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Measures and LMIs for optimal control of piecewise-affine systems

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Abstract— **This paper considers the class of deterministic continuous-time optimal control problems (OCPs) with piecewise-affine (PWA) vector field, polynomial Lagrangian and semialgebraic input and state constraints. The OCP is first relaxed as an infinite-dimensional linear program (LP) over a space of occupation measures. This LP is then approached by an asymptotically converging hierarchy of linear matrix inequality (LMI) relaxations. The relaxed dual of the original LP returns a polynomial approximation of the value function that solves the Hamilton-Jacobi-Bellman (HJB) equation of the OCP. Based on this polynomial approximation, a suboptimal policy is developed to construct a state feedback in a sample-and-hold manner. The results show that the suboptimal policy succeeds in providing a suboptimal state feedback law that drives the system relatively close to the optimal trajectories and respects the given constraints.**

I. INTRODUCTION

Piecewise-affine (PWA) systems are a large modeling class for nonlinear systems. It can naturally arise from linear systems in the presence of state saturation or from simple hybrid systems with state-based switching where the continuous dynamics in each regime are linear or affine [9]. Many engineering systems fall in this category, like power electronics converters for example. In addition, common electrical circuits components as diodes and transistors are naturally modeled as piecewise-linear elements. PWA systems are also used to approximate large classes of nonlinear systems as in [10], and [3], and then used to pose the controller design problem of the original nonlinear system as a robust control problem of an uncertain nonlinear system as suggested in [19]. Problems of piecewise-affine systems are known to be challenging. The problems have a complex structure of regions stacked together in the state-space with each region containing an affine system. Any approach must identify the behavior in each region and then link them together to form a global picture of the dynamics. In [2], it has been shown that even for some simple PWA systems the problem of analysis or design can be either NP-hard or undecidable.

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The motivation behind this paper is the need of optimal control synthesis methods for continuous-time PWA systems with input and state constraints. Additionally, there is often a need to find suitable tools for the design of stabilizing feedback controllers under state and input constraints. Over the last few years, there were several attempts addressing the synthesis problem for continuous-time PWA systems. The techniques are based on analysis methods and use convex optimization to construct a state-based switched linear or affine controller. For example, in [7] a piecewise-linear state-feedback controller synthesis is done for piecewise-linear systems by solving convex optimization problem involving LMIs. The method is based on constructing a globally quadratic Lyapunov function such that the closed-loop system is stable. Similarly, in [17] a quadratic performance index is suggested to obtain lower and upper bounds for the optimal cost using any stabilizing controller. However, the optimal controller is not computed. The methods assume a piecewise-affine controller structure which can be shown not to be always optimal (see section V). In [18], the work done in [7] is extended to obtain dynamic output feedback stabilizing controllers for PWA systems. It formulates the search for a piecewise-quadratic Lyapunov function and a PWA controller as a nonconvex Bilinear Matrix Inequality (BMI), which is solved only locally by convex optimization methods. More recently in [11], a nonconvex BMI formulation is used to compute a state feedback control law. For constrained PWA systems in discrete-time with polyhedral partitions and constraints, [1] combines multi-parametric programming, dynamic programming and polyhedral manipulation to solve OCP for linear or quadratic performance indices[1]. The resulting solution when applied in receding horizon fashion guarantees stability of the closed-loop system.

To the best of the authors' knowledge there are no available guaranteed methods for synthesis of optimal controllers in continuous-time for PWA systems that consider general semi-algebraic state-space partitions, or that do not restrict the controller to be PWA, or do not require the performance index to be quadratic or piecewise-quadratic. The technique presented in this paper provides a systematic approach, inspired by [15], [12], to synthesize a suboptimal state feedback control law for continuous-time PWA systems based on a polynomial approximation of the value function. The OCP is first formulated as an infinite-dimensional LP over a space of occupation measures. The PWA structure of the dynamics and the state-space partition are then used to decompose the

[1]We are grateful to Michal Kvasnica for pointing out this reference.

