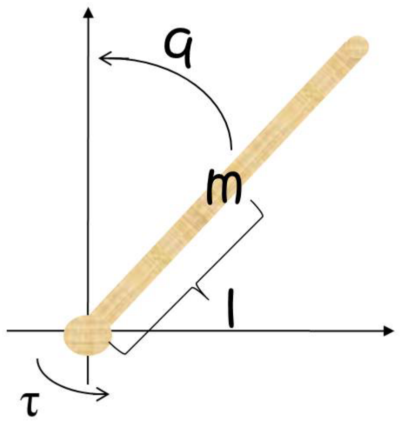

Figure 1. The one-link pendulum system.

m is the mass of the pendulum, l is the distance to the center of mass, J is moment of inertia of the pendulum about the center of mass, g is the gravity acceleration, τ is the control torque. The friction force F is described by

$$F(\dot{q}) = \rho_v \dot{q} + \rho_c sign(\dot{q}). \tag{40}$$

where $\rho_v > 0$ denotes the viscous friction coefficient and $\rho_c > 0$ denotes the Coulomb friction level. Suppose that the uncertainty term $\delta(t, q, \dot{q})$ is bounded by growing terms as in (27).

The control objective is to drive the one-link pendulum to a known trajectory in exact finite time, *i.e.*

$$q(t) - r(t) = 0 \tag{41}$$

where $r(t) = \frac{1}{8} sin(t)$ even in the presence of an admissible external disturbance (27). Let $x = q$ and $y = \dot{q}$, then equation (39) can be written in the state space form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{1}{(ml^2 + J)} \Big(mglsin(x) - \rho_v y + \rho_c sign(y) \\ &+ \tau + \delta(t, x, y) \Big) \end{aligned} \tag{42}$$

Let the tracking error given by

$$e(t) = x(t) - r(t). \tag{43}$$

Using equations (42), the error dynamics are described by

$$\begin{aligned} (ml^2 + J)\ddot{e} &= mglsin(x) - \rho_v y + \rho_c sign(y) + \tau \\ &+ \delta(t, x, y) - (ml^2 + J)\ddot{r}. \end{aligned} \tag{44}$$

Let us choose the control in the form with $\alpha = \frac{1}{3}$

$$\begin{aligned} \tau &= \Big(ml^2 + J\Big) \ddot{r} - k_1 |e|^{\frac{\alpha}{2-\alpha}} sign(e) - k_2 |\dot{e}|^{\alpha} sign(\dot{e}) \\ & \quad -mglsin(x) + \rho_v y \end{aligned} \tag{45}$$

with $\alpha = 1/5$ and where (29) are satisfied. Parameters of a real laboratory one-link pendulum system are considered: the mass of the pendulum is $m = 0.5234kg$, the length of the link $l = 0.108m$, and the inertia about the center of the mass $j = 0.006kg \cdot m^2$. The Coulomb friction is given by $\rho_v = 0.00053N \cdot m \cdot s/rad$ and the viscous friction as $\rho_c = 0.05492N \cdot m$.

The initial conditions for the pendulum, selected for all experiments, are fixed as $\theta(0) = \pi \ rad$ and $\theta(0) = 0 \ rad/seg$ for the position and velocity, respectively . In Figure 2 shows the dynamics of one-link pendulum system in closed loop affected by the bounded external perturbations. The simulations of the modified algorithm (continuous line) using fixed gains as in [3],*i.e.* $k_1 = 4$, $k_2 = 2.5$ as gains. Simulations of the proposed design (dashed line) use $k_1 = 6$, $k_2 = 3.5$ as gains. Both exercises are affected by $\delta(t, x, y) = 3|x|^{\frac{\alpha}{2-\alpha}} + |y|^{\alpha} \ N \cdot m$ as an uncertainty term.

As fig. 2 shows, only the closed loop system with the proposed design achieves the control objective, the pendulum follows the desired trajectory. Indeed, the algorithm with other design (see [3]) is not able to compensate growing perturbations denoted as (27). Moreover, The strict Lyapunov function gives an estimation of convergence time. Indeed, using $\gamma_c = 1$, $\gamma_c = 0.58$, $\gamma_c = 1.2$, then $W(\pi, 0) = 261.1890$. The estimation of convergence time using equation (30) gives

$$t_{reach} \leq 433.6159 \ \ sec \tag{46}$$

while in practice the convergence is in 4 seconds.

IV. CONCLUSIONS

A generalization of a second-order sliding mode controller “Twisting” is tuned such as global finite time exact convergence with respect to the growing perturbations is shown. With this aim a non-smooth strict Lyapunov function is proposed allowing an estimation of the upper bound of the convergence time. The performance of the proposed algorithm was shown by solving the tracking control problem of a one-link pendulum in spite of bounded external and parametric perturbations. The closed loop mechanical system showed to be robust and provide nice performance in spite of unknown but bounded uncertainties. For future work, this result can be easily generalized for multidimensional case.

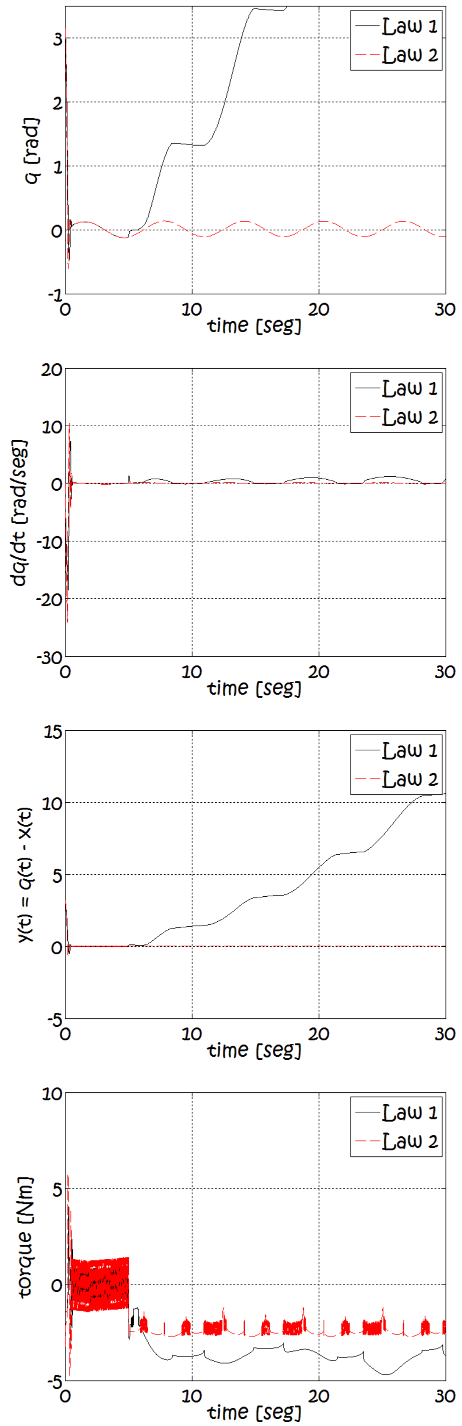

Figure 2. Tracking stabilization of a one-link pendulum.

Moreover, it can be extended when a state variable is not available for measurement, then a finite time observer can be applied such as super-twisting algorithm.

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