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Approximate Solutions to a Class of Nonlinear Differential Games Using a Shared Dynamic Extension

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Abstract— **A class of nonzero-sum differential games is considered and a dynamic state feedback control law that approximates the solution of the differential game is proposed. The control law relies upon the solution of algebraic equations in place of partial differential equations or inequalities and makes use of dynamics *shared* by the players, thus relaxing the structural assumption required in [1]. The idea is firstly illustrated by the two-player case and then extended to the N-player case. A simple numerical example completes the paper.**

I. Introduction

With roots in the work of von Neumann and Morgernstern in the late 1940's, game theory is the study of multi-player decision making [2]–[5] and it introduces the notion of *strategic behaviour* to control theory. The field has a vast range of applications, ranging from economics and management to military and defense [3], [6], [7]. With multi-agent systems receiving an increased level of attention in recent years, it is likely that game theoretic methods will be of interest to such systems as well, see for instance [8] where the optimal monitoring problem is studied.

There is a close relationship between optimal control and differential games: in optimal control one seeks to find the best strategy for *one* player attempting to optimise a performance criterion subject to the dynamic of the state variable [9], [10]. Differential games can be thought of as an extension of this, where there are *several* players attempting to optimise their own performance criteria. Solving a differential game entails finding *equilibrium strategies* for all players [2], [3], [7], [11], [12].

In the well-known zero-sum, two-player games, which have been extensively studied by Isaacs, the situation where there are two players with opposite goals is considered. In these games a gain for one player results in an identical loss for the second player, as seen for instance in standard pursuit-evasion games [13]. In these cases the game is defined by the dynamical system and a single cost functional which one player attempts to minimise and the other tries to maximise [5].

In several situations the players' goals are not necessarily opposite, *i.e.* a gain for one player need not imply a similar loss for the other player. Such scenarios cannot be formulated as zero-sum differential games and a more general class of games, namely *nonzero-sum differential games*, must be considered. Nonzero-sum games include situations where there may be more than two players and each player seeks to optimise its own, individual, cost functional [7], [14]. Determining the players' equilibrium strategies relies upon the solution of the Hamilton-Jacobi-Isaacs (HJI) partial differential equations (PDE's) associated with the problem [2]. These are a generalisation of the Hamilton-Jacobi-Bellman equation encountered in optimal control problems. Note that in many cases it is not viable to obtain closed-form solutions of the HJI PDE's. Consequently it may be necessary to settle for approximate solutions.

We consider a class of nonzero-sum, differential games, focusing on *Nash equilibria*. In [1], approximate solutions to such games are determined by introducing a dynamic extension for each player. The method therein requires strict structural assumptions on the dynamical system. Here, we propose a new method that allows a relaxation of the structural assumptions in [1]. The proposed method makes use of a *shared* dynamic extension and the notion of *algebraic $\bar{P}$ solutions* introduced in [15].

The remainder of the paper is organised as follows. The method and conditions for finding approximate solutions for the case where there are two players with different cost-functionals is presented in Section II. The results are extended to the N-player case in Section III, before a numerical example is presented in Section IV and conclusions and directions for future work are presented in Section V.

II. The 2-Player case

In this section we consider a dynamical system, with state x, and the case in which two players have different, possibly conflicting, objectives [2]. Consider the case in which the system is input-affine, *i.e* the

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dynamics of the state are given by[1]

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2\,, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u_1(t) \in \mathbb{R}^{m_1}$, and $u_2(t) \in \mathbb{R}^{m_2}$.

Assumption 1: The origin is an equilibrium point of the vector field f, *i.e.* $f(0) = 0$.

For the remainder of this paper it is assumed that Assumption 1 holds.

A consequence of Assumption 1 is that there exist a continuous, possibly not unique, matrix-valued function $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $f(x) = F(x)x$ for all $x \in \mathbb{R}^n$. Note that $F(0)$ is unique.