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occupation measure of the trajectory into a combination of local occupation measures, one measure for each partition cell. Then, the LP formulation can be written in terms of the moments these measures (countably many variables). This allows for a numerical solution via a hierarchy of convex LMI relaxations with vanishing conservatism which can be solved using off-the-shelf Semidefinite Programming (SDP) solvers. The relaxations give a nondecreasing sequence of lower bounds on the optimal value. An important feature of the approach is that state constraints as well as any input constraints are very easy to handle. They are simply reflected into constraints on the supports of the occupation measures. It turns out that the dual formulation of the original infinite-dimensional LP problem on occupation measures can be written in terms of Sum-of-Squares (SOS) polynomials, that yields a polynomial subsolution of the Hamilton-Jacobi-Bellman (HJB) equation. This gives a good polynomial approximating value function along optimal trajectories that can be used to synthesize a suboptimal, yet admissible, control law. The idea exploits the structure of the HJB equation. The right-hand side of the HJB equation is iteratively minimized to construct a state feedback in a sample-and-hold manner with suboptimality guarantees.

II. The piecewise-affine optimal control problem

In this section, we first introduce the piecewise-affine continuous-time model, and then formulate the OCP.

A. Piecewise-affine systems

We consider exclusively continuous-time PWA systems. The term PWA is to be understood as PWA in the system state x. The state-space is assumed to be partitioned into a number of cells X_i such that the dynamics in each cell takes the form $\dot{x} = A_i x + a_i + B_i u, \quad \text{for} \quad x \in X_i, \quad i \in I$, where $I = \{1, \ldots, r\}$ is the set of cell indices, and the union of all cells is $X = \cup_{i \in I} X_i \subset \mathbb{R}^n$. The global dynamics of the system depends on both the cells and the corresponding local dynamics. The matrices A_i, a_i, and B_i are time independent. In general, the geometry of the partition X_i can be arbitrary. However, to arrive at useful results we assume the cells to be compact basic semi-algebraic sets (intersection of polynomial sublevel sets) with disjoint interiors. They are allowed to share boundaries as long as these boundaries have Lebesgue measure zero in X. There are many notions of solutions for PWA systems with different regularity assumptions on the vector field, [5],[9]. The concern here is to ensure the uniqueness of the trajectories. We assume that the PWA system is well-posed in the sense that it generates a unique trajectory for any given initial state.

B. Problem formulation

Optimal control problems of PWA systems are usually Lagrange problems where the state-space is partitioned into a finite number of cells. In addition to the dynamics, the cost functional can be also defined locally in each cell. The objective is to find optimal trajectories starting from an initial set and terminating at a target set that minimize the running cost and respect some input and state constraints. Consider the following general free-time PWA OCP with both state and control constraints

$$
\begin{aligned}
v^*(x_0) = \inf_{T,\, u} \quad & L_T(x_T) + \sum_{i=1}^{r} \int_0^T L_i(x(t), u(t)) \mathbb{I}_{X_i}(x(t)) dt \\
\text{s.t.} \quad & \dot{x} = f_i(x, u) = A_i x(t) + a_i + B_i u(t), \\
& i = 1, \ldots, r \quad t \in [0, T],\ x(t) \in X_i \\
& x(0) = x_0 \in X_0 \subset \mathbb{R}^n, \\
& x(T) = x_T \in X_T \subset \mathbb{R}^n, \\
& (x(t), u(t)) \in X \times U \subset \mathbb{R}^n \times \mathbb{R}^m.
\end{aligned} \tag{1}
$$

where the scalar mapping $\mathbb{I}_{X_i} : X_i \to \{0, 1\}$ is the indicator function defined as follows:

$$
\mathbb{I}_{X_i}(x(t)) := \begin{cases} 1, & \text{if} \quad x(t) \in X_i \\ 0, & \text{if} \quad x(t) \notin X_i. \end{cases}
$$

The infimum in (1) is sought over all admissible functions $u(\cdot)$ with free terminal time T. The dynamic programming approach of optimal control reduces the problem to the problem of solving the following system of Hamilton-Jacobi-Bellman (HJB) equations for the value function v^*:

$$
\begin{aligned}
& \inf_{u \in U} \{\nabla v^*(x) \cdot f_i(x, u) + L_i(x, u)\} = 0 \\
& \qquad \forall (x, u) \in X_i \times U, \quad \forall i = 1, \ldots, r
\end{aligned} \tag{2}
$$

with the terminal condition $v^*(x(T)) = L_T(x^*(T))$. In full generality, solving the HJB equations is very hard and the value function is not necessarily differentiable. Therefore, solutions must be interpreted in a generalized sense. To proceed, we adopt the following assumptions:

