

# An Algebraic Approach for Robust Fault Detection of Input-Output Elastodynamic Distributed Parameter Systems

Riccardo M. G. Ferrari, Thomas Parisini, and Marios M. Polycarpou

**Abstract**—This paper deals with the problem of designing a robust fault detection methodology for a class of input-output, uncertain dynamical distributed parameter systems, namely mechanical elastodynamic systems, which are representative of a whole class of problems related to on-line health monitoring of mechanical and civil engineering structures. The proposed approach does not require full state measurements and is robust to measuring, modeling and numerical errors, thanks to a time varying detection threshold. In order to avoid the problems associated with classical discretization techniques for distributed parameter systems, which can lead to numerical errors difficult to bound a priori, and thus higher thresholds, a suitable structure-preserving algebraic approach, called *Cell Method*, will be employed. This method consists in writing the equations of a distributed parameter system directly in discrete form, avoiding the usual discretization process and leading to a symplectic, that is energy preserving, numerical scheme.

## I. INTRODUCTION

Recently there has been significant research activity about Fault Detection and Isolation (FDI) methods for large-scale, distributed systems [1], [2], [3], [4], [5], [6], [7]. While in these papers the possible applications ranged from large-scale industrial plants, to autonomous vehicle formations, one kind of systems that received less attention are distributed parameter systems. The latter arise naturally in the modeling of many physical systems whose dimension is such that propagation phenomena cannot be neglected, for instance in vibration analysis in mechanics, or in systems transmitting or receiving electromagnetic waves, or in large multi-phase chemical reactors dominated by advection.

The need for FDI methods for this kind of systems is testified, for instance, by the crescent use in civil and mechanical engineering structures of permanent diagnostic transducers. Being embedded in the structure, they continuously check its health status, by monitoring the amount of vibrations, that can be inborn or purposely excited [8]. The signals captured by the transducers are classified as either *faulty* or *healthy* by comparing them to known historical recordings taken in controlled conditions, thus implementing a so called *process history-based* FDI method [9]. When trying to apply more advanced *model-based* methods [10] to such elastodynamic distributed parameter systems, an evident obstacle lies in the fact that models obtained by describing the distributed

dynamics with a Partial Differential Equations (PDE) system are infinite dimensional [11]. In order to make the problem tractable in practice, a finite-dimensional approximate model needs to be used: conventional approaches consist in approximating the solution with an expansion into a finite number of modes, in relying on the Method Of Moments (MoM) [12], or in discretizing the PDEs on a finite-dimensional space-time grid such as in the FEM, FDTD or FVM methods [13], [14]. The modal expansion method leads to drawbacks when fault parameters are to be estimated, as pointed out in [15] and the references contained therein, while the MoM is in practice limited to about the first ten moments only [16]. On the other hand, PDE's discretization can lead to numerical errors such as spurious solutions, numerical dissipation or dispersion errors [17], especially if the discretization is not structure-preserving (e.g. topological relations such as the Stokes' Theorem may not be exactly conserved [18]).

A fundamental problem that must be solved in an effective model-based FDI method is the design of adaptive detection and isolation thresholds which are robust to modeling, measuring and numerical errors. While in the case of the modal expansion technique the robustness to model and measuring uncertainties was addressed in [19], [20], to the best of the authors knowledge no such result was obtained for the PDE discretization technique. PDE discretization is an attractive approach, as it benefits from a large number of choices regarding the approximation order and the kind of discretization grid, and the adoption of methods like FEM is widespread in the whole scientific community. The numerical integration schemes used to solve PDEs are usually implicit, leading to long computation times that are seldom compatible with the real-time requirements of online FDI, and to numerical errors, especially dissipation, difficult to bound a priori, thus leading to very conservative thresholds.

In order to exploit the PDE discretization advantages, while avoiding the problems of implicit integration schemes, in this paper we will employ a structure-preserving, algebraic modeling approach called *Cell Method* (CM) [21], [22]. The CM, which is based on the so-called Algebraic Formulation (AF), leads to an explicit, symplectic [17] time marching scheme which can be proved to be devoid of numerical dissipation. Its dispersion error as well as the effects of parametric uncertainties can be characterized analytically, thus leading to tighter thresholds. The paper is organized as follows:

- 1) As the proposed fault detection methodology for distributed-parameter systems falls into the category of model-based ones, as already enhanced in the introduction, a key aspect is the specific structure-preserving algebraic formulation that plays a fundamental enabling

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Riccardo M.G. Ferrari is with Danieli Automation S.p.A., Buttrio, Italy. (r.ferrari@dca.it)

T. Parisini is with Imperial College London (UK) and University of Trieste (I). (t.parisini@gmail.com)

M. M. Polycarpou is with the KIOS Research Center for Intelligent Systems and Networks, Dept. of Electrical and Computer Engineering, University of Cyprus, Nicosia 1678, Cyprus. (mpolycar@ucy.ac.cy)

role. The basic aspects of the CM method is thus addressed in Section II.

- 2) The paper is focused on a specific class of distributed-parameter systems, namely, elastodynamic ones. Therefore, the CM technique has to be customized to this class of systems and this is carried out in Section III.
- 3) The proposed fault detection approach is then presented in Section IV, where the features of the CM-based analytical redundancy approach are enhanced by designing a special class of robust adaptive detection thresholds, taking into account numerical errors.
- 4) The effectiveness of the proposed fault detection technique is shown in Section V, where thorough simulation trials are reported.

