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A simplified Goddard problem in the presence of a nonlinear media resistance and a bounded thrust

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***Abstract*— We consider a problem of maximization of the distance traveled by a material point in the presence of a nonlinear friction under a bounded thrust and fuel expenditure. Using the maximum principle we obtain the form of optimal control and establish conditions under which it contains a singular subarc. This problem seems to be the simplest one having a mechanical sense in which singular subarcs appear in a nontrivial way.**

INTRODUCTION

We consider the following optimal control problem:

$$\begin{cases} \dot{s} = x, & s(0) = 0, & s(T) \to \max, \\ \dot{x} = u - \varphi(x), & x(0) = 0, & x(t) \text{ is free} \\ \dot{m} = -u, & m(0) = m_0, & m(T) \geqslant m_T, \\ 0 \leqslant u \leqslant C. \end{cases} \qquad (1)$$

Here $s(t)$ and $x(t)$ are one-dimensional position and velocity of a vehicle, $m(t)$ describes the total mass of vehicle's body and fuel, $u(t)$ is the rate of fuel expenditure, $\varphi(x)$ is a twice smooth function describing the "friction" (media resistance) depending on the velocity. We assume that $\varphi(0) = 0$, $\varphi'(0) \geqslant 0$, and $\varphi''(x) > 0$ for all $x > 0$. This object can be considered as a material point moving along a horizontal track and being forced by a nonnegative thrust. Our aim is to maximize the distance passed by the object in a given time T under a fuel limitation. Here $m_T \in (0, m_0)$ is the mass of "empty" vehicle without fuel.

This problem can be also considered as a simplification of the Goddard problem [1], where, first, the object moves in a horizontal, not in a vertical direction (which mathematically means that if the speed is zero and the thrust is not applied, the speed keeps zero value on), and second, the change of the mass of the object is not taken into account in the equation for acceleration.

It is well-known that a typical feature of optimal trajectories in the Goddard problem is the presence of singular arcs. However, in the original Goddard problem it is hardly possible to investigate optimality conditions analytically, and so, even qualitative properties of optimal trajectories are obtained by using numerical calculations (see, e.g. [6]–[8]).

Problem (1) includes rather simple equations, which allow one to determine the form of optimal trajectories analytically. At the same time, these trajectories still may contain singular subarcs for some typical forms of the friction function φ. Moreover, we show that in the case of linear friction function the optimal control is always bang-bang. Thus, our simplification leads to the "threshold form" of the considered control system: the considered function $\varphi(x)$ is the simplest one which may induces the bang-singular trajectory.

Probably, this problem is the simplest optimal control problem with an underlying mechanical sense in which singular subarcs appear in a nontrivial way.

I. MAXIMUM PRINCIPLE FOR PROBLEM (1)

Let $s(t), x(t), m(t), u(t),\ t \in [0,T]$ be an optimal process. According to the Pontryagin Maximum Principle (MP), there exist constants $(\alpha_0, \alpha, \beta_s, \beta_x, \beta_m)$, not all identically zero, and Lipschitz functions $\psi_s(t)$, $\psi_x(t), \psi_m(t)$, that generate *the endpoint Lagrange function*

$$l = -\alpha_0 s(T) - \alpha(m(T) - m_T) + \beta_s s(0) + \\ + \beta_x x(0) + \beta_m (m(0) - m_0) \qquad (2)$$

and *the Pontryagin function*

$$H(s,x,m,u) = \psi_s\, x + \psi_x\, (u - \varphi(x)) - \ \psi_m\, u, \qquad (3)$$

such that the following conditions are satisfied:

(1) nonnegativity condition: $\alpha_0 \geqslant 0,\ \ \alpha \geqslant 0$,
(2) nontriviality condition: $(\alpha_0, \alpha, \beta_s, \beta_x, \beta_m) \neq (0,0,0,0,0)$,
(3) complementarity slackness condition:

$$\alpha(m(T) - m_T) = 0, \qquad (4)$$

(4) costate (adjoint) equations

$$\begin{cases} -\dot{\psi}_s = H_s = 0, \\ -\dot{\psi}_x = H_x = \psi_s - \psi_x \varphi'(x), \\ -\dot{\psi}_m = H_m = 0. \end{cases} \qquad (5)$$

(5) transversality conditions:

$$\begin{cases} \psi_s(0) = \beta_s, & \psi_s(T) = \alpha_0, \\ \psi_x(0) = \beta_x, & \psi_x(T) = 0, \\ \psi_m(0) = \beta_m, & \psi_m(T) = \alpha. \end{cases} \qquad (6)$$

(6) the "energy conservation law": $H(s,x,m,u) \equiv const$,
(7) and the maximality condition: for almost all t

$$\max_{0 \leqslant u' \leqslant 1} H(s(t), x(t), m(t), u') = \\ = H(s(t), x(t), m(t), u(t)). \qquad (7)$$

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According to (5)–(6), in order to simplify further computations we set $\psi_x \equiv \alpha_0$, $\psi_m \equiv \alpha$ and write $\psi(t)$ instead of $\psi_x(t)$. Then the maximality condition (7) gives the optimal control in the form

$$u(t) \in \operatorname{Sign}^+_C(\psi - \alpha), \tag{8}$$

where Sign^+_C is the set-valued function

$$\operatorname{Sign}^+_C(z) = \begin{cases} \{C\}, & z > 0, \\ [0, C], & z = 0, \\ \{0\}, & z < 0, \end{cases}$$

and the costate $\psi(t)$ is determined by the equation

$$\dot{\psi} = -\alpha_0 + \psi\,\varphi'(x) \tag{9}$$

with the terminal condition $\psi(T) = 0$.