- The terminal time T is finite and the control functions $u(\cdot)$ are measurable.
- The PWA system is well-posed in the sense that it generates a unique trajectory from every initial state.
- The Lagrangians and the terminal cost are polynomial maps, namely $L_i \in \mathbb{R}[x, u]\ \forall i$ and $L_T \in \mathbb{R}[x]$.
- The cells X_i, the control set U, the sets X_0 and X_T are compact basic semi-algebraic sets defined as follows:

$$
\begin{aligned}
X_i \times U = \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \,|\, p_{i,k}(x, u) \geq 0, \\
\forall k = 1, \ldots, m_i\},\ i = 1, \ldots, r
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
X_0 = \{x \in \mathbb{R}^n \,|\, p_{0,k}(x) \geq 0, \forall k = 1, \ldots, m_0\}, \\
X_T = \{x \in \mathbb{R}^n \,|\, p_{T,k}(x) \geq 0, \forall k = 1, \ldots, m_T\}.
\end{aligned} \tag{4}
$$

- The cells X_i have disjoint interior and they are allowed to share boundaries as long as these boundaries have Lebesgue measure zero in X.

III. The moment approach

In this section, we formulate the nonlinear and nonconvex PWA OCP (1) into a convex infinite-dimensional LP over the state-action occupation measure, which is approached by an asymptotically converging hierarchy of LMI relaxations.

A. Occupation measures

To illustrate the measures formulation, first consider the uncontrolled autonomous dynamic system defined by the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ and the differential equation $\dot{x} = f(x), \quad x(0) = x_0$. We think of the initial state x_0 as a random variable in $\mathbb{R}^n$ modeled by a probability measure ξ_0 supported on the compact set X_0, i.e. a nonnegative measure ξ_0 such that $\xi_0(X_0) = 1$. Then at each time instant t, the state $x(t)$ can also be seen as a random variable ruled by a nonnegative probability measure ξ_t. Solving the ODE for the state trajectory $x(t)$ yields a family of trajectories starting in X_0 and ending at a final set X_T. The measure ξ_t of a set can then be thought of to be the ratio of the volume of trajectory points that lie inside that set at time t to the total volume of points at time t. The family of measures ξ_t can thus be thought of as a density of trajectory points and satisfies the following linear first-order continuity Partial Differential Equation (PDE) in the nonnegative probability measures space $\frac{\partial \xi_t}{\partial t} + \nabla \cdot (f \xi_t) = 0$. This equation is known as Liouville's equation or advection PDE, in which $\nabla \cdot (f\xi_t)$ denotes the divergence of measure $f\xi_t$ in the sense of distributions. It describes the linear transport of measures from the initial set to the terminal set.

The occupation measure of the solution over some subset $\mathcal{X}$ in the σ-algebra of X is simply defined as the time integration of ξ_t as follows

$$\xi(\mathcal{X}) := \int_0^T \xi_t(\mathcal{X}) dt = \int_0^T \mathbb{I}_{\mathcal{X}}(x(t)) dt, \tag{5}$$

where the last equality is valid when $x(t)$ is a unique solution. It is important to note that when the initial condition x_0 is deterministic, the occupation measure ξ is the time spent by the solution $x(t)$ in the subset $\mathcal{X}$. We see that the occupation measure can indicate when the solution is within a given subset.

B. The primal formulation

In the Borel σ-algebra (smallest σ-algebra that contains the open sets) let $\mathcal{M}(X)$ denote the space of signed Borel measures supported on a compact subset X of the Euclidean space, and let $\mathcal{C}(X)$ be the space of bounded continuous functions on X, equipped with the supremum norm. Then, the space $\mathcal{M}(X)$ is the dual space of $\mathcal{C}(X)$ with a duality bracket $\langle v, \mu \rangle = \int_X v d\mu, \quad \forall (v, \mu) \in \mathcal{C}(X) \times \mathcal{M}(X)$. Now consider the PWA OCP (1). We define the state-action local occupation measure (including sets on the space of control inputs) associated with the cell X_i to be $\mu_i(X_i \times U) = \int_0^T \mathbb{I}_{X_i \times U}(x(t), u(t)) dt$. The local occupation measure $\mu_i(X_i \times U)$ encodes the trajectories in the sense that it measures the total time spent by the trajectories $(x(t), u(t))$ in the admissible set $X_i \times U$. Furthermore, define the global state-action occupation measure for the trajectory $(x(t), u(t))$ to be $\mu(X \times U) = \sum_{i=1}^r \mu_i(X_i \times U)$. The initial and terminal occupation measures are defined as probability measures supported on X_0 and X_T, respectively.