## II. ALGEBRAIC FORMULATION BACKGROUND

The Algebraic Formulation, on which the CM is based, aims to avoid in first place the use of PDEs, by answering the following question: “*Is it possible to avoid any discretization process of differential equations and to directly write physical laws in a discrete, i.e. algebraic, form?*”. The answer relies on the use of global physical variables that are associated to space-time elements of finite size, called *cells*. Based on this approach, balance and constitutive physical equations can be written in an algebraic, that is discrete, and explicit way, without any requirement on continuity, smoothness or differentiability as in the differential setting. While many approaches exist in the literature that share common traits with the Algebraic Formulation (AF), such as Discrete Exterior Calculus (DEC) [23] and structure-preserving discretization of distributed-parameter port-Hamiltonian systems [24], Mimetic Discretization (MD) [18], [25] or Finite Integration Technique (FIT) [26], the unique feature of the AF is that it completely builds from the ground up a discrete version of physical theories, achieving discrete operators analogous to *grad*, *div* and *curl* that feature structural properties by definition, instead of inheriting them from differential operators thanks to a proper discretization.

The usual three ingredients of a physical theory are: *variables*, *equations* and a *space* where the first two are defined. The AF makes use of *global variables*, that is, variables associated not to a point, like field variables in differential theories, but to a finitely-sized domain. For instance, a global variable in electromagnetism is the magnetic flux through a surface, which is measured in Wb, whereas a field variable is the magnetic flux density at a point on the surface, measured in  $\text{Wb m}^{-2}$ . Global variables can be divided in two main categories<sup>1</sup>:

- **Configuration variables:** variables that define the state of a system (e.g. position and velocity of a mechanical particle)
- **Source variables:** variables that describe all the causes that can change the configuration of a system (e.g. forces acting on a mechanical particle)

Equations, instead, may be divided into:

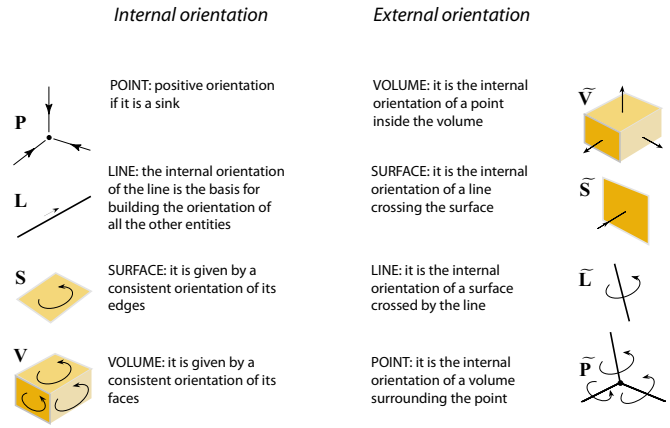


Fig. 1. Internal and external orientation of geometric elements.

- **Definition equations:** define a variable in terms of other variables of the same category (e.g. velocity is the time rate of change of position)
- **Balance equations:** impose conservation of a variable (e.g. the sum of forces on a particle is zero)
- **Constitutive equations:** link a configuration and a source variable, thus defining the behavior of a system (e.g. the elastic force, a source, is proportional to the displacement, a configuration)

The space-time reference framework where these variables and equations are defined is a *complex* ([27], [28]) of oriented finite geometric elements, called *p-cells*, where  $p \in \{0, \dots, 3\}$  is the geometric dimension<sup>2</sup>. The generic elements are denoted with the letters **P** (points), **L** (lines), **S** (surfaces), **V** (volumes), **I** (time instants), **T** (time intervals) and can be endowed with an *internal* orientation, or an *external* one, in which case a tilde is used (Fig. 1).

Each variable is *associated* with a geometric element (e.g., a surface force is associated with the surface where it is acting) and this is denoted by placing the symbol of the elements in square brackets after the variable, as in  $a[\mathbf{P}, \mathbf{I}]$ , which reads:  $a$  is associated to a point and an instant with internal orientations. Such association is formalized by the use of *co-chains* for computing the value of a variable on a given  $p$ -cell (see Definition 11). It is worth noting that this topological information is the key to the definition of a structure-preserving numerical method. In contrast, the differential formulation makes all the variables collapse to points, because of the limit operation usually found in the definition of field variables. Classic numerical schemes many times fail to recover the original topological association of variables when discretizing the equations, thus leading to unavoidable, intrinsic numerical errors.

In the AF, two different cell complexes are employed: the *primal*, made of cells with internal orientation to which configuration variables are associated; and the *dual*, made with cells with external orientation and devoted to source variables [21]. Usually the primal complex is obtained by a Delaunay triangulation of the computation domain, in which

<sup>1</sup>A third category, *energy variables*, is not used in this paper.

<sup>2</sup>Here we do not address how such a complex is obtained by discretizing a compact manifold in  $\mathbb{R}^p$ , the interested reader may refer to [29].

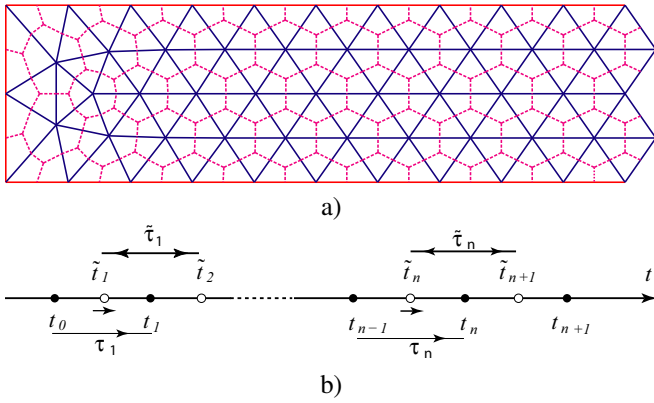


Fig. 2. a) A two-dimensional primal simplex (—) and its barycentric dual (- -); b) a subdivision of the time-line into primal and dual time instants  $t_i$  and intervals  $\tau_i$ .

case it is called a *simplicial complex*, the other being its Voronoi or barycentric dual (see Fig. 2).