Note that in the further considerations it is enough to set the upper bound of the control as $C = 1$.

Recall that by definition $\Delta m = m_0 - m_T > 0$. If $\Delta m \geqslant T$, then the optimal control is obviously $u \equiv 1$. So, in further considerations we assume that

$$0 < \Delta m < T. \tag{10}$$

II. ANALYSIS OF THE MAXIMUM PRINCIPLE

Consider first the abnormal case $\alpha_0 = 0$. Then equation (11) for $\psi(t)$ restricts to a homogeneous one, and condition $\psi(T) = 0$ yields $\psi(t) \equiv 0$. Hence $\beta_x = 0$ (see (6)) and nontriviality condition gives $\alpha > 0$. Then (8) yields $u(t) \equiv 0$, and from equations (1) we have $m(t) = const = m_0$, which contradicts complementarity slackness condition (4). Hence the normal case $\alpha_0 > 0$ is realised and we may take $\alpha_0 = 1$. Thus, equation (9) reads

$$\dot{\psi} = -1 + \psi\,\varphi'(x). \tag{11}$$

Proposition 1: $\psi(t) > 0$ for all $t < T$.

Proof: According to (11), $\dot{\psi}(T) = -1$. Since $\dot{\psi}$ is continuous, $\psi(t) > 0$ in a left neighborhood of T. Suppose there exists $t' < T$ such that $\psi(t') = 0$ and $\psi(t) > 0$ on (t', T). From (11) we have $\dot{\psi}(t') = -1$ again, which contradicts the previous inequality. ■

Proposition 2: Any trajectory satisfying the maximum principle is globally optimal in problem (1).

Proof: Since $\psi(t) \geqslant 0$ for all t, the Pontryagin function (3) is concave w.r.t. the pair (x, u). Moreover, the endpoint constraints are linear, the control set $[0, 1]$ is convex, and the multiplier at the cost $\alpha_0 = 1$. It is well known that in this case the maximum principle guarantees the global optimality. ■

Proposition 3: $\alpha > 0$.

Proof: Suppose that $\alpha = 0$. Then from (8) we have $u \equiv 1$ for a.a. t. Hence $\Delta m = T$, which contradicts (10). ■

From the last proposition and (8) it follows that there exists $t_2 < T$ such that $u = 0$ for a.a. $t \in (t_2, T)$. Moreover, since $\alpha > 0$, condition (4) gives $m(T) = m_T$, and hence

$$\int_0^T u\,dt = \Delta m > 0. \tag{12}$$

Remark 1: If the friction is linear: $\varphi(x) = \gamma x$ $(\gamma > 0)$, the analysis of MP is quite simple. Then (11) determines $\psi(t) = \left(1 - e^{\gamma(t-T)}\right)/\gamma$ which is positive on $[0, T]$ and decreases monotonically from $\psi(0) > 0$ to $\psi(T) = 0$. In this case condition (8) gives that the optimal control always has a bang-bang form $u = (1, 0)$ on $((0, \Delta m),\, (\Delta m, T))$. Such a case is not interesting, and this is why we assume that $\varphi(x)$ is strictly convex.

Now, define the set $M_0 = \{t : \psi(t) = \alpha\}$. Obviously, M_0 is closed. Moreover, it is not empty (otherwise $\psi < \alpha$ on $(0, T)$, hence $u \equiv 0$, which contradicts (12)).

Proposition 4: The set M_0 is connected.

Proof: Suppose the opposite. Then there exists an interval $\omega = (t', t'')$ such that $\psi(t') = \psi(t'') = \alpha$, and either i) $\psi(t) < \alpha$ on ω, or ii) $\psi(t) > \alpha$ on ω.

Consider the case i). Since $\dot{\psi}(t') \leqslant 0$ and $\dot{\psi}(t'') \geqslant 0$, from (11) it follows that $\varphi'(x(t')) \leqslant \varphi'(x(t''))$ and hence $x(t') \leqslant x(t'')$ by the strict monotonicity of φ'. But $u \equiv 0$ on ω, so $\dot{x} = -\varphi(x)$ by (1), whence $x(t)$ cannot increase along ω. Hence, $x(t) = const$ on ω, and since $\varphi(x) > 0$ for $x > 0$, we get $x(t) \equiv 0$ on ω. Since $\varphi'(0) = b \geqslant 0$, equation (11) reads $\dot{\psi} = -1 + b\psi$. Its solution is either increasing or decreasing function, which cannot have equal values at t' and t'', a contradiction. Case ii) is analysed similarly. ■

Thus, M_0 is a segment $[t_1, t_2]$ with possible $t_1 = t_2$.

Proposition 5: $M_0 \subset (0, T)$, i.e. $t = 0$ and $t = T$ do not belong to M_0.