The cost functionals which the first and second player seek to minimise are

$$J_1(x(0), u_1, u_2) \triangleq \frac{1}{2}\int_0^\infty q_1(x(t)) + \|u_1(t)\|^2 - \|u_2(t)\|^2 \mathrm{d}t\,, \tag{2}$$

and

$$J_2(x(0), u_1, u_2) \triangleq \frac{1}{2}\int_0^\infty q_2(x(t)) + \|u_2(t)\|^2 - \|u_1(t)\|^2 \mathrm{d}t\,, \tag{3}$$

respectively, with $q_i : \mathbb{R}^n \to \mathbb{R}_+$, $i = 1, 2$, positive definite running costs, namely such that $q_i > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $q_i(0) = 0$. Roughly speaking, each player tries to achieve a desired goal, while minimising its own control action and maximising the effort of the opponent.

To streamline the presentation of the forthcoming result we introduce the linearised problem. The linearisation of the system (1) around the origin is given by

$$\dot{x} = Ax + B_1u_1 + B_2u_2\,, \tag{4}$$

with

$$A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=0} = F(0), \quad B_1 \triangleq g_1(0), \quad B_2 \triangleq g_2(0)\,,$$

while the cost functionals are defined as in (2) and (3) with q_i replaced by the quadratic term $x^\top Q_i x$, where

$$Q_i \triangleq \frac{1}{2}\left.\frac{\partial^2 q_i}{\partial x^2}\right|_{x=0}, \quad \text{for } i = 1, 2\,.$$

Problem 1: Consider the system (1) and the cost functionals (2) and (3). The problem of solving the non-cooperative differential game consists in determining a pair of feedback strategies $(u_1^*,\, u_2^*)$ such that

$$J_1^* \triangleq J_1(x(0), u_1^*, u_2^*) \le J_1(x(0), u_1, u_2^*)\,,$$
$$J_2^* \triangleq J_2(x(0), u_1^*, u_2^*) \le J_2(x(0), u_1^*, u_2)\,,$$

for all $u_1 \neq u_1^*$ and $u_2 \neq u_2^*$, and the zero-equilibrium of system (1) in closed-loop with u_1^* and u_2^* is asymptotically stable.

Note that the pair $(u_1^*,\, u_2^*)$ constitutes a Nash equilibrium solution for the 2-player game, while $(J_1^*,\, J_2^*)$ is the corresponding Nash equilibrium outcome [2]. Suppose that we are able to solve the system of coupled Hamilton-Jacobi-Isaacs partial differential equations given by

$$\frac{\partial V_1}{\partial x}f(x) - \frac{1}{2}\frac{\partial V_1}{\partial x}g_1(x)g_1(x)^\top\frac{\partial V_1}{\partial x}^\top + \frac{1}{2}q_1(x) - \frac{1}{2}\frac{\partial V_2}{\partial x}g_2(x)g_2(x)^\top\frac{\partial V_2}{\partial x}^\top - \frac{\partial V_1}{\partial x}g_2(x)g_2(x)^\top\frac{\partial V_2}{\partial x}^\top = 0\,, \tag{5}$$

and

$$\frac{\partial V_2}{\partial x}f(x) - \frac{1}{2}\frac{\partial V_2}{\partial x}g_2(x)g_2(x)^\top\frac{\partial V_2}{\partial x}^\top + \frac{1}{2}q_2(x) - \frac{1}{2}\frac{\partial V_1}{\partial x}g_1(x)g_1(x)^\top\frac{\partial V_2}{\partial x}^\top - \frac{\partial V_1}{\partial x}g_1(x)g_1(x)^\top\frac{\partial V_1}{\partial x}^\top = 0\,, \tag{6}$$

with $V_i(0) = 0$, $i = 1, 2$, where $V_i : \mathbb{R}^n \to \mathbb{R}_+$ denotes the value function of the i^{th} player. Then the optimal policy for each player is provided by the static state feedback

$$u_i = -g_i(x)^\top\frac{\partial V_i}{\partial x}^\top\,. \tag{7}$$

In this paper we provide a solution for a modified problem, which *approximates* the differential game introduced in Problem 1. We consider a dynamic extension, which is shared by the players, yielding a dynamic state feedback control law. The players therefore exploit additional dynamics, the state of which is denoted by $\xi(t) \in \mathbb{R}^n$. We then show how to construct value functions, for each player, defined in the extended state-space (x, ξ), that solve partial differential inequalities in place of equations. The interpretation of this relaxation entails that we allow for additional running costs in the functionals (2) and (3) that are imposed on x and ξ.