Now some formal definitions from algebraic topology will be given (see, for instance, [30], [27], [31], [28]).

**Definition 1:** A  $p$ -cell  $\sigma^p \subseteq \Sigma^p$  is defined as the convex hull of an ordered set of at least  $p+1$  non-collinear points in  $\mathbb{R}^p$ , where  $\Sigma^p$  is the set of all  $p$ -cells. If a  $p$ -cell has exactly  $p+1$  vertices, it is called a  $p$ -simplex.

**Definition 2:** A *face* of a  $p$ -cell  $\sigma^p \subseteq \Sigma^p$  is a  $(p-1)$ -cell  $\sigma^{p-1} \subset \partial(\sigma^p)$ , where  $\partial(\sigma^p)$  is the *boundary* of the cell  $\sigma^p$ .

**Definition 3:** A *coface* of a  $p$ -cell  $\sigma^p \subseteq \Sigma^p$  is a  $(p+1)$ -cell  $\sigma^{p+1}$  such that  $\sigma^p \subset \partial(\sigma^{p+1})$ . The set of all the cofaces of  $\sigma^p$  is called the *coboundary* of  $\sigma^p$ .

**Definition 4:** A (*primal*) *cell complex*  $K^n \subseteq \mathbb{R}^n$  is a set of cells with dimensions  $p \leq n$  such that:

- 1) For each  $\sigma^p \in K^n$ , each of its faces are in  $K^n$ .
- 2) For every two  $\sigma_i^p$  and  $\sigma_j^p$  in  $K^n$ , their intersection is either empty or a common face.

**Definition 5:** The *dual cell complex*  $\tilde{K}^n \subseteq \mathbb{R}^n$  of a cell complex  $K^n$  is a complex whose elements are duals of the elements of  $K^n$ .

**Definition 6:** The *duality map*  $\sim: K^n \mapsto \tilde{K}^n$  is a bijective relation that maps a primal  $p$ -cell  $\sigma^p$  to its dual  $(n-p)$ -cell  $\tilde{\sigma}^{n-p}$ , that is  $\tilde{\sigma}^{n-p} = \sim(\sigma^p)$ .

**Definition 7:** The *incidence number*  $a_{h,k}$  between a  $p+1$  and a  $p$ -cell is denoted as  $a_{h,k} \equiv [\sigma_h^{p+1} : \sigma_k^p]$ , and is defined as

$$a_{h,k} \triangleq \begin{cases} 1 & \text{if } \sigma_k^p \text{ is a face of } \sigma_h^{p+1} \text{ and have} \\ & \text{compatible orientations} \\ -1 & \text{if } \sigma_k^p \text{ is a face of } \sigma_h^{p+1} \text{ and have} \\ & \text{incompatible orientations} \\ 0 & \text{otherwise} \end{cases}$$

**Definition 8:** The *incidence matrix*  $A^{p+1} \in \mathbb{R}^{N_{p+1}} \times \mathbb{R}^{N_p}$  of a complex  $K^n$  containing  $N_{p+1}$   $(p+1)$ -cells and  $N_p$   $p$ -cells is the matrix  $A^{p+1} = [a_{h,k}]$ .

In this paper we will denote with special symbols the

following incidence matrices:  $G \triangleq A^1$ ,  $C \triangleq A^2$ ,  $D \triangleq A^3$  for the primal complex, and  $\tilde{G} \triangleq \tilde{A}^1 = -D^\top$ ,  $\tilde{C} \triangleq \tilde{A}^2 = C^\top$ ,  $\tilde{D} \triangleq \tilde{A}^3 = -G^\top$ , as they allow to compute the algebraic analogues of the operators *grad*, *curl* and *div*.

**Definition 9:** A  $p$ -chain  $c^p \in C^p(K^n)$ , where  $C^p(K^n)$  is the chain group on  $K^n$ , is a formal linear combination of  $p$ -cells of  $K^n$  with integer coefficients  $c^p \triangleq \sum_{i=1}^{N_p} c_i \sigma_i^p$ ,  $c_i \in \mathbb{Z}$ ,  $\sigma_i^p \in K^n$ .

**Definition 10:** The *boundary operator*  $\partial^p$  maps a  $p$ -chain  $c^p = \sum_i c_i \sigma_i^p$  to a  $(p-1)$ -chain  $b^{p-1} = \sum_i b_i \sigma_i^{p-1}$  such that  $b_i \triangleq \sum_j c_j [\sigma_j^p : \sigma_i^{p-1}]$ .

Now, the fact that a global variable is associated to a  $p$ -cell, that is a  $p$ -dimensional sub-manifold of an  $n$ -dimensional manifold, means that at each cell or chain one can associate a value of the variable. Such a mapping is called a *cochain*.

**Definition 11:** A  $p$ -cochain  $f^p \in \Omega^p(K^n, \mathbb{G})$  from the chain group  $C^p(K^n)$  on  $K^n$  to the additive group  $\mathbb{G}$  is a linear homomorphism such that

$$\begin{aligned} f^p(c^p + b^p) &= f^p(c^p) + f^p(b^p), \\ f^p(\alpha c^p) &= \alpha f^p(c^p), \forall c^p, b^p \in C^p(K^n), \forall \alpha \in \mathbb{Z} \end{aligned}$$

From the definition of cochain it follows that a cochain evaluated on a generic chain  $c^p = \sum_i c_i \sigma_i^p$  can be written as the sum of the cochain values on the cells as  $f^p(c^p) = \sum_i c_i f^p(\sigma_i^p)$ .

Finally, the *coboundary operator* is introduced, which allows to express balance, circuital and difference equations when applied to 3-, 2-, 1-dimensional cochains, respectively. It is the algebraic analogue of the exterior differentiation on exterior  $p$ -forms and leads to typical operators such as *grad*, *curl* and *div*, along with time derivatives<sup>3</sup> [27].