Proof: Since $\psi(T) = 0 < \alpha$, the right end $T \notin M_0$. So, we just need to show that $0 \notin M_0$. Taking into account that M_0 is a segment $[t_1, t_2]$, suppose first that $M_0 = \{0\}$. Then $\psi < \alpha$ on $(0, T)$, which yields $u \equiv 0$ for a.a. $t < T$ and contradicts (12). Now suppose $M_0 = [0, t_2]$, where $0 < t_2 < T$. Then along $[0, t_2]$ we have $x = const = x(0) = 0$, $u = \varphi(x) = 0$, hence $u \equiv 0$ along the whole $[0, T]$, which again contradicts (12). ■

Proposition 6: $\psi(t) > \alpha$ on $(0, t_1)$.

Proof: Suppose this is wrong. Then since $\psi(t) \neq \alpha$ on $(0, t_1)$, we have $\psi < \alpha$, which yields $u \equiv 0$ on $(0, t_1)$, whence also $x \equiv 0$ and $\varphi(x) \equiv 0$. Since obviously $\dot{\psi}(t_1) = 0$, from (11) we have $\varphi'(0) = 1/\alpha$. Thus, on $(0, t_1)$ we get $\dot{\psi} = -1 + \psi/\alpha < 0$ since $\psi < \alpha$. Hence, $\psi < \alpha$ and decreases on $(0, t_1)$, so $\psi(t_1) < \alpha$, which contradicts the relation $t_1 \in M_0$. ■

The next two propositions hold for any function $\psi(t)$, not only for the "true" costate function from MP. (We will need such an extension below, in a numerical algorithm.)

Proposition 7: Let $\psi(t)$ satisfy (11) on an interval $(0, t_1)$ where $u = 1$, and moreover, $\psi(t_1) = \alpha > 0$ and $\dot{\psi}(t_1) \leqslant 0$. Then $\psi(t)$ strictly decreases on $(0, t_1)$.

Proof: First, we show that $\psi(t) > \alpha$ in a left neighborhood of t_1. If $\dot{\psi}(t_1) < 0$, this is obvious. If $\dot{\psi}(t_1) = 0$, then

$$\ddot{\psi}(t_1) = \psi\,\varphi''(x)(1 - \varphi(x)) > 0,$$

since $\varphi(x(t)) < 1$ for all $t \geqslant 0$, and the required property again holds true.

Now, if ψ does not strictly decrease on $(0, t_1)$, one can show that there exist $t' < t'' < t_1$ such that $\psi(t') = \psi(t'') = c > \alpha$ and $\psi(t) \geqslant c$ on (t', t''). Then $\dot{\psi}(t') \geqslant 0$ and $\dot{\psi}(t'') \leqslant 0$, which in view of (11) means that $\psi(t')\,\varphi'(x(t')) \geqslant 1$ and $\psi(t'')\,\varphi'(x(t'')) \leqslant 1$, and hence, $\varphi'(x(t')) \leqslant \varphi'(x(t''))$. But since $x(t') < x(t'')$ (because $u = 1$) and $\varphi'(x)$ strictly increases in x, we obtain a contradiction. ■

Proposition 8: Let $\psi(t)$ satisfy (11) on an interval $[t_2, T]$ where $u = 0$, and let $\psi(t_2) > 0$ with $\dot{\psi}(t_2) \leqslant 0$. Then $\psi(t)$ strictly decreases on $[t_2, T]$.

Proof: Since $u = 0$, we have $\dot{x} = -\varphi(x)$. In view of (11), $\ddot{\psi} = \dot{\psi}\varphi' - \psi\varphi''\varphi$ is a continuous function. Since $\dot{\psi}(t_2) \leqslant 0$, we have $\ddot{\psi}(t_2) < 0$, so $\ddot{\psi} < 0$ in a right neighborhood of t_2. Hence, in this neighborhood, $\dot{\psi} < 0$, so $\psi < \psi(t_2)$, and then, since x decreases, it follows from (11) that these inequalities hold on the whole interval $(t_2, T]$. ■

Thus, when it comes to the true costate function ψ (i.e. satisfying MP), it has the following form. First, ψ decreases on $(0, t_1)$ from $\psi(0)$ to $\psi(t_1) = \alpha > 0$. Then $\psi(t) = \alpha$ on $[t_1, t_2]$. Finally, $\psi(t) < \alpha$ on (t_2, T), decreasing from $\psi(t_2) = \alpha$ to $\psi(T) = 0$.

To describe the control function, we use for convenience the notation $u = (u_1, u_2, ...)$ on $(\Delta_1, \Delta_2, ...)$, where $\Delta_1, \Delta_2, ...$ are some intervals, if $u(t) = u_1$ on Δ_1, $u(t) = u_2$ on Δ_2, etc.

The following two different cases are possible:

i) ψ crosses the value α only at time t_1. Here, $u = (1, 0)$ on $((0, t_1),\, (t_1, T))$ is a bang-bang control.

ii)ψ stays at the value α along an interval $[t_1, t_2]$ with $t_1 < t_2$. Here the control is bang-singular-bang, and, as usual, the maximality condition doesn't allow to find the singular control on $[t_1, t_2]$ directly. However, differentiating the equality $\psi(t) \equiv \alpha$, we obtain $\dot{\psi} = -1 + \alpha\varphi'(x) \equiv 0$. Since φ' strictly increases in x, we get $x(t) = const$, whence $\dot{x} = u - \varphi(x) = 0$, and so,

$$u_{sing}(t) = \varphi(x(t)) \qquad \text{on} \quad (t_1, t_2). \tag{13}$$

Thus, here we get $u = (1, \varphi(x(t_1)), 0)$ on $((0, t_1),\, (t_1, t_2),\, (t_2, T))$ with a singular subarc (t_1, t_2).