In [1], a similar approach is pursued. In particular, each player is given its own dynamic extension and it is shown that approximate solutions can be found provided certain structural assumptions are satisfied. The rationale behind the alternative approach pursued in this paper is that the players *share* the state of the dynamic extension as they *share* the state of the system, which allows for a relaxation of the aforementioned structural assumptions. The above discussion is summarised in the following definition.

Problem 2: Consider the system (1) and the cost functionals (2) and (3). The problem of solving the *approximate dynamic non-cooperative differential game* consists in determining a pair of *dynamic feedback strategies* $(S_1,\, S_2)$, where the strategy $S_i = (\dot{\xi}\,, u_i)$, $i = 1, 2$, is a dynamical system described by equations of the form

$$\begin{aligned} \dot{\xi} &= \alpha(x,\, \xi)\,, \\ u_i &= \beta_i(x,\, \xi)\,, \end{aligned} \tag{8}$$

[1]All mappings are assumed to be sufficiently smooth.

with $\alpha(0,0)=0$, $\beta_i(0,0)=0$, $i=1,2$ and α, β_i smooth mappings, and non-negative functions $c_i:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ such that the following holds.

1) For $u_i\neq\beta_i$,

$$\hat{J}_1((x(0),\,\xi(0)),\beta_1,\beta_2)\leq\hat{J}_1((x(0),\,\xi(0)),u_1,\beta_2)\,,$$
$$\hat{J}_2((x(0),\,\xi(0)),\beta_1,\beta_2)\leq\hat{J}_2((x(0),\,\xi(0)),\beta_1,u_2)\,,$$

where the extended cost functionals $\hat{J}_1$ and $\hat{J}_2$ are defined as

$$\begin{aligned}\hat{J}_1 &\triangleq \frac{1}{2}\int_0^\infty\Big(q_1(x(t))+\|u_1(t)\|^2-\|u_2(t)\|^2\\&\qquad\qquad+c_1(x(t),\,\xi(t))\Big)\mathrm{d}t\\ \hat{J}_2 &\triangleq \frac{1}{2}\int_0^\infty\Big(q_2(x(t))+\|u_2(t)\|^2-\|u_1(t)\|^2\\&\qquad\qquad+c_2(x(t),\,\xi(t))\Big)\mathrm{d}t\,.\end{aligned}\tag{9}$$

2) The zero-equilibrium of the closed-loop system (1), (8) is asymptotically stable.

The following definition provides a different notion of solution of the system of partial differential equations (5) and (6) and is instrumental in the construction of a dynamic strategy that solves Problem 2.

Definition 1: Consider the system (1) and the cost functionals (2) and (3). Let $\sigma_i(x)\triangleq x^\top\Sigma_i(x)x>0$, with $\Sigma_i:\mathbb{R}^n\to\mathbb{R}^{n\times n}$, $\Sigma_i(0)=\bar{\Sigma}_i=\bar{\Sigma}_i^\top>0$. Continuously differentiable mappings $P_i:\mathbb{R}^n\to\mathbb{R}^{1\times n}$, $P_i(0)=0$, $i=1,2$, are said to be *$\mathcal{X}$-algebraic $\bar{P}$ solutions*[2] of the equations (5) and (6) provided the following conditions hold.

(i) For all $x\in\mathcal{X}\subseteq\mathbb{R}^n$ and $i=1,2$, $j=1,2$, $i\neq j$

$$\begin{aligned}&P_i(x)f(x)-\frac{1}{2}P_i(x)g_i(x)g_i(x)^\top P_i(x)^\top+\frac{1}{2}q_i(x)\\&-\frac{1}{2}\left(P_j(x)+2P_i(x)\right)g_j(x)g_j(x)^\top P_j(x)^\top+\sigma_i(x)=0\,.\end{aligned}\tag{10}$$

(ii) P_i is tangent at $x=0$ to the symmetric positive definite solution of the *coupled Riccati equations*[3]

$$\begin{aligned}&\bar{P}_iA+A^\top\bar{P}_i-\bar{P}_iB_iB_i^\top\bar{P}_i-\bar{P}_iB_jB_j^\top\bar{P}_j\\&-\bar{P}_jB_jB_j^\top\bar{P}_i-\bar{P}_jB_jB_j^\top\bar{P}_j+Q_i+2\bar{\Sigma}_i=0\,,\end{aligned}\tag{11}$$

for $i=1,2$, $j=1,2$, $i\neq j$, namely $\left.\dfrac{\partial P_i(x)^\top}{\partial x}\right|_{x=0}=\bar{P}_i$.