**Definition 12:** The *coboundary operator*  $\delta^p$  maps a  $p$ -cochain  $f^p$  to a  $(p+1)$ -cochain  $g^{p+1}$  such that

$$g^{p+1}(\sigma_i^{p+1}) = \sum_j [\sigma_i^{p+1} : \sigma_j^p] f^p(\sigma_j^p) \triangleq \delta^p f^p(\sigma_j^p)$$

for every  $(p+1)$ -cell  $\sigma_i^{p+1}$  and every  $p$ -cell  $\sigma_j^p$  in  $K^n$ .

This definition leads to the following, fundamental

**Theorem 1 (Discrete Stokes Theorem [31]):** The value of a  $p$ -cochain  $f^p$  on the  $p$ -dimensional boundary of a  $(p+1)$ -dimensional chain  $c^{p+1}$  is equal to the value of the coboundary of  $f^p$  on  $c^{p+1}$ .

$$f^p(\partial c^{p+1}) = \delta^p f^p(c^{p+1})$$

□

Recalling Definition 8, it follows easily that  $\delta^0$ ,  $\delta^1$  and  $\delta^2$  can be obtained by multiplication by the matrices  $G$  (*grad*),  $C$  (*curl*) and  $D$  (*div*), or their dual equivalents. In the following we will use the notation  $\delta^{\mathbf{X}}$ , with  $\mathbf{X} \in \{\mathbf{P}, \mathbf{L}, \mathbf{S}, \mathbf{V}, \mathbf{I}, \mathbf{T}, \tilde{\mathbf{P}}, \tilde{\mathbf{L}}, \tilde{\mathbf{S}}, \tilde{\mathbf{V}}, \tilde{\mathbf{I}}, \tilde{\mathbf{T}}\}$ , to denote over which kind of element the coboundary is taken.

<sup>3</sup>While the coboundary can be used to express balance law which are purely topological, operators analogous to *grad*, *curl* and *div* require the definition of a metric.

### III. ELASTODYNAMIC DISTRIBUTED PARAMETER SYSTEMS

In this section, the algebraic approach addressed in Section II is specialized to the important class of distributed parameter systems constituted by one-dimensional elastodynamic mechanical systems [32]. This problem is casted on a primal complex with  $N_0$  0-cells and  $N_1$  1-cells, whose dual complex has  $N_0$  1-cells and  $N_1+1$  0-cells, with  $N_0 = N_1+1$ . Each cell must be thought of as a small portion of a bar-shaped domain, so that each cell is endowed with a volume, a length and a cross-section area.

We will now introduce the problem variables and the related constitutive equations. We start with the nodal displacement  $u[\mathbf{P}, \mathbf{I}] \in \Omega^0 \times \Omega^0$ , which is a configuration variable, that leads to the definition of the elastic strain  $\epsilon[\mathbf{L}, \mathbf{I}] \triangleq \frac{\delta^{\mathbf{P}}(u[\mathbf{P}, \mathbf{I}])}{|\mathbf{L}|} \in \Omega^0 \times \Omega^0$ . The elastic stress  $\sigma[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}] \triangleq \frac{F_E[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}]}{|\tilde{\mathbf{S}}|} \in \tilde{\Omega}^2 \times \tilde{\Omega}^1$ , a source variable, is linked to the strain by the elastic constitutive relation

$$\sigma[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}] = -E[\mathbf{L}, \mathbf{I}] \cdot \epsilon[\mathbf{L}, \mathbf{I}] \triangleq f_{\epsilon, \sigma}(\epsilon[\mathbf{L}, \mathbf{I}]),$$

where  $F_E[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}]$  are the elastic forces acting on the dual surfaces  $\tilde{\mathbf{S}}$ . The space-time elements involved are such that  $\tilde{\mathbf{S}} = \sim(\mathbf{L})$ ,  $\tilde{\mathbf{T}} = \sim(\mathbf{I})$  and  $f_{\epsilon, \sigma} : \Omega^1 \times \Omega^0 \mapsto \tilde{\Omega}^2 \times \tilde{\Omega}^1$  is a function that maps a primal<sup>4</sup>  $(1, 0)$ -cochain into a dual  $(2, 1)$ -cochain. In particular,  $f_{\epsilon, \sigma}$  describes the elastic behavior of the material, thanks to the possibly space and time varying Young modulus  $E[\mathbf{L}, \mathbf{I}]$ .

We need also to introduce the nodal velocities  $v[\mathbf{P}, \mathbf{T}] = \frac{\delta^1(u[\mathbf{P}, \mathbf{I}])}{|\mathbf{T}|} \in \Omega^0 \times \Omega^1$  and the relative velocities  $\nu[\mathbf{L}, \mathbf{T}] \triangleq \frac{\delta^{\mathbf{P}}(v[\mathbf{P}, \mathbf{T}])}{|\mathbf{L}|} \in \Omega^1 \times \Omega^1$ , which are responsible for a Kelvin-Voigt viscous stress contribution  $\sigma_R[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}] \triangleq \frac{F_R[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}]}{|\tilde{\mathbf{S}}|} \in \tilde{\Omega}^2 \times \tilde{\Omega}^1$ , with  $F_R[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}]$  being the viscous forces. The viscous constitutive relation is defined by

$$\sigma_R[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}] = -\mu_R[\mathbf{L}, \mathbf{T}] \cdot \nu[\mathbf{L}, \mathbf{T}] \triangleq f_{\nu, \sigma_R}(\nu[\mathbf{L}, \mathbf{T}]),$$

where  $f_{\nu, \sigma_R} : \Omega^1 \times \Omega^1 \mapsto \tilde{\Omega}^2 \times \tilde{\Omega}^1$  describes the material damping through the coefficient  $\mu_R[\mathbf{L}, \mathbf{I}]$ .