Note that, for a given starting point t_1 of singular subarc, the corresponding end point is uniquely determined as

$$t_2 \;=\; t_1 + \frac{m_0 - t_1 - m_T}{\varphi\,(x(t_1))}\,. \tag{14}$$

This can be easily obtained from equations (1). Indeed, since $u \equiv 1$ on $(0, t_1)$, we have $m(t_1) = m_0 - t_1$. Since $u \equiv 0$ on $[t_2, T]$, we get $m(t_2) = m_T$. On (t_1, t_2) we have $u = \varphi\,(x(t_1))$, which leads to $m(t_1) - m(t_2) = \varphi\,(x(t_1))\,(t_2 - t_1)$. The last relation implies (14).

Let us establish conditions under which each of these cases is realised.

III. NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMAL CONTROL TO BE OF THE BANG-BANG FORM

Let $\hat{x}(t)$ be the trajectory corresponding to the bang-bang control $\hat{u}(t) = 1$ for $0 \leqslant t \leqslant \gamma = \Delta m$ and $\hat{u}(t) = 0$ for $\gamma < t < T$. Thus, $\hat{x}(t)$ satisfies the relations $\dot{\hat{x}}(t) = 1 - \varphi(\hat{x}(t))$ on $[0, \gamma]$ with $\hat{x}(0) = 0$ and $\dot{\hat{x}}(t) = -\varphi(\hat{x}(t))$ on $[\gamma, T]$.

The optimality of this trajectory is equivalent to the existence of a function $\psi(t)$ satisfying (11) such that $\psi(t) > \psi(\gamma) = \alpha$ for $t \in [0, \gamma)$ and $\psi(t) < \alpha$ for $t \in (\gamma, T]$ with $\psi(T) = 0$.

According to Propositions 7 and 8, this is equivalent to that $\psi(t)$ satisfies (11) with

$$\psi(\gamma) = \alpha > 0, \qquad \dot{\psi}(\gamma) \leqslant 0, \qquad \text{and} \quad \psi(T) = 0.$$

The second condition means that $\psi(\gamma)\,\varphi'(\hat{x}(\gamma)) \leqslant 1$, i.e.

$$\alpha = \varphi(\gamma) \;\leqslant\; \alpha_{\max} \;= 1/\varphi'(\hat{x}(\gamma)).$$

This is an upper bound for α. Taking any $\alpha \leqslant \alpha_{\max}$, we obtain a unique $\psi = \psi(\alpha, t)$ as a solution to (11) with the initial value $\psi(\gamma) = \alpha$, and then we can determine $T = T(\alpha)$ such that $\psi(\alpha, T) = 0$. Let us establish the dependence of T on α.

Proposition 9: $\dfrac{\partial\psi(\alpha, t)}{\partial\alpha} > 0$ for all t.

Proof: Note that (11) is a linear (nonhomogeneous) equation. If α obtains an increment $\bar{\alpha}$, then the corresponding increment $\bar{\psi}(t) = \psi(\alpha + \bar{\alpha},\, t) - \psi(\alpha, t)$ satisfies the linear homogeneous equation

$$\dot{\bar{\psi}}(t) = \bar{\psi}(t)\,\varphi'(\hat{x}), \qquad \bar{\psi}(\gamma) = \bar{\alpha}.$$

If $\bar{\alpha} > 0$, then obviously $\bar{\psi}(t) > 0$ for all t. ■

Hence, if $\alpha \leqslant \alpha_{\max}$, then, in particular, $\forall\, t \geqslant \gamma$ we have $\psi(\alpha, t) \leqslant \psi(\alpha_{\max}, t)$ and therefore, $T(\alpha) \leqslant T(\alpha_{\max}) = T_{\max}$. If $\alpha = \psi(\gamma)$ decreases from $\alpha_{\max}$ to +0, then $\dot{\psi}(\gamma) = -1 + \alpha\,\varphi'(\hat{x}(\gamma))$ decreases from 0 to $-1 + 0$, and the corresponding $T(\alpha)$ decreases from $T_{\max}$ to $\gamma + 0$.

Thus, for any $T \in (\gamma,\, T_{\max}]$ there is a unique $\alpha \leqslant \alpha_{\max}$, such that the function $\psi(\alpha, t)$ fulfils the MP for the trajectory $\hat{x}(t)$ on $[0, T]$. If $T \leqslant \gamma$, then, as was already noted above, $\hat{x}(t)$ with $\hat{u} \equiv 1$ is obviously optimal, and if $T > T_{\max}$, then the corresponding $\alpha \leqslant \alpha_{\max}$ does not exist, and so the bang-bang $\hat{x}(t)$ is not optimal.

Thus, we proved the following

Theorem 1: The optimal control in problem (1) is of bang-bang form, which is described above, if and only if $T \leqslant T_{\max}$.

Examples of application of Theorem 1 are presented in section VI below.