If $x\in\mathbb{R}^n$, *i.e.* $\mathcal{X}=\mathbb{R}^n$, then the P_i, $i=1,2$ are said to be *algebraic $\bar{P}$ solutions*.

Remark 1: In what follows we assume the existence of *algebraic $\bar{P}$ solutions*[4], *i.e.* we assume $\mathcal{X}=\mathbb{R}^n$. Note that all the statements can be modified accordingly if $\mathcal{X}\subset\mathbb{R}^n$.

Remark 2: Arbitrary mappings P_1 and P_2 satisfying (10) and (11) can be found. It follows that, unlike the solutions to (5) and (6), it is not a requirement for the *algebraic $\bar{P}$ solutions* to be gradient vectors.

Exploiting the notion of algebraic $\bar{P}$ solutions, consider the extended value functions

$$\begin{aligned}V_1(x,\xi)&=P_1(\xi)x+\frac{1}{2}\|x-\xi\|^2_{R_1}\,,\\V_2(x,\xi)&=P_2(\xi)x+\frac{1}{2}\|x-\xi\|^2_{R_2}\,,\end{aligned}\tag{12}$$

where $\xi\in\mathbb{R}^n$ and $R_i=R_i^\top>0$, $i=1,2$. Let $\Phi_i:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$ denote a continuous matrix-valued function such that $P_i(x)-P_i(\xi)=(x-\xi)^\top\Phi_i(x,\xi)^\top$. Moreover let $\Psi_i:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ denote the Jacobian matrix of the mapping $P_i(\xi)$ and define

$$A_{cl}(x)\triangleq F(x)-g_1(x)g_1(x)^\top N_1(x)-g_2(x)g_2(x)^\top N_2(x)\,,\tag{13}$$

where $N_i:\mathbb{R}^n\to\mathbb{R}^{n\times n}$, $i=1,2$ is a continuous matrix-valued function such that $P_i(x)=x^\top N_i(x)^\top$ for all $x\in\mathbb{R}^n$. Interestingly, the vector field $A_{cl}(x)x$ describes the closed-loop nonlinear system when only the *algebraic inputs* $u_i=-g_i(x)^\top P_i(x)^\top$, $i=1,2$, are applied.

Remark 3: The mappings Φ_i and N_i, $i=1,2$ always exist.

Theorem 1: Consider the system (1) and the cost functionals (2) and (3). Let P_i, $i=1,2$, be algebraic $\bar{P}$ solutions of the equations (5) and (6). Assume that

$$\bar{P}_i\left(\bar{P}_1+\bar{P}_2\right)+\left(\bar{P}_1+\bar{P}_2\right)\bar{P}_i>0\,,\tag{14}$$

$i=1,2$. Then there exist $R_i=R_i^\top>0$, $i=1,2$, a constant $\bar{k}\geq0$ and a set $\Omega\subseteq\mathbb{R}^n\times\mathbb{R}^n$ such that the functions V_i, $i=1,2$, in (12) solve the system of partial differential inequalities

$$\begin{aligned}&\frac{\partial V_i}{\partial x}f(x)+\frac{\partial V_i}{\partial\xi}\dot{\xi}-\frac{1}{2}\frac{\partial V_i}{\partial x}g_i(x)g_i(x)^\top\frac{\partial V_i}{\partial x}^\top+\frac{1}{2}q_i(x)\\&-\frac{1}{2}\frac{\partial V_j}{\partial x}g_j(x)g_j(x)^\top\frac{\partial V_j}{\partial x}^\top-\frac{\partial V_i}{\partial x}g_j(x)g_j(x)^\top\frac{\partial V_j}{\partial x}^\top\leq0\,,\end{aligned}\tag{15}$$

$i=1,2$, $j=1,2$, $i\neq j$, with $\dot{\xi}=-k\left(\dfrac{\partial V_1}{\partial\xi}+\dfrac{\partial V_2}{\partial\xi}\right)^\top$, for all $k>\bar{k}$ and for all $(x,\,\xi)\in\Omega$. Hence the dynamical systems

$$\begin{aligned}\dot{\xi}&=-k\sum_{i=1}^{2}\left(\Psi_i(\xi)^\top x-R_i(x-\xi)\right)\,,\\u_i&=-g_i(x)^\top\left(P_i(x)+(R_i-\Phi_i(x,\,\xi))(x-\xi)\right)\,,\end{aligned}\tag{16}$$

$i=1,2$, yield dynamic strategies that solve the approximate dynamic non-cooperative differential game defined in Problem 2.