Now, the motion of the material will be described by relying to the discrete formulation of particle mechanics. We start by introducing the linear momentum, that is a source variable, and its constitutive equation:

$$p[\tilde{\mathbf{V}}, \tilde{\mathbf{I}}] = m[\tilde{\mathbf{V}}, \mathbf{I}] \cdot v[\mathbf{P}, \mathbf{T}] \triangleq f_{v, p}(v[\mathbf{P}, \mathbf{T}])$$

The linear momentum is related to the impulse of force  $I[\tilde{\mathbf{V}}, \tilde{\mathbf{T}}]$  thanks to Newton's second law of motion, which is a balance equation:

$$I[\tilde{\mathbf{V}}, \tilde{\mathbf{T}}] = \delta^{\tilde{\mathbf{I}}}(p[\tilde{\mathbf{V}}, \tilde{\mathbf{I}}])$$

The impulse  $I$  can be written as the sum of different terms:

$$I[\tilde{\mathbf{V}}, \tilde{\mathbf{T}}] = I_E[\tilde{\mathbf{V}}, \tilde{\mathbf{T}}] + I_R[\tilde{\mathbf{V}}, \tilde{\mathbf{T}}] + I_V[\tilde{\mathbf{V}}, \tilde{\mathbf{T}}] \quad (1)$$

where  $I_E[\tilde{\mathbf{V}}, \tilde{\mathbf{T}}] = \delta^{\tilde{\mathbf{S}}}(F_E[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}]) \cdot |\tilde{\mathbf{T}}|$  is the impulse due to the balance on the boundary of the volume  $\tilde{\mathbf{V}}$  of the

<sup>4</sup>In the following notation the first index refer to the space element dimension, and the second to the time element dimension.

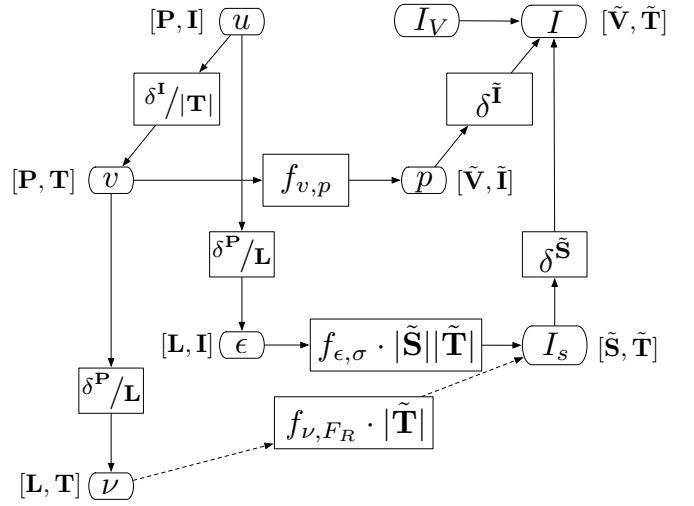


Fig. 3. Tonti diagram of one-dimensional elastodynamics with internal viscous damping.

elastic forces  $F_E$ . Similarly,  $I_R[\tilde{\mathbf{V}}, \tilde{\mathbf{T}}] = \delta^{\tilde{\mathbf{S}}}(F_R[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}]) \cdot |\tilde{\mathbf{T}}|$  is due to the balance of the viscous forces  $F_R$ , while  $I_V[\tilde{\mathbf{V}}, \tilde{\mathbf{T}}]$  represents the impulse of the volume forces  $F_V$ , which account for forces such as gravity, or electromagnetic attraction or repulsion.

*Remark 1:* The computation of the balance on  $\tilde{\mathbf{V}}$  of the surface forces acting on  $\tilde{\mathbf{S}} = \partial\tilde{\mathbf{V}}$  is an example of the application of Th. 1.

By expanding the terms in eq. (1), it can be written as

$$\left\{ \delta^{\tilde{\mathbf{S}}}(F_E[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}] + F_R[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}]) + F_V[\tilde{\mathbf{V}}, \tilde{\mathbf{T}}] \right\} \cdot |\tilde{\mathbf{T}}| = \delta^{\tilde{\mathbf{I}}}(p[\tilde{\mathbf{V}}, \tilde{\mathbf{I}}])$$

and, by further expansion, as:

$$\delta^{\tilde{\mathbf{S}}} \left[ \left( f_{\epsilon, \sigma} \left( \frac{\delta^{\mathbf{P}}(u[\mathbf{P}, \mathbf{I}])}{|\mathbf{L}|} \right) + f_{\nu, \sigma_r} \left( \frac{\delta^{\mathbf{P}}(v[\mathbf{P}, \mathbf{T}])}{|\mathbf{L}|} \right) \right) \cdot |\tilde{\mathbf{S}}| \right] + F_V[\tilde{\mathbf{V}}, \tilde{\mathbf{T}}] = \frac{\delta^{\tilde{\mathbf{I}}}(f_{v, p}(\frac{\delta^1(u[\mathbf{P}, \mathbf{I}])}{|\mathbf{T}|}))}{|\tilde{\mathbf{T}}|}. \quad (2)$$

Eq. (2) can be conveniently represented by its *Tonti diagram* [21] (fig. 3). If we consider, for a while, space and time invariant constitutive relations, a uniform material, and a complex uniform in space and time, Eq. (2) becomes

$$-\frac{|\tilde{\mathbf{S}}|}{|\mathbf{L}|} \delta^{\tilde{\mathbf{S}}} \delta^{\mathbf{P}} \left( E \cdot u[\mathbf{P}, \mathbf{I}] + \mu_R \frac{\delta^1(u[\mathbf{P}, \mathbf{I}])}{|\mathbf{T}|} \right) + F_V[\tilde{\mathbf{V}}, \tilde{\mathbf{T}}] = \frac{m}{|\mathbf{T}|^2} \delta^{\tilde{\mathbf{I}}} \delta^1(u[\mathbf{P}, \mathbf{I}]) \quad (3)$$