IV. GEOMETRICAL ARGUMENTS FOR THE EXISTENC OF SINGULAR ARC

In this section we obtain a sufficient condition for the presence of singular arc along an optimal trajectory by using geometrical considerations. Since $s(t) = \int_0^t x(\tau)d\tau$, our problem (1) consists in finding such a bang-singular

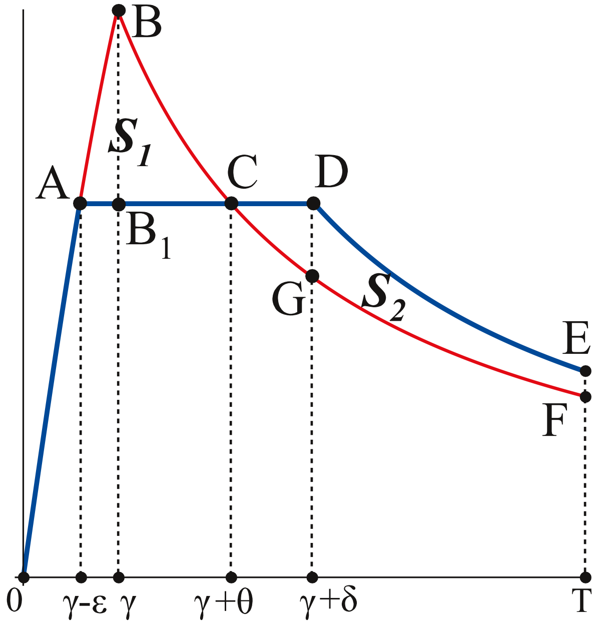


Fig. 1. Trajectory OBF is corresponds to bang-bang control; trajectory $OADE$corresponds to bang-singular-bang control.

switching time moment $t_1 \in [0, \gamma = \Delta m]$ that the square under the graph of corresponding $x(t)$ is maximal.

Take a small $\varepsilon > 0$ and compare squares under the graphs of bang-bang trajectory $\hat{x}(t)$ with $\hat{u} = (1, 0)$ on $((0, \gamma), (\gamma, T))$ corresponding to line OBF, and bang-singular-bang trajectory $x(t)$ with $u = (1,\, \varphi(\hat{x}(\gamma - \varepsilon),\, 0)$ on $((0, \gamma - \varepsilon),\, (\gamma - \varepsilon,\, \gamma + \delta(\varepsilon)),\, (\gamma + \delta(\varepsilon),\, T))$ corresponding to line $OADE$ in Fig 1.

From geometrical considerations it is easily seen that, if we obtain $S_2(\varepsilon) > S_1$, where S_1 is the square of ABC and $S_2(\varepsilon)$ is the square of $CDEF$, then the bang-bang trajectory is not optimal.

To simplify further formulas, define $x_\gamma = \hat{x}(\gamma)$, $\varphi_\gamma = \varphi(x_\gamma)$ and $p_\gamma = 1 - \varphi_\gamma$. One can show that, up to terms of order ε^2,

$$S_1 = \frac{p_\gamma}{2\varphi_\gamma}\,\varepsilon^2, \qquad \delta(\varepsilon) = \frac{p_\gamma}{\varphi_\gamma}\left(\varepsilon + \frac{\varphi'_\gamma}{\varphi_\gamma}\varepsilon^2\right), \tag{15}$$

$$|DG| = x(\gamma + \delta) - \hat{x}(\gamma + \delta) = \Delta x = \frac{p_\gamma \varphi'_\gamma}{2\varphi_\gamma}\,\varepsilon^2.$$

The last relation gives that $S_{GCD} \leqslant \frac{1}{2}|CD| \cdot |DG| = o(\varepsilon^2)$. Hence,

$$S_2(\varepsilon) = S_{DGFE} + o(\varepsilon^2) = \int_{\gamma+\delta}^{T} (\hat{x}(\tau) - x(\tau))\, d\tau + o(\varepsilon^2).$$

On the interval on $[\gamma + \delta,\, T]$, we have

$$\begin{cases} \dot{\hat{x}} = -\varphi(\hat{x}), \\ \hat{x}(\gamma + \delta) = x(\gamma - \varepsilon) - \Delta x, \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{x} = -\varphi(x), \\ x(\gamma + \delta) = x(\gamma - \varepsilon), \end{cases}$$

whence $\bar{x}(t) = x(t) - \hat{x}(t)$ satisfies (up to terms of order ε^2)

$$\dot{\bar{x}} = -\varphi'\left(\hat{x}(t)\right)\bar{x}, \qquad \bar{x}(\gamma + \delta) = \Delta x,$$

and now our aim is to find $S_2(\varepsilon) = \int_{\gamma+\delta}^{T} \bar{x}(\tau) d\tau$. We will use the following well-known property of linear ODE systems.

Lemma 1: Let $\bar{x}(t)$ be defined by the equation $\dot{\bar{x}} = A(t)\bar{x}$, where $A(t)$ is an integrable function. Then the linear functional

$$\bar{J} = l\bar{x}(T) + \int_{t_0}^{T} c(t)\bar{x}(t)dt,$$

where $c(t)$ is an integrable function and l is a real number, can be represented in the form $\bar{J} = -\bar{\psi}(t_0)\bar{x}(t_0)$, where the function $\bar{\psi}(t)$ is determined by the equation