[2] Provided $0\in\mathcal{X}$.

[3] *Coupled Riccati equations* appear in linear quadratic differential games, see [16]–[20]. Note that these may have non-unique positive definite solutions.

[4] If a solution to (5) and (6) exists, then its gradient vector is an *algebraic $\bar{P}$ solution*. Hence, this assumption is a relaxation of the assumption that the HJI equations can be solved.

Remark 4: If the running costs $q_1(x)$ and $q_2(x)$ are allowed to be positive semi-definite, asymptotic stability of the equilibrium $(x, \xi) = (0, 0)$ is guaranteed provided the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) - g_1(x)g_1(x)^\top \frac{\partial V_1}{\partial x}^\top - g_2(x)g_2(x)^\top \frac{\partial V_2}{\partial x}^\top, \\ \dot{\xi} &= -k\left(\frac{\partial V_1}{\partial \xi} + \frac{\partial V_2}{\partial \xi}\right)^\top, \end{aligned} \tag{17}$$

is zero-state detectable with respect to the output $y = q_1(x) + q_2(x)$, which is the case if $q_1(x) + q_2(x) > 0$.

Remark 5: Alternative conditions ensuring the existence of R_i, $i = 1, 2$, and $\bar{k} > 0$ such that the partial differential inequalities (15), $i = 1, 2$, are satisfied, which are stronger than those given in (14), are

$$(\Psi_i \Psi_j^\top + \Psi_j \Psi_i^\top) > 0, \tag{18}$$

and

$$(R_i R_j + R_j R_i) - \frac{1}{2}\Upsilon^\top (\Psi_i \Psi_j^\top + \Psi_j \Psi_i^\top)^{-1} \Upsilon \geq 0, \tag{19}$$

with $\Upsilon = (\Psi_i R_j + \Psi_j R_i)$. These ensure that the cross terms $\frac{\partial V_i}{\partial \xi}\frac{\partial V_j}{\partial \xi}^\top$ in (15) are non-negative for $i = 1, 2$, $j = 1, 2$ and $j \neq i$,.

Remark 6: If the algebraic solution P_i is linear, then $\Psi_i = N_i = \bar{P}_i$.

III. The N-Player case

Consider now the case of N players and the input-affine system

$$\dot{x} = f(x) + \sum_{i=1}^{N} g_i(x) u_i, \tag{20}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system and $u_i(t) \in \mathbb{R}^{m_i}$, $i = 1, \ldots, N$ is the control input of the i^{th} player. Each player attempts to minimise its own cost funtional given by

$$J_i \triangleq \frac{1}{2}\int_0^\infty q_i(x(t)) + \|u_i(t)\|^2 - \sum_{j=1, j\neq i}^{N} \|u_j(t)\|^2 \mathrm{d}t, \tag{21}$$

for $i = 1, \ldots, N$. The standing assumptions and definitions introduced in the 2-player case can be naturally extended to the N-player case. For example, the matrix valued function (13) becomes

$$A_{cl}(x) = F(x) - \sum_{i=1}^{N} g_i(x) g_i(x)^\top N_i(x). \tag{22}$$

The notion of algebraic $\bar{P}$ solutions for the N-player case is extended by replacing the algebraic equations (10) and (11) with

$$\begin{aligned} &P_i(x)f(x) - \frac{1}{2}P_i(x)g_i(x)g_i(x)^\top P_i(x)^\top + \frac{1}{2}q_i(x) + \sigma_i(x) \\ &- \frac{1}{2}\sum_{j=1, j\neq i}^{N} (P_j(x) + 2P_i(x))\, g_j(x)g_j(x)^\top P_j(x)^\top = 0, \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\bar{P}_i A + A^\top \bar{P}_i - \bar{P}_i B_i B_i^\top \bar{P}_i + Q_i + 2\bar{\Sigma}_i \\ &- \sum_{j=1, j\neq i}^{N} \left(\bar{P}_i B_j B_j^\top \bar{P}_j + \bar{P}_j B_j B_j^\top \bar{P}_i + \bar{P}_j B_j B_j^\top \bar{P}_j\right) = 0, \end{aligned} \tag{24}$$

respectively. The following theorem extends Theorem 1 to the case where an approximate solution of a nonzero-sum differential game with N players is sought.