By recalling that the operator  $\delta^{\tilde{\mathbf{S}}}$  is analogous to the differential *div*,  $\delta^{\mathbf{P}}$  is analogous to *grad* and  $\delta^{\tilde{\mathbf{I}}}$  and  $\delta^1$  are analogous to  $\partial/\partial t$ , eq. (3) can be seen as the discrete analogous, modulus a different sign convention, of a damped hyperbolic partial differential equation:

$$\nabla^2(E \cdot u + \mu_R \frac{\partial u}{\partial t}) + f_V = m \frac{\partial^2 u}{\partial t^2}.$$



$$\begin{cases} \hat{x}(n+1) = A\hat{x}(n) + f(\hat{x}(n), u(n)) \\ \quad \quad \quad + L(y(n) - \hat{y}(n)) \\ \hat{y}(n) = C\hat{x}(n) \end{cases} \quad (6)$$

where the output error gain matrix  $L \in \mathbb{R}^{2N_0 \times m}$  is chosen such that  $A^0 \triangleq A - LC$  is stable. A fault will be detected when the absolute value of at least one component  $\epsilon_{y,k}(n)$  of the residual  $\epsilon_y(n) \triangleq y(n) - \hat{y}(n)$  crosses the corresponding robust, adaptive threshold [33]

$$\bar{\epsilon}_{y,k}(n) \triangleq \sum_{h=0}^{n-1} \alpha_k \delta_k^{n-1-h} [\bar{\Delta}f(h) + \bar{\eta}_x(h)] + \alpha_k \delta_k^n \bar{\epsilon}_x(0) + \bar{\eta}_{y,k}(n), \quad (7)$$

where  $\alpha_i$  and  $\delta_i$  are such that  $\|C_k(A^0)^n\| \leq \alpha_k \delta_k^n \leq \|C_k\| \|A^0\|^n$ ,  $\alpha_k > 0$ ,  $0 < \delta_k \leq 1$ , with  $C_k$  being the  $k$ -th row of  $C$ . Furthermore,  $\bar{\epsilon}_x(0)$  is a known upper bound on  $\epsilon_x(0) \triangleq x(0) - \hat{x}(0)$ ,  $\bar{\Delta}f(n) \triangleq f(x(n), u(n)) - f(\hat{x}(n), u(n))$  is the error due to the computation of the function  $f$  with the estimate  $\hat{x}$  in lieu of the true value  $x$  as an argument, and  $\bar{\eta}_x$  and  $\bar{\eta}_y$  are known upper bounds on the modeling and measuring uncertainties.

*Remark 3:* A condition that must be met by system (5) is that  $(A, C)$  must be an observable pair [33]. This can be translated, by using the Popov-Belevitch-Hautus test [34], in the condition  $Cv \neq 0$  for each right eigenvector  $v \neq 0$  of  $A$ . It is interesting to note that, in the case of linear elasticity, this can be interpreted as the requirement that the sensors used to measure the output must not be all placed in a *node*<sup>7</sup> of the same eigen-mode, which is quite intuitive.

#### A. Modeling uncertainty and numerical accuracy

An important point to address, is how to determine non-over-conservative bounds on the state uncertainty  $\eta_x$ , which are needed for computing the fault detection threshold (7). We start by noticing that, in the simplifying hypothesis of constant geometric and material parameters and zero viscous coefficients, the nominal equation (3) can be written, for a generic node  $k$  and in terms of nodal displacements alone, as

$$\begin{aligned} u_k(n+1) &= 2u_k(n) - u_k(n-1) + \left(\frac{\tau}{L}c^*\right)^2 \\ &\quad \times (u_{k+1}(n) - 2u_k(n) + u_{k-1}(n)) \\ &\triangleq f_k^*(n) \end{aligned} \quad (8)$$

which admits as solutions travelling waves of the type

$$u_k^*(n) = e^{j\omega(n\tau + k\frac{L}{c^*})}$$

where  $\omega$  is the angular frequency,  $c^* \triangleq \sqrt{\frac{ESL}{m}}$  is the nominal phase speed,  $L$  and  $\tau$  are the cell complex space and time step,  $E$  and  $S$  are the Young modulus and the section of the material, and  $m$  the mass contained in each dual cell. An uncertainty in the material parameters will lead to the actual equation being

$$u_k(n+1) = f_k^*(n) + \eta_k(n), \quad (9)$$

<sup>7</sup>That is, a point in the physical domain where the mode shape is null.

where  $\eta_k(n) \triangleq \left(\frac{\tau}{L}\sqrt{c^2 - (c^*)^2}\right)^2 (u_{k+1}(n) - 2u_k(n) + u_{k-1}(n))$  is a model uncertainty term, and  $c$  is the phase speed computed with the actual physical parameters. By comparing (9) with (5), it follows that the state uncertainty is equal to  $\eta_x = [0_{N_0}^T \text{col}(\frac{\eta_k}{\tau}, k=1, \dots, N_0)^T]^T$ . In practice, the presence of the modeling uncertainty leads to an estimation error that is obtained by filtering through the fault detection estimator the following solution, which represents the *beating* between two waves with different phase speed

$$\check{u}_k(n) = e^{j\omega(n\tau + kL/c^*)} - e^{j\omega(n\tau + kL/c)}$$

and that we call, albeit in an imprecise way, the *uncertainty wave*. The fact that the effect of the uncertainty can be written in the form of a solution of the system allows to compute an upper bound on its magnitude by using the same kind of model used to simulate the nominal dynamics. This point will be used in the next section to compute on-line a non-over-conservative bound on the uncertainties.