The objective function of problem (1) can be rewritten in terms of these measures to get the linear cost $J(x_0, u(t)) = \sum_{i=1}^r \langle L_i, \mu_i \rangle + \langle L_T, \mu_T \rangle$. If the Lagrangian is the same for all the cells, say L, it can be written in terms of the global state-action occupation measure as $J(x_0, u(t)) = \langle L, \mu \rangle + \langle L_T, \mu_T \rangle$. The next step is to write the measure transport equation that encodes the PWA dynamics in the measure space. Define a compactly supported global test function $v \in \mathcal{C}^1(X)$. For $i = 1, \ldots, r$ we define a linear map $F_i : \mathcal{C}^1(X_i) \to \mathcal{C}(X_i \times U)$, $F_i(v) := -\dot{v} = -\nabla v \cdot (A_i x + a_i + B_i u)$. Integration gives $\int_0^T dv = -\sum_{i=1}^r \langle F_i(v), \mu_i \rangle = \langle v, \mu_T \rangle - \langle v, \mu_0 \rangle$. Define the $(r+1)$-tuple, $\nu := (\mu_1, \ldots, \mu_r, \mu_T)$. The OCP (1) is equivalent to the following infinite-dimensional LP:

$$\begin{aligned} p^* = \quad & \inf_{\nu} \quad \sum_{i=1}^r \langle L_i, \mu_i \rangle + \langle L_T, \mu_T \rangle \\ & \text{s.t.} \quad \sum_{i=1}^r \langle F_i(v), \mu_i \rangle + \langle v, \mu_T \rangle \quad = \langle v, \mu_0 \rangle \\ & \qquad\qquad \forall v \in \mathcal{C}^1(X). \end{aligned} \tag{6}$$

Define the linear map $\mathcal{L} : \mathcal{C}^1(X) \to \prod_{i=1}^r \mathcal{C}(X_i) \times \mathcal{C}(X_T)$, as $\mathcal{L}(v) = (F_1(v), \ldots, F_r(v), v)$. The constraint in (6) is then written as $\langle (F_1(v), \ldots, F_r(v), v), \nu \rangle = \langle \mathcal{L}(v), \nu \rangle = \langle v, \mathcal{L}^*(\nu) \rangle = \langle v, \mu_0 \rangle, \quad \forall v \in \mathcal{C}^1(X)$. This defines the adjoint map $\mathcal{L}^* : \prod_{i=1}^r \mathcal{M}(X_i) \times \mathcal{M}(X_T) \to \mathcal{M}^*(X)$. The measure transport equation [6] is then given by [6] $\mathcal{L}^*(\nu) = \mu_0 = \sum_{i=1}^r \nabla \cdot (f_i \mu_i) + \mu_T$. Finally, define the tuple $c := (L_1, \ldots, L_r, L_T)$, and associate (1) to the following infinite-dimensional LP

$$\begin{aligned} p^* = \quad & \inf_{\nu} \quad \langle c, \nu \rangle \\ & \text{s.t.} \quad \mathcal{L}^*(\nu) = \mu_0, \quad \nu \succeq 0. \end{aligned} \tag{7}$$

This is the primal formulation of the OCP in terms of occupation measures of the trajectory $(x(t), u(t))$. The PWA OCP (1) is reformulated as an infinite-dimensional LP.

C. Moments and LMI relaxations

The moments of the occupation measure μ are defined by integration of monomials with respect to μ . The α-th moment of μ over X is given by $y_\alpha = \int_X x^\alpha d\mu$, $\forall \alpha \in \mathbb{N}^n$ where $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$. Noting that $d\mu(x) = \mu(dx)$ and using the definition (5) of μ, moments can be rewritten as $y_\alpha = \int_0^T [x(t)]^\alpha dt$, $\forall \alpha \in \mathbb{N}^n$ where $x(t)$ denotes the solution of the ODE starting at x_0. Therefore, if we can find the moments by handling the representation conditions, solving the moments gives the solution of the ODE because the infinite (but countable) number of moments uniquely characterize a measure (on a compact set).