Theorem 2: Consider the system (20) and the cost functionals (21). Let P_i, $i = 1, \ldots, N$ be algebraic $\bar{P}$ solutions. Assume

$$\bar{P}_i \left(\sum_{l=1, l\neq i}^{N} \bar{P}_l\right) + \left(\sum_{l=1, l\neq i}^{N} \bar{P}_l\right) P_i > 0, \tag{25}$$

for $i = 1, \ldots, N$. Then there exists $R_i = R_i^\top > 0$, $i = 1, \ldots, N$, a constant $\bar{k} > 0$ and a set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ such that the extended value functions V_i, $i = 1, \ldots, N$ solve the system of extended partial differential inequalities

$$\begin{aligned} \mathcal{HJ}_i(x, \xi) =& \frac{\partial V_i}{\partial x} f(x) + \frac{\partial V_i}{\partial \xi}\dot{\xi} - \frac{1}{2}\frac{\partial V_i}{\partial x} g_i(x)g_i(x)^\top \frac{\partial V_i}{\partial x}^\top \\ &+ \frac{1}{2}q_i(x) - \sum_{j=1, j\neq i}^{N} \frac{1}{2}\frac{\partial V_j}{\partial x} g_j(x)g_j(x)^\top \frac{\partial V_j}{\partial x}^\top \\ &- \sum_{j=1, j\neq i}^{N} \frac{\partial V_i}{\partial x} g_j(x)g_j(x)^\top \frac{\partial V_j}{\partial x}^\top \leq 0, \end{aligned} \tag{26}$$

$i = 1, \ldots, N$ with $\dot{\xi} = -k\sum_{i=1}^{N} \frac{\partial V_i}{\partial \xi}$, for all $k > \bar{k}$ and for all $(x, \xi) \in \Omega$. Suppose additionally that

$$\sum_{i=1}^{N}\left(\frac{N-2}{2}\frac{\partial V_i}{\partial x} g_i(x)g_i(x)^\top \frac{\partial V_i}{\partial x}^\top - c_i(x, \xi)\right) \leq 0, \tag{27}$$

where $c_i(x, \xi) = -2\mathcal{HJ}_i(x, \xi)$. Then, the dynamical systems

$$\begin{aligned} \dot{\xi} &= -k\sum_{i=1}^{N}\left(\Psi_i(\xi)^\top x - R_i(x - \xi)\right), \\ u_i &= -g_i(x)^\top \left(P_i(x) + (R_i - \Phi_i(x, \xi))(x - \xi)\right), \end{aligned} \tag{28}$$

$i = 1, \ldots, N$, yield dynamic strategies that solve the approximate dynamic non-cooperative differential game defined by the dynamics (20) and the cost functionals (21).

Remark 7: Similarly to Remark 5, it is possible to derive alternative conditions to (25) ensuring that the cross terms $\frac{\partial V_i}{\partial \xi}\frac{\partial V_j}{\partial \xi}^\top$ are non-negative.

IV. EXAMPLE

In this section an example with accompanying simulations is presented. The example involves a dynamical system which does not satisfy the structural assumptions of [1] and illustrates how approximate solutions to a wider range of differential games can be found using a *shared* dynamic extension.