$$\dot{\bar{\psi}} = -\bar{\psi}A(t) + c(t), \qquad \bar{\psi}(T) = -l.$$

Proof: We can write $\bar{J} = l\bar{x}(T) + \int_{t_0}^{T} \left(c\bar{x} + \bar{\psi}\left(\dot{\bar{x}} - A\bar{x}\right)\right) dt$. Integrating $\bar{\psi}\dot{\bar{x}}$ by parts we get $\int_{t_0}^{T} \bar{\psi}\dot{\bar{x}}dt = \bar{\psi}(T)\bar{x}(T) - \bar{\psi}(t_0)\bar{x}(t_0) - \int_{t_0}^{T} \dot{\bar{\psi}}\bar{x}dt$, so we obtain

$$\bar{J} = \left(l + \bar{\psi}(T)\right)\bar{x}(T) - \bar{\psi}(t_0)\bar{x}(t_0) +$$
$$+ \int_{t_0}^{T} \left(c - \left(\dot{\bar{\psi}} + \bar{\psi}A\right)\right)\bar{x}dt = -\bar{\psi}(t_0)\bar{x}(t_0).$$

■

(Note that Lemma 1 is valid also in the case where $\bar{x}$ is a vector function.)

In our case $A(t) = -\varphi'(\hat{x}(t))$, $l = 0$, $c(t) \equiv 1$, $\bar{\psi}(t) = -\psi(t)$, so $S_2(\varepsilon) = \psi(\gamma)\frac{p_\gamma\varphi'_\gamma}{2\varphi_\gamma}\,\varepsilon^2$, and condition $S_2(\varepsilon) > S_1$ in view of (15) is equivalent to

$$\psi(\gamma + \delta(\varepsilon))\,\varphi'(x(\gamma + \delta(\varepsilon))) > 1.$$

The existence of such $\varepsilon > 0$ that the last inequality takes place is guaranteed if $\psi(\gamma)\varphi'(x_\gamma) > 1$, which allows us to formulate a sufficient condition for the existence of singular arc.

Theorem 2: Let $\hat{x}(t)$ be a bang-bang trajectory (may be not optimal one) with a switching time $\gamma = \Delta m$, and let $\psi(t)$ be determined according to (11) on $[\gamma, T]$. If

$$\psi(\gamma)\,\varphi'(\hat{x}(\gamma)) > 1 \qquad (\text{i.e.} \quad \alpha > \alpha_{\max})$$

then the optimal trajectory in problem (1) contains a singular subarc.

This theorem, obtained by geometrical arguments, is in accordance with the above Theorem 1.

V. ALGORITHM TO FIND SINGULAR SUBINTERVAL

Suppose that, for a given T the optimal control in (1) is of bang-singular-bang form. The below algorithm allows one to find the start and end times of the singular subarc.

Given $\Delta m = m_0 - m_T$, a function $\varphi(x)$, and accuracy $\varepsilon > 0$, define $\gamma = \Delta m$ and a function $\hat{x}(t)$ on $[0, \gamma]$ from the equation $\dot{x} = 1 - \varphi(x)$, $x(0) = 0$. Taking $k = 0$ and $t_1^0 \in (0, \hat{t})$, go to step 1 of the following algorithm.

1) Compute $x_1^k = \hat{x}(t_1^k)$ and set $t_2^k = t_1^k + \left(m_0 - t_1^k - m_T\right)/\varphi(x_1^k)$.
2) If $t_2^k \geqslant T$, increase t_1^k and go to step 1. Else, go to step 3.

3) Find $\psi^k(T)$ from the following IVP on $[t_2^k, T]$:

$$\begin{cases} \dot{\psi}^k(t) = -1 + \psi^k(t)\varphi'\left(x(t)\right), & \psi^k(t_2^k) = 1/\varphi'(x_1^k), \\ \dot{x}(t) = -\varphi(x(t)), & x(t_2^k) = x_1^k. \end{cases}$$

4) (a) If $|\psi^k(T)| \leqslant \varepsilon$, finish the computations with the resulting $t_1^* = t_1^k$.
   (b) If $\psi^k(T) > \varepsilon$, take $t_1^{k+1} > t_1^k$ and go to step 1.
   (c) If $\psi^k(T) < -\varepsilon$, take $t_1^{k+1} < t_1^k$ and go to step 1.

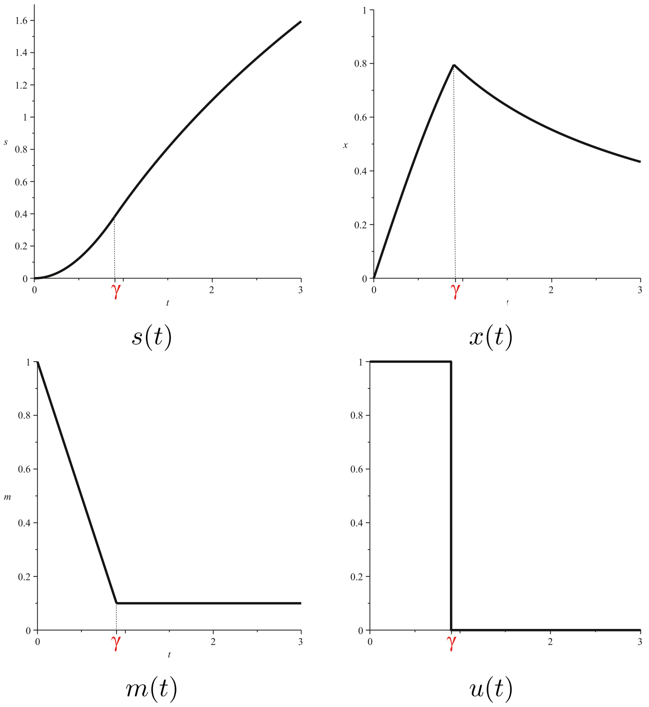
Note that the increase of t_1^k corresponds to the decrease of α^k. As a result, we obtain that the singular subarc of optimal trajectory begins at t_1^* and ends at $t_2^* = t_1^* + \alpha^*\,(m_0 - m_T - t_1^*)$, where $\alpha^* = 1/\varphi(\hat{x}(t_1^*))$. The described algorithm uses properties of $\psi^k(t)$ which are given in Propositions 7 – 9.