Note that LPs (6) and (7) are equivalent. To proceed numerically we restrict the continuously differentiable functions to be polynomial functions of the state. In other words, we consider $v \in \mathbb{R}[x] \subset \mathcal{C}^1(X)$. By this restriction, we obtain an instance of the generalized moment problem (GMP), i.e. an infinite-dimensional linear program over moments sequences corresponding to the occupation measures. It turns out [14, Ch3] that if the supports of the measures are compact basic semi-algebraic sets, the GMP can be approached using an asymptotically converging hierarchy of LMI relaxations. To write the semidefinite relaxation, let $y_i = (y_{i_\alpha})$, $\alpha \in \mathbb{N}^{n+m}$ be the moments sequence corresponding to the local occupation measure μ_i, $i = 1, \dots, r$, let $y_0 = (y_{0_\beta})$ and $y_T = (y_{T_\beta})$ with $\beta \in \mathbb{N}^n$ be the moment sequences corresponding to μ_0 and μ_T respectively. Given any infinite sequence $y = (y_\alpha)$ of real numbers with $\alpha \in \mathbb{N}^n$, define the linear functional $\ell : \mathbb{R}[x] \to \mathbb{R}$ as follows $p(x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha \;\; \mapsto \;\; \ell_y(p) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha y_\alpha$. In each LMI relaxation we truncate the infinite moment sequence to a finite number of moments. The LMI relaxation of order d,of the GMP instance (7) can be formulated by taking test functions $v = x^\alpha$ with $\alpha \in \mathbb{N}^n$, such that deg $v = 2d$, as

$$
\begin{aligned}
p_d^* = \inf_{y_1,\dots,y_r,y_T} \quad & \sum_{i=1}^{r} \ell_{y_i}(L_i) + \ell_{y_T}(L_T) \\
\text{s.t.} \quad & \sum_{i=1}^{r} \ell_{y_i}(F_i(v)) + \ell_{y_T}(v) = \ell_{y_0}(v), \\
& M_d(y_i) \succeq 0, \forall i \\
& M_d(p_{i,k}\, y_i) \succeq 0, \forall i, \forall k = 1, \dots, m_g, \\
& M_d(y_T) \succeq 0, \\
& M_d(p_{T,k}\, y_T) \succeq 0, \forall k = 1, \dots, m_T.
\end{aligned} \tag{8}
$$

The minimum relaxation order has to allow the enumeration of all the moments appearing in the objective function and the linear equality constraint. The matrices $M_d(y_i)$ and $M_d(y_T)$ are called moment matrices of the local occupation measure and terminal probability measures, respectively. Each moment matrix is defined to be a square matrix of dimension $\binom{d+n}{n}$ filled with the first $2d$ moments corresponding to the representing measure. They are linear in the moments. The matrices $M_d(p_{i,k}\, y_i)$ and $M_d(p_{T,k}\, y_T)$ are linear in the moments and are called localizing matrices. The equality constraint represents the PWA dynamics, and the LMIs force the moments to represent the measures on on the given support sets defined in (3),(4) (see [14] for details).

D. The dual formulation

The duality between finite measures and compactly supported bounded continuous functions is captured by convex analysis. The dual of the LP (7) is formulated over the space of positive bounded continuously differentiable functions as

$$
\begin{aligned}
d^* = \sup_{v \in \mathcal{C}^1(X)} \quad & \langle v, \mu_0 \rangle \\
\text{s.t.} \quad & z = c - \mathcal{L}(v), \; z \geq 0
\end{aligned} \tag{9}
$$

where z is a continuous functions vector, and more explicitly

$$
\begin{aligned}
d^* = \sup_{v \in \mathcal{C}^1(X)} \quad & \int_{X_0} v d\mu_0 \\
\text{s.t.} \quad & \nabla v(x) \cdot f_i + L_i(x,u) \geq 0, \\
& \qquad \forall (x,u) \in X_i \times U, \forall i = 1, \dots, r \\
& L_T - v(x_T) \geq 0, \forall x \in X_T.
\end{aligned} \tag{10}
$$

By conic complementarity, along the optimal trajectory (x^*, u^*), it holds $\langle z^*, \nu^* \rangle = 0$. Therefore, for the optimal dual function v^*, the following holds:

$$
\begin{aligned}
\nabla v^*(x^*) \cdot f_i + L_i(x^*, u^*) = 0, \\
\forall (x^*, u^*) \in X_i \times U, \; \forall i
\end{aligned} \tag{11}
$$

$$
\text{and in addition, } v^*(x^*(T)) = L_T(x^*(T)). \tag{12}
$$

This is an important result. It shows the following:

1) We can identify what we have in (11) to be the HJB PDE of the PWA OCP satisfied along optimal trajectories, with the terminal conditions given by (12).
2) The optimal dual function $v^*(x)$ is equivalent to the value function of the optimal control problem, hence the notation. The maximizer function $v^*(x)$ of the dual LP in equation (9) solves, globally, the HJB equation of the PWA OCP along optimal trajectories.