Another source adding to the uncertainty  $\eta_k$ , apart from material or geometrical uncertainties, is due to numerical errors, which can be easily analyzed thanks to eq. (8). First of all, numerical stability can be investigated by relying on the Von Neumann method [17], which imposes a solution of the form

$$u_k(n) = \xi^n e^{jk\gamma L},$$

with  $\gamma \triangleq \frac{\omega}{c}$  being the wave number. By substituting this solution in (8), the stability condition  $|\xi| \leq 1$  is obtained with  $\kappa \leq 1$ , where  $\kappa \triangleq \frac{\tau c}{L}$  is the Courant number.

Numerical dissipation and dispersion errors<sup>8</sup> can be analyzed by imposing the solution  $u_k(n) = e^{j(\omega n\tau + \gamma kL)}$ , which substituted in (8) leads to  $\cos(\omega\tau) = (1 - \kappa^2) + \kappa^2 \cos(\gamma L)$ . This admits a real angular frequency  $\omega$  for every  $\kappa \leq 1$ , which means that the stability condition brings also a complete lack of dissipation errors, a result we did expect as the Verlet-Leapfrog scheme is symplectic [17]. On the other hand, the resulting dispersion relation is

$$c = \frac{\omega}{\gamma} = \frac{c^*}{\kappa\gamma L} \arccos(1 - \kappa^2 + \kappa^2 \cos(\gamma L)),$$

which reduces to the correct one,  $c = c^*$ , only for  $\kappa = 1$ .

This means that, in order to avoid numerical errors, the density of the discretization along time and space must be such that  $\kappa = 1$  or, put into another way, such that the length of the cells is equal to the space travelled by a physical wave during one time step. Should this be not possible, for instance for constraints on the available computation power,  $\kappa$  must be anyway chosen as to be smaller than one, and the extent of the uncertainties  $\eta_x$  due to numerical dispersion errors should be assessed thanks to eq. (9) and (5), so that an appropriate bound  $\bar{\eta}_x$  on the state uncertainty can be computed.

## V. SIMULATION RESULTS

In this section, we illustrate the effectiveness of the proposed approach by simulating a 1-D elastodynamic system. The modelled domain is 20 m long and is subdivided into 100

<sup>8</sup>Numerical dissipation leads to numerical waves dissipating more, or less energy than physical counterparts. Similarly, numerical dispersion leads to numerical waves propagating at the wrong speed, for a given wave number.

primal cells, and nominally has an elastic constant equal to  $E = 1.01 \text{ Pa}$ , a linear mass density  $\rho_m = 0.90 \text{ kg m}^{-1}$  and a viscous coefficient of  $\mu_R = 0.12 \text{ Pa s}$ . These coefficients are known with an uncertainty of  $0.02 \text{ Pa}$ ,  $0.15 \text{ kg m}^{-1}$  and  $0.2 \text{ Pa s}$ , respectively; in fact the actual values are assumed to be  $E^* = 1.00 \text{ Pa}$ ,  $\rho_m^* = 1.00 \text{ kg m}^{-1}$  and  $\mu_R^* = 0.10 \text{ Pa s}$ .

The system has a Neumann condition at the left boundary, where it is excited by a gaussian force impulse lasting  $10 \text{ s}$ , and a Dirichlet condition at the right boundary, where displacement and velocity are forced to be null by making all the elements of the last row of  $A$  equal to zero. The modeled fault is a reduction of the elastic constant in a small part of the domain, that at time  $T_0 = 50 \text{ s}$  will drop to  $E^f = 0.15 \text{ Pa}$  for each point between  $l_0 = 6 \text{ m}$  and  $l_1 = 8 \text{ m}$ .

Only a single point of measure is assumed to be available, at  $l_2 = 8.2 \text{ m}$ , that is slightly to the right of the portion that will fail, where displacements and velocities are measured. In order to implement the fault detection estimator, one Cell Method model is used to simulate the nominal dynamics, while other  $N_\eta = 8$  models are used to estimate on-line the time-varying bound on the uncertainty  $\eta_x$  by solving for the uncertainty wave.

In Fig. (4) it is possible to see a series of snapshots of the displacement and the velocity profile along the simulation domain, and their corresponding state residuals and thresholds<sup>9</sup>. What is important to notice, here, is that the presence of the measuring point (at cell number 41) has the effect of pinning down the threshold and, before the fault, the residual too. It should also be noted that, for points where there is no measure available, the thresholds grow quickly to very large values reflecting the lack of information.

In Fig. (5) and (6), instead, a time plot of the output residual and the threshold for the displacement and velocity at the single measure point is visible. It can be noticed that the fault is detected slightly after the fault time ( $50 \text{ s}$ ), thanks to the velocity residual crossing its threshold.

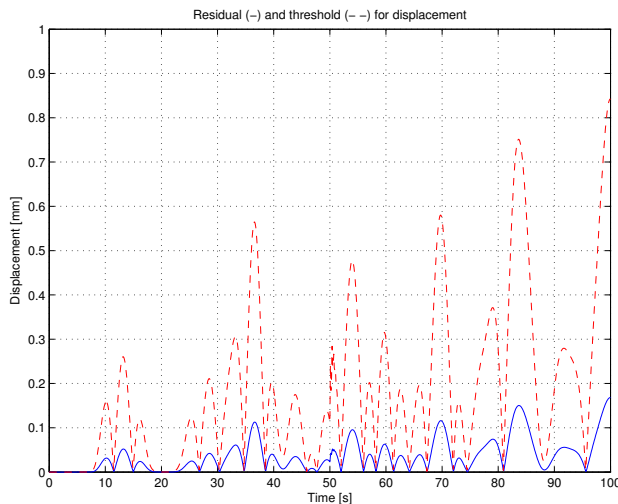


Fig. 5. Displacement output residual and threshold.