VI. NUMERICAL EXPERIMENTS

In this section we consider the case of simple $\varphi(x) = x^2/2$ with $m_0 = 1$ and $m_T = 0.1$. Thus, we have $\gamma = \Delta m = 0.9$. Choose the accuracy $\varepsilon = 10^{-6}$.

1) Find $\alpha_{\max}$ and the "threshold value" $T_{\max} = T(\alpha_{\max})$ as was shown in Sec. 5. Easy computations give $\alpha = 1.087745$ and $T_{\max} = 3.414422$.

2) Consider $T = 3 < T_{\max}$. According to Theorem 1, the optimal control is of bang-bang form: $u = 1$ on $(0, \gamma)$ and $u = 0$ on (γ, T).

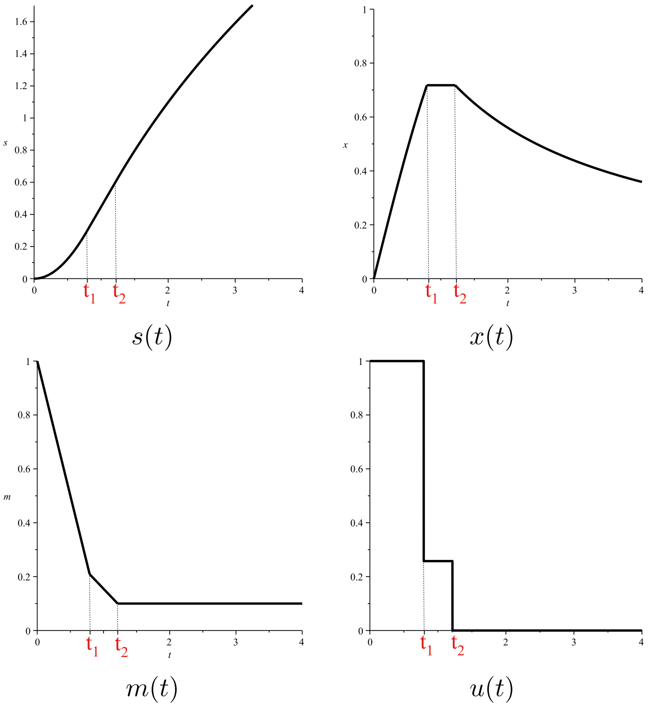


$s(t)$ $x(t)$

$m(t)$ $u(t)$

Fig. 2. Graphs of the optimal trajectory and control for $T = 3$.

3) Consider $T = 4 > T^*$. Applying the algorithm in Sec. 7, we obtain that the singular subarc starts at $t_1^* = 0.7912296$ and ends at $t_2^* = 1.2133897$.

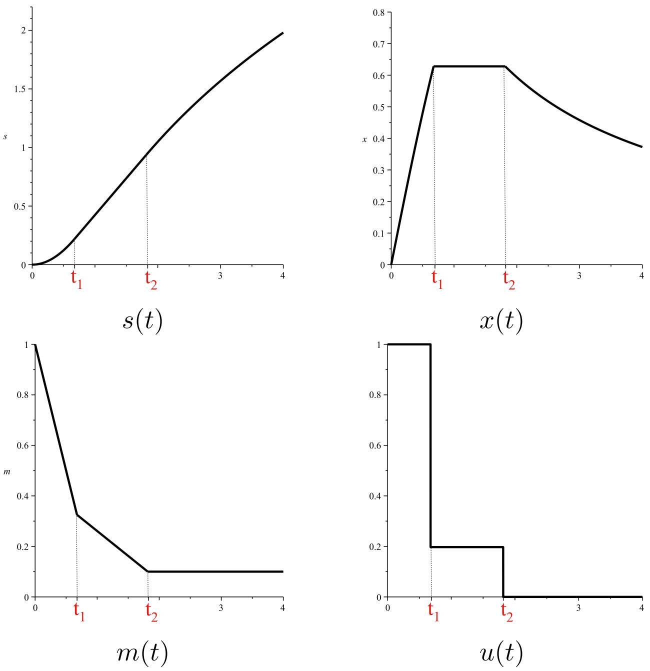


$s(t)$ $x(t)$

$m(t)$ $u(t)$

Fig. 3. Graphs of the optimal trajectory and control for $T = 4$

4) Consider $T = 5 > T^*$. Applying the algorithm in Sec. 7, we obtain that the singular subarc starts at $t_1^* = 0.6746996$ and ends at $t_2^* = 1.818053$.



$s(t)$ $x(t)$

$m(t)$ $u(t)$

Fig. 4. Graphs of the optimal trajectory and control for $T = 5$.

All computations were performed on a computer with Pentium(R) Dual-Core CPU 2.2 GHz, 2 GB RAM under 32-bit Windows 7 operating system.