Consider a differential game with two players, dynamics

$$\dot{x} = \begin{bmatrix} 0.1x_1 - 10x_1x_2^2 \\ 0.1x_2 - 20x_1^2x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 + 10x_1^2 \end{bmatrix} u_1 + \begin{bmatrix} 2 + 20x_2^2 \\ 0 \end{bmatrix} u_2 \,, \tag{29}$$

and cost functionals (2) and (3), with running costs

$$q_1(x) = 47.6x_1^2 + 0.8x_2^2 \,, \tag{30}$$

and

$$q_2(x) = 15.6x_1^2 + 10x_2^2 \,. \tag{31}$$

The mappings $P_1(x) = \begin{bmatrix} 2x_1(1+x_2^2) & 1.1x_2(1+x_1^2) \end{bmatrix}$ and $P_2(x) = \begin{bmatrix} 2.1x_1(1+x_2^2) & 5.1x_2(1+x_1^2) \end{bmatrix}$ are algebraic $\bar{P}$ solutions for which $\bar{P}_1 = \text{diag}\{2, 1.1\}$ and $\bar{P}_2 = \text{diag}\{2.1, 5.1\}$ are solutions to the coupled algebraic Riccati equations (11) with $\bar{\Sigma}_1(x) = \text{diag}\{1.62, 0.095\}$ and $\bar{\Sigma}_2(x) = \text{diag}\{0.81, 0.705\}$. Furthermore, $\bar{P}_1$ and $\bar{P}_2$ satisfy the conditions (14), $i = 1, 2$. The dynamic control laws are then given according to (16).

The linear-quadratic approximation of the problem is given by the linear system (4) with $A = \text{diag}\{0.1, 0.1\}$, $B_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$ and $B_2 = \begin{bmatrix} 2 & 0 \end{bmatrix}^\top$ and the quadratic costs with $Q_1 = \text{diag}\{47.6, 0.8\}$ and $Q_2 = \text{diag}\{15.6, 10\}$. The corresponding coupled algebraic Riccati equations are given by (11) with $\bar{\Sigma}_i = 0$, $i = 0, 1$, for which the solutions are $\hat{P}_1 = \text{diag}\{2, 1\}$ and $\hat{P}_2 = \text{diag}\{2, 5\}$. The linear control strategies are $u_1^l = -B_1\hat{P}_1x$ and $u_2^l = -B_2\hat{P}_2x$.

Simulations of the differential game with dynamic (29) and running costs (30) and (31) with the control actions u_i^d and u_i^l, $i = 1, 2$, have been run, starting from a range of initial conditions, x_0, positioned on or inside the unit circle. For the dynamic controller $k = 6$ and $\xi_0 = \begin{pmatrix} 0.2 & -0.8 \end{pmatrix}^\top$ are used. Furthermore, $R_1 = \bar{P}_1$ and $R_2 = \bar{P}_2$ are selected in accordance with the proof of Theorem 1.

Figure 1 shows the difference between the convex combinations of $J_1(u_1^d, u_2^d)$ and $J_2(u_1^d, u_2^d)$, and, $J_1(u_1^l, u_2^l)$ and $J_2(u_1^l, u_2^l)$, *i.e.* it shows $L(\alpha, u_1^d, u_2^d, u_1^l, u_2^l) = \alpha\left(J_1(u_1^d, u_2^d) - J_1(u_1^l, u_2^l)\right) + (1-\alpha)\left(J_2(u_1^d, u_2^d) - J_2(u_1^l, u_2^l)\right)$ as a function of $\alpha \in [0, 1]$ for several initial conditions x_0. This gives an indication of how well *both* players perform for the different strategies, where α determines the relative weighting between the two cost functionals. The solid black line at $L = 0$ is added for reference. The dashed and dotted lines indicate that the convex combinations are positive or negative, respectively. As can be seen, for most of the initial conditions, x_0, the difference between the convex combinations is negative for most values of α, meaning the dynamic strategies perform better than the linear strategies in these cases. Note that these simulations have been run with the same ξ_0, and it would be possible to further improve the performance of the dynamic feedback by selecting the initial states of the dynamic variable appropriately for each individual value of x_0.