The dual convex relaxation, dual of LMI (8), is formulated over positive polynomials. Putinar's Positivstellensatz [16] is used to enforce positiveness. The unknown dual variables are the coefficients of the polynomial v and several SOS polynomials that deal with the polynomial positivity conditions of the constraints:

$$
\begin{aligned}
d_d^* = \sup_{v_d, s} \quad & \int_{X_0} v_d d\mu_0 \\
\text{s.t.} \quad & L_i - F_i(v_d) = s_{i,0} + \sum_{k=1}^{m_i} p_{i,k}\, s_{i,k} \\
& \forall (x,u) \in X_i \times U, \quad \forall i = 1, \dots, r, \\
& L_T - v_d(x) = s_{T,0} + \sum_{k=1}^{m_T} p_{T,k}\, s_{T,k}, \forall x \in X_T.
\end{aligned} \tag{13}
$$

in which the degree of v_d is d. The polynomials $s_{i,0}$, $s_{i,k}$, $s_{T,0}$ and $s_{T,k}$ are positive. The polynomials $p_{i,k}$, $p_{T,k}$ define $X_i \times U$ and X_T respectively, see equ. (3)–(4).

Using weak-star compactness arguments similar to those in the proof of [13, Theorem 2], we can show that $p^* = d^*$. Moreover, the equality $p^* = v^*(x_0)$ holds if we allow relaxed controls (enlarging the set of admissible control functions to probability measures) and chattering phenomena in OCP (1), see e.g. the discussion in [13, Section 3.2] and references therein. Note that in [15], the identity $p^* = d^* = v^*(x_0)$ was shown under stronger convexity assumptions.

IV. Suboptimal control synthesis

Assume that the analytical value function $v^*(x)$ of a general optimal control problem is available by solving the HJB PDE. The optimal feedback $k^*(x(t))$ can then be selected such that it generates an optimal control trajectory $u^*(t)$ that satisfies the optimal necessary and sufficient conditions (11). The resulting optimal feedback $k^*(\cdot)$ generates the optimal control trajectory starting from any initial value x_0. Therefore we get the solution of the OCP as a feedback strategy, namely, if the system is at state x, the control is adjusted to $k^*(x)$. This approach has the difficulty that even if the value function is smooth there would be in general no continuous optimal state feedback that satisfies the optimality conditions for every state. More recently it was shown that asymptotic controllability of a system is necessary and sufficient for state feedback stabilization without insisting on the continuity of the feedback. The synthesis of such discontinuous feedbacks was described in [4], together with a definition of a solution concept for an ODE with discontinuous dynamics, namely the sample-and-hold implementation. It turns out that this concept is very convenient for PWA systems. The first paper to deal with sampled-date PWA systems in the form used here can be found in reference [20]. The suboptimal trajectory of the control system is defined by a partition, call it π, of the time set $[0, T_\pi]$ as follows: define $\pi = \{t_j\}_{0 \leq j \leq p}$ with a given diameter $d(\pi) := \sup_{0 \leq j \leq p-1}(t_j, t_{j+1})$ such that $0 = t_0 < t_1 \cdots < t_p = T_\pi$. The sequence t_j for $1 \leq j \leq p$ depends on the evolution of the trajectory and the diameter of the partition $d(\pi)$. Starting at an initial point t_j, the suboptimal trajectory is the classical solution of the Lipschitz ODE $\dot{x}(t) = A_i x(t) + a + B_i k^*(x_j)$, $x(t) \in X_i$ and $x_j = x(t_j)$, for $t_j \leq t \leq t_{j+1}$ such that t_{j+1} depends on the evolution of the trajectory.

The generated suboptimal trajectory corresponds to a piecewise constant open-loop control (point-wise feedback). The suboptimal trajectory converges to the optimal trajectory when $d(\pi) \to 0$. In this case the generated trajectory corresponds to a fixed optimal feedback. This can be shown as follows: take a partition π with $d(\pi) = \delta$, we can use the mean value theorem to write $v^*(x_{j+1}) - v^*(x_j) = [\nabla v^*(x(\tau)) \cdot f_i(x(\tau), u_j^*(x_j))](t_{j+1} - t_j)$, where $\delta_j = (t_{j+1} - t_j) <= \delta$, and $\tau \in [0, \delta_j]$. Then we note that when $\delta_j \to 0$, $\dot{v}^*(x_j) \to -L_i(x_j, u_j^*(x_j))$, and the algorithm converges to the optimal trajectories. It is then clear that for every initial condition x_0, some prescribed $g > 0, \epsilon > 0$; there exist some $\delta > 0$, $T_\pi > 0$ such that if $d(\pi) < \delta$ the generated trajectories starting at x_0 satisfy $J(x_0, u(t)) - v^*(x_0) < g$ (suboptimality gap), $\|(x(T_\pi) - x_T)\| \leq \epsilon$ (tolerance). In the special case of having $\delta = 0$, we have no gap due to the algorithm and $T_\pi = T$. Assuming that a polynomial approximation of the value function is available, the proposed suboptimal feedback policy $[0, T) \times \mathbb{R}^n \to U$ is constructed using closed-loop sampling and Algorithm 1.