VII. ON A TWO-STAGED VARIANT OF THE PROBLEM 1 IN THE CASE OF A QUADRATIC MEDIA RESISTANCE

In this section we discuss rather briefly the following variant of our problem, where our “rocket” consists of two stages and the first stage disconnects at time t_1 when it is empty.

$$\begin{cases} \dot{s} = x, & s(0) = 0, \quad s(T) \to \max\limits_{0 \leqslant u \leqslant C}, \\ \dot{x} = u - \varphi(x), & x(0) = 0, \quad x(T) \text{ is free}, \\ \dot{m} = -u, & m(0) = m_0, \quad m(T) \geqslant m_T, \\ m(t_1 - 0) = m_1 & \in (0, m_0), \\ m(t_1 + 0) = m_2 & \in (0, m_1), \end{cases} \tag{16}$$

where the friction function $\varphi(x)$ still satisfies the conditions $\varphi(0) = 0$, $\varphi'(0) \geqslant 0$, $\varphi''(x) > 0$ for all $x > 0$. Using the version of Maxumum principle (see, for example, [7]) for the problems with terminal constraints that include points lying “inside” the time interval, we can easily verify that the optimal trajectory may contains no more than one singular subarc along each of time intervals $(0, t_1)$ and (t_1, T), and the singular subarc, if exists, follows and precedes the subarcs corresponding to $u = C$ and $u = 0$, respectively.

The case of “general” $\varphi(x)$ is one of the goals of the future investigations. Here we discuss the very simple case of $C = 1$, $\varphi(x) = x^2/2$. In this case it is rather easy to verify that the singular subarc may takes place only on the interval (t_1, T), i.e. the optimal control has the form $u = (1, 1, u_s, 0)$ along the time intervals $(0, t_1 = m_0 - m_1)$, (t_1, t_s^1), (t_s^1, t_s^2), (t_s^2, T), respectively. Moreover, the goal functional in this case can be found analytically. Thus, we obtain the following restricted problem which allows us to find the singular subarc (t_s^1, t_s^2):

Maximize

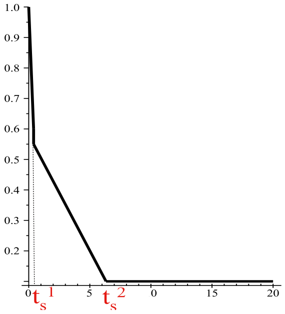
$$s(T) = 2\ln\left(1 + \frac{T - t_s^2}{\sqrt{2}} \tanh\frac{t_s^1}{\sqrt{2}} + \frac{t_s^1 - (m_0 - m_1)}{\sqrt{2}\tanh\frac{t_s^1}{\sqrt{2}}} - \right.$$
$$\left. - \frac{m_2 - m_T}{\sqrt{2}\tanh\frac{t_s^1}{\sqrt{2}}}\right) + 2\ln\frac{e^{t_s^1\sqrt{2}} + 1}{2} - \sqrt{2}t_s^1 +$$
$$+ \sqrt{2}\left(t_s^1 - (m_0 - m_1)\right)\tanh\frac{t_s^1}{\sqrt{2}} -$$
$$- \frac{\sqrt{2}\left(t_s^1 - (m_0 - m_1) - (m_2 - m_T)\right)}{\tanh\frac{t_s^1}{\sqrt{2}}}$$

Subject to

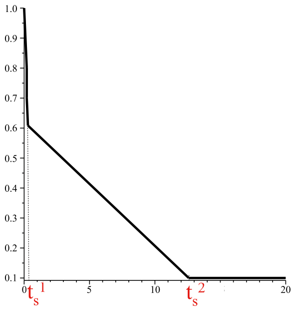
$$\begin{cases} t_s^1 \leqslant (m_0 - m_1 + m_2 - m_T), \\ t_s^2 = t_s^1 + \dfrac{m_2 - m_T - t_s^1 + m_0 - m_1}{\tanh^2\frac{t_s^1}{\sqrt{2}}} \leqslant T. \end{cases}$$

1) Taking $m_0 = 1, m_1 = 0.6, m_2 = 0.5, m_T = 0.1$ and $T = 20$, we obtain $t_s^1 = 0.4, t_s^2 = 5.92737$
2) Taking $m_0 = 1, m_1 = 0.8, m_2 = 0.7, m_T = 0.1$ and $T = 20$, we obtain $t_s^1 = 0.291625, t_s^2 = 12.295763$
3) Taking $m_0 = 1, m_1 = 0.5, m_2 = 0.4, m_T = 0.1$ and $T = 10$, we obtain $t_s = 0.8 = m_0 - m_1 + m_2 - m_T$, i.e. the optimal control is of bang-bang form.

The graphs of corresponding $m(t)$ are the following:



$m(t)$ for the case 1)



$m(t)$ for the case 2)

CONCLUSION

For a simplified Goddard problem the form of optimal control is obtained from the Pontryagin Maximum Principle. analytically. Necessary and sufficient conditions of bang-bang form of this control are formulated and numerical algorithm to find bounds of the singular arc is suggested. A method to find “threshold value” T^* such that for all $T > T^*$ optimal trajectory in problem (1) contains singular subarc is also described. Our future plans include modification of described algorithm to work with the “multi-staged” version of the problem.

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