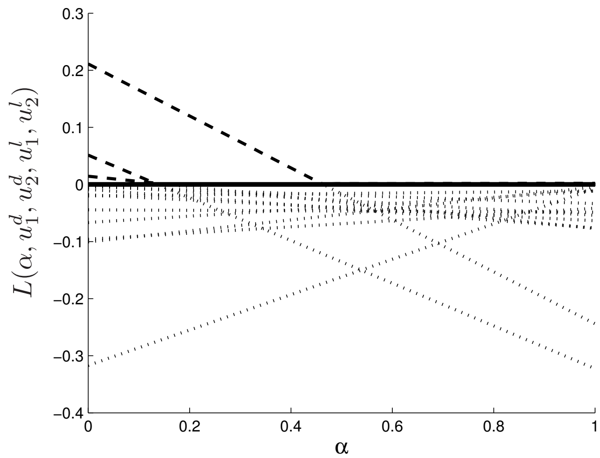

Fig. 1. The difference between the convex combinations of the two cost functionals with dynamic and linear strategies, *i.e.* $L(\alpha, u_1^d, u_2^d, u_1^l, u_2^l)$ for relative weightings $\alpha \in [0, 1]$ for different x_0. Dashed and dotted lines indicate $L > 0$ and $L < 0$, respectively.

Considering the case where $x_0 = \begin{pmatrix} -0.2 & -0.5 \end{pmatrix}^\top$, Figure 2 shows the time histories of the states, x_1 (top) and x_2 (bottom) under the action of the controls u_1^d and u_2^d (black line) and u_1^l and u_2^l (gray line), whereas Figure 3 shows the time histories of the dynamic extension, ξ_1 (solid line) and ξ_2 (dashed line), associated with the case where both players adopt the dynamic controls. When the dynamic strategies are implemented by both players, the cost functionals J_1 and J_2 converge to 0.1196 and 0.5448 respectively. When the linear strategies are used instead, the same values converge to 0.1309 and 0.6614, indicating that both players perform better when the dynamic strategies are used.

Note that it cannot be concluded that either of the strategies are Nash equilibria. However, the results suggest that the dynamic approximate solution yields better performances than the solution obtained using the linear approximation.

V. CONCLUSIONS AND FUTURE WORK

2-player nonzero-sum differential games, as described by (1) and the cost functionals (2) and (3) are studied to obtain approximate solutions. The results are then extended to the general case where there are N participating players. Approximate solutions are

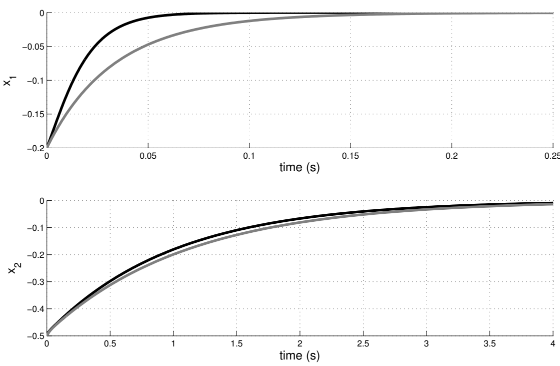

Fig. 2. Time histories of x_1 (top) and x_2 (bottom) with the controls u_1^d and u_2^d (black line) and u_1^l and u_2^l (gray line).

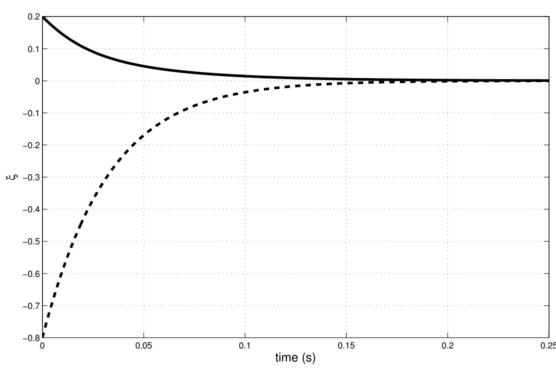

Fig. 3. Time histories of the dynamic extensions ξ_1 (solid line) and ξ_2 (dashed line) associated with the dynamic approximation.

sought using a *dynamic extension* which is *shared* by the players. This approach is somewhat different from what presented in [1] and allows for relaxation on the structural assumptions on the system used in [1] and thus apply to a larger class of differential games. Augmenting the system with a dynamic variable and considering an extended value functions for each of the players, approximate solutions to the original problem are found by identifying *algebraic $\bar{P}$ solutions* satisfying certain conditions. The theory is illustrated by an example which does not satisfy the structural assumptions of [1]. It is shown that the dynamic approximation results in better performance for both players than the linear approximation of the problem for several initial conditions.

Future work includes considering finite horizon differential games and exploring different applications of the theory.

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