Algorithm 1 Algorithm for Suboptimal Synthesis

given: $d(\pi)$, ϵ, $x_0 \in X_i$, x_T, and poly. approx. of $v^*(\cdot)$
initialization: $t = t_0 = 0$, $x(0) = x_0$, $j = 0$.

while $\|x(t) - x_T\| > \epsilon$ **do**
solve the static polynomial optimization problem

$$u_j^*(x_j) = \underset{u \in U}{\operatorname{argmin}} \; \nabla v(x_j) \cdot f_i(x_j, u) + L_i(x_j, u)$$

solve $\dot{x}(t) = A_i x(t) + a_i + B_i u_j^*(x_j)$, starting at t_j up to $t = t_j + d(\pi)$ or $x(t) \notin X_i$.

set $j \leftarrow j + 1$ {next interval}
set $t_j \leftarrow t$, and $x_j \leftarrow x(t)$. {initials for next interval}
determine new region X_k for $x(t)$ and set $i \leftarrow k$
end while

$p \leftarrow j$ {number of intervals in π}
return $\pi = \{t_j\}_{0 \leq j \leq p}$, $x(t)$, and $u(t)$ with $0 < t \leq T_\pi$.

V. Numerical Example

Consider the following PWA OCP:

$$\begin{aligned} v^*(x_0) = \quad & \inf_{T,u} \quad \int_0^T (2(x-1)^2 + u^2) dt \\ & \text{s.t.} \quad \dot{x} = \begin{cases} f_1 = -x + 1 + u, & x \in X_1 \\ f_2 = x + 1 + u, & x \in X_2 \end{cases} \\ & \qquad x(0) = -1, \quad x(T) = +1 \\ & \qquad U = \{u \in \mathbb{R} \,|\, u^2 \leq 25\} \\ & \qquad X = \{x \in \mathbb{R} \,|\, x^2 \leq 1\}. \end{aligned} \tag{14}$$

The state space is partitioned into two regions $X = X_1 \cup X_2$ where, $X_1 = \{x \in X \,|\, 0 \leq x \leq 1\}$, and $X_2 = \{x \in X \,|\, -1 \leq x \leq 0\}$. The control Hamiltonian

$$H(x, u, \nabla v^*) = \begin{cases} \nabla v^* f_1 + (2(x-1)^2 + u^2), x \in X_1 \\ \nabla v^* f_2 + (2(x-1)^2 + u^2), x \in X_2 \end{cases} \tag{15}$$

is used to write the HJB equation. Since the terminal time T is subject to optimization, the control Hamiltonian vanishes along the optimal trajectory. This gives the HJB PDE

$$\inf_{u \in \mathbb{R}} H(x(t), u(t), \nabla v^*(x(t))) = 0, \; \forall x(t), \; t \in [0, T]. \tag{16}$$

Ignoring the state and control constraints, an analytical optimal solution can be obtained by solving the HJB equations corresponding to each cell. This results in a state feedback

$$k^*(x) = \begin{cases} (1 - \sqrt{3})(x - 1) & \text{if} \quad x \geq 0 \\ -x - 1 + \sqrt{2(x-1)^2 + (x+1)^2} & \text{if} \quad x \leq 0 \end{cases} \tag{17}$$

that satisfies the HJB PDE (16) for all $t \in [0, T]$ and x. The optimal control trajectory $u^*(t) = k^*(x^*(t))$, $\forall t \in [0, T]$ is obtained when the optimal state feedback is applied starting at the given initial condition. The partial derivative of the value function with respect to the state is a viscosity solution of the HJB PDE (16), see e.g. [6].