<sup>9</sup>They are shown here for analysis purposes only, but they are not used by the detection scheme as only the output residuals and thresholds can be computed.

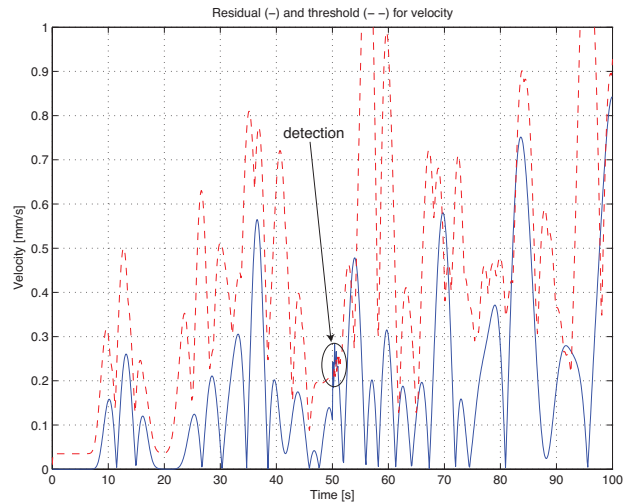


Fig. 6. Velocity output residual and threshold.

## VI. CONCLUDING REMARKS

In this paper, a model-based fault detection methodology for a class of uncertain distributed parameter systems has been proposed, based on a structure-preserving numerical method, the Cell Method, which relies on an Algebraic Formulation of physical equations. While the problem of fault diagnosis for such systems has been addressed previously in the literature by relying on different numerical approaches, to the best of the authors knowledge this is the first contribution that deals with both the following fundamentals issues in the application of model based fault diagnosis schemes to distributed parameter systems, namely: i) the availability of input/output measurements only and ii) the computation of time varying detection thresholds that are robust to measuring, modeling and numerical errors. To this end the AF approach was used for modeling the healthy system, and the effects of both the fault and the uncertainties. The validity of the approach is investigated thanks to the simulation of a one-dimensional elastodynamic problem, which is representative of many problems found in health monitoring of mechanical and civil structures.

Future research efforts will be devoted to the important issue of fault isolation and identification, and to the application to higher-dimensional distributed-parameters problems.

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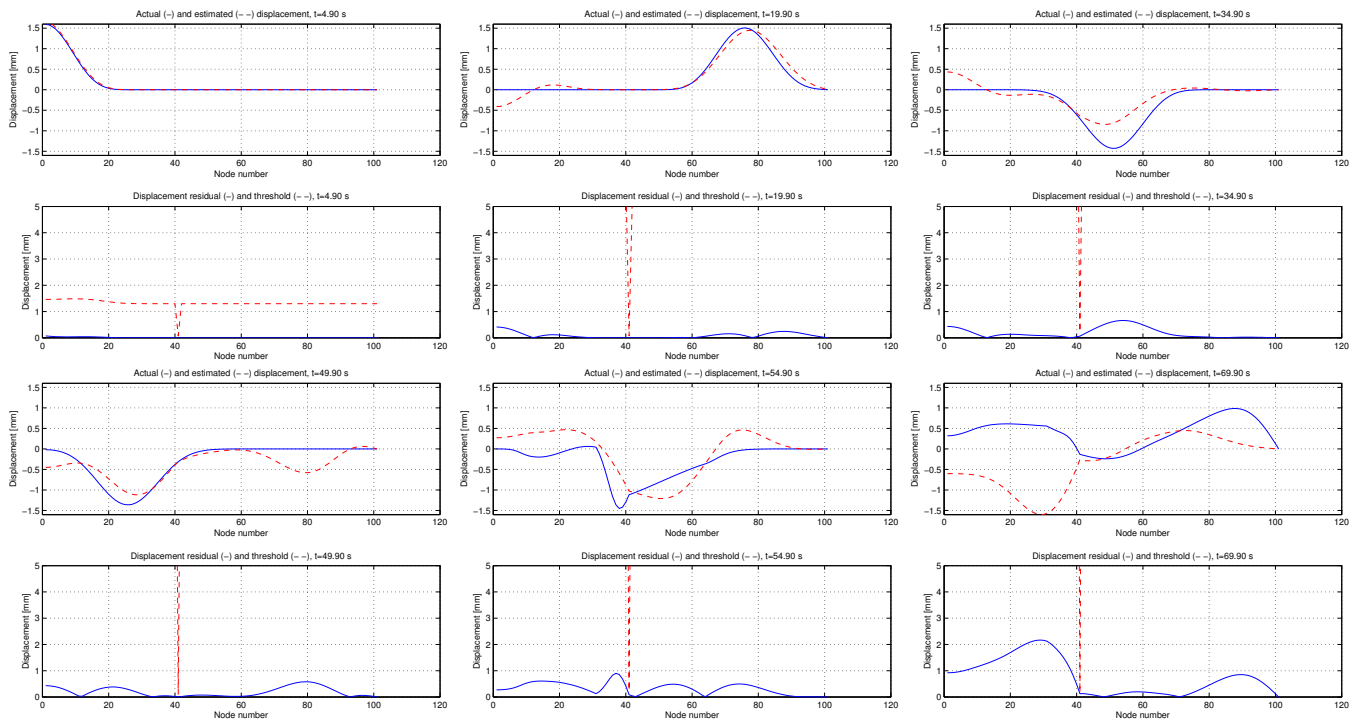


Fig. 4. Actual and estimated velocity profile, and corresponding state residual and threshold. The fault occurs at  $T_0 = 50$  s.

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