The initial and terminal occupation measures are Dirac measures, since the initial and final states are known. The global occupation measure $\mu \in \mathcal{M}(X \times U)$ is a combination of two local occupation measures $\mu = \mu_1 + \mu_2$ such that the support of μ_1 is $\{(x, u) \,|\, x \in X_1, u \in U\}$ and the support of μ_2 is $\{(x, u) \,|\, x \in X_2, u \in U\}$. The OCP is formulated as an infinite-dimensional LP in measure space

$$\begin{aligned} p^* = & \inf_{\mu_1, \mu_2} \langle L, \mu_1 \rangle + \langle L, \mu_2 \rangle \\ & \text{s.t.} \langle \nabla v \cdot f_1, \mu_1 \rangle + \langle \nabla v \cdot f_2 d\mu_2 \rangle = \langle v, \mu_0 \rangle - \langle v, d\mu_T \rangle, \forall v \end{aligned} \tag{18}$$

where μ_0 and μ_T are the initial and final measures, supported on $X_T = \{+1\}$, $X_0 = \{-1\}$, respectively, and $L = 2(x-1)^2 + u^2$ is the Lagrangian. The functions v are functions of the state x and belong to the space of continuously differentiable functions. The Matlab toolbox GloptiPoly [8] is used to formulate this infinite-dimensional LP on measures (primal) as a truncated GMP instant as in (8). The numerical problem is then solved, and the optimal dual variables are obtained. Figure 1 shows the obtained

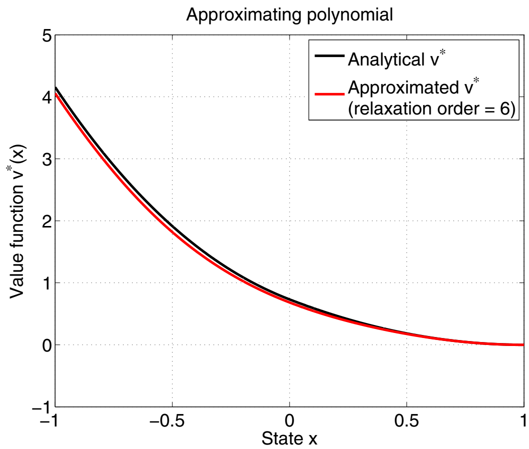

Fig. 1. PWA system. Approximating value function for $d = 6$.

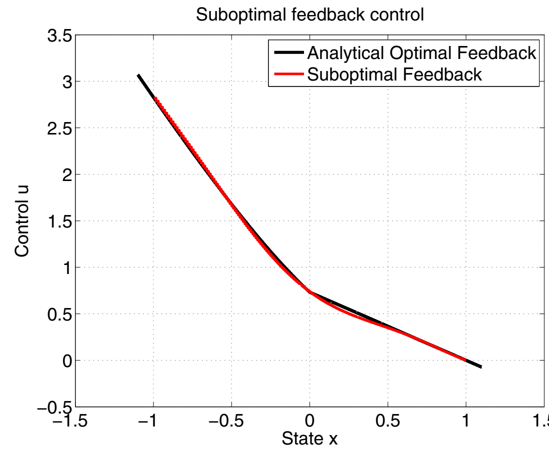

Fig. 2. PWA system. Suboptimal feedback for $d = 6$.

approximation of the value function with LMI relaxation order $d = 6$. The obtained approximation is a polynomial of the state x with degree equal to $2d$. Based on this approximation, we employ algorithm 1 from section IV. The resulting suboptimal feedback is shown in figure 2 in comparison with the analytical optimal feedback. The algorithm gives a suboptimal feedback close to the optimal.

VI. Conclusions

The focus of this paper is the synthesis of suboptimal state feedback controllers for continuous-time optimal control problems (OCP) with piecewise-affine (PWA) dynamics and piecewise polynomial cost functions. Both state constraints and input constraints are considered in a very convenient way and they do not pose additional complexity. The problem is formulated as an abstract infinite-dimensional linear program over a space of occupation measures which is then solved via a converging hierarchy of LMI problems. By restricting the dual variables in the dual of the original infinite-dimensional program to be monomials, we obtain a polynomial representation of the value function of the OCP in terms of upper envelope of subsolutions to a system of HJB equations corresponding to the OCP. By fixing the degree of the monomials, the same dual program can be relaxed and written, using Putinar's Positivstellensatz, as a polynomial sum-of-squares (SOS) program, which can be transformed and solved as an LMI problem. As soon as the polynomial approximation of the value function is available, one can systematically generate a suboptimal, yet admissible, feedback control. The suboptimal control strategy is based on a closed-loop sampling implementation which is very convenient for PWA systems. The method generates an approximate control signal which is piecewise constant, and near optimal trajectories that respect the constraints.

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