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Stabilization of nonlinear discrete-time dynamics in strict-feedforward form

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Abstract— **The paper deals with stabilization of nonlinear discrete-time dynamics in strict-feedforward form. The proposed design procedure is based on the concept of average passivity: a concept recently introduced by the authors to overcome a well known pathology which occurs in defining passivity in discrete time. For such dynamics it is possible to set an iterative procedure which mimics the continuous-time design approach alternating, at each step, coordinates change and average passivity based control design. The complete controller is derived at the last step and a Lyapunov function for the whole control system is described. An example concludes the paper.**

I. Introduction

The forwarding technique has been introduced in continuous time as a Lyapunov stabilizing strategy for nonlinear input-affine dynamics admitting the strict feedforward form

$$\begin{aligned}\dot{x}_1(t) &= f_1(x_2(t),...,x_n(t))+g_1(x_2(t),...,x_n(t))u(t)\\ &\cdots\\ \dot{x}_{n-1}(t) &= f_{n-1}(x_n(t))+g_{n-1}(x_n(t))u(t)\\ \dot{x}_n(t) &= g_n u(t).\end{aligned}$$

A central role is played by the concept of passivity; as a matter of facts the whole design procedure can be interpreted as the iterative computation of coordinates changes and passivity based controllers. One begins with the stabilization of the last equation $\dot{x}_n(t)=u(t)$ under feedback $u_n(x_n(t))$. Then, one looks for a coordinates change $z=\Phi(x)$, setting $z_n=x_n$ and $z_{n-1}=x_{n-1}+\Phi_{n-1}(x_n)$, in such a way that $\dot{z}_{n-1}(t)$, under the action of $u_n(t)$, be drift free. At this stage, a stabilizing passivity based controller $u_{n-1}=u_n+v_{n-1}$ is designed on the two-dimensional dynamics $(\dot{z}_{n-1},\dot{z}_n)'$. The procedure is iterated and ends with the design of the complete controller $u(t)=u_n+\sum_{i=1}^{n-1}v_i(t)$ which ensures global asymptotic and local exponential stabilization properties. The forwarding technique has been developed in several text books [18] and extended to many different contexts (see [4], [5], [6], [9] and the references therein) in the continuous-time context.

The extension of these results to the discrete-time context is far from being trivial. As well known, a first difficulty to face in discrete time depends on the definition itself of passivity. Passivity concepts and nonlinear stabilization in discrete time have been addressed in several papers in the last twenty years under specific restrictions on the nonlinearities [1], [7], [8], [2] or for sampled-data dynamics in an approximated context [16], [17], [12].

A first elegant solution to stabilize discrete-time dynamics which exhibit feedforward like forms has been given in [10]. In the present work we propose a solution to global asymptotic stabilization of discrete-time nonlinear dynamics in strict feedforward form which mimics the iterative continuous-time procedure. This is possible making use of the representation of discrete-time dynamics as coupled difference and differential equations [11] and thanks to the notion of average passivity introduced by the authors in [13]. The controller is characterized as the solution of a set of algebraic equations. Even if its explicit computation can be a difficult task, an ad hoc procedure can be worked out with reference to special structures of the equations, as polynomial ones, or performing approximations in the model and/or in the design. Sampled-data dynamics, not discussed in this work, represent an interesting case which deserves ad hoc investigations. Following [12], iterative algorithms can be set in such a case to compute the expansions in powers of the sampling interval of the sampled-data solutions.

We consider in this paper a single-input strict-feedforward discrete-time dynamics on R^n exhibiting the following lower triangular structure

$$\begin{aligned}x_{1k+1} &= x_{1k}+f_1(x_{2k},...,x_{nk},u_k)\\ x_{2k+1} &= x_{2k}+f_2(x_{3k},...,x_{nk},u_k)\\ &\cdots\\ x_{n-1k+1} &= x_{(n-1)k}+f_{n-1}(x_{nk},u_k)\\ x_{nk+1} &= x_{nk}+f_n(u_k)\end{aligned}\tag{1}$$

where the f_i's are R-valued functions of x and $u\in R$. Such dynamics have invertible drift term $x_i+f_i(x_{i+1},..,x_n,0)$ with Jacobian matrix equal to the identity and $x=0$ is an equilibrium; i.e. $f_i(0,0)=0$. Specific cases should include linearity in the control u or polynomial dynamics f_i's.

As in the continuous-time case, a first stabilizing controller u_n is computed for the last state component $x_{nk+1}=x_{nk}+f_n(u_k)$. Then, one looks for a coordinates change $z=\Phi(x)$ setting $z_n=x_n$ and $z_{n-1}=x_{n-1}+\Phi_{n-1}(x_n)$ in such a way that the dynamics of z_{n-1} under u_n be drift free, $z_{n-1k+1}=z_{n-1k}$. At this time, a stabilizing controller $u_{n-1}=u_n+v_{n-1}$ is designed on the two-dimensional dynamics $(z_{n-1},z_n)'$ through average passivity arguments. The procedure is iterated up to the last step to provide the complete controller $u=u_n+\sum_{i=1}^{n-1}v_i$ achieving global asymptotic stabilization. The

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main difficulty remains the control computation since each v_j is implicitly described at each step of the procedure as the solution of an algebraic equation. Numerical solutions can be proposed especially when considering polynomial cases or polynomial approximations of the involved dynamics.

The paper is organized as follows. The system class and its equivalent differential difference representation are described in section 2. Average dissipativity is recalled in section 2 as well as the related nonlinear stabilizing or damping controller. The stabilizing forwarding strategy is developed in section 3 for a dynamics in R^2; it is generalized in section 3 for a n-dimensional dynamics. An example illustrates the computational aspects.

II. The class of systems under study

We show in this section that lower triangular dynamics of the form (1) can be described by Differential Difference Representations - denoted in the sequel as (F_0,G) representations or DDR's - which exhibit a similar triangular structure. Such (F_0,G) representations, introduced in [15] and assumed at the basis of our investigation, have been demonstrated a powerful tool for overcoming the conceptual and somehow technical obstructions when developing a comparative analysis or extending results from continuous-time to discrete-time systems in the nonlinear context [14] .

A. The Differential Difference Representation

As in [15], the following couple of equations are used to describe a nonlinear discrete-time dynamics

$$x^+ = F_0(x) \quad (2)$$

$$\frac{\partial x^+(u)}{\partial u} = G(x^+(u),u) \quad \text{with} \quad x^+(0)=x^+ \quad (3)$$

where F_0 is a R^n-valued smooth map and $G(.,u)$ is a vector field on R^n, parameterized by $u \in U$ and assumed complete. When the initial condition $x^+(0)$ is fixed, the completeness of the parametrized vector field $G(.,u)$ ensures integrability of (3) so recovering the usual representation in the form of a map

$$x_{k+1} = x^+(u_k) = F(x_k,u_k). \quad (4)$$

For any pair $(x_k,u_k) \in R^n \times U$, denoting by $x^+(u_k)$ any curve in R^n parameterized by $u \in U$, one gets

$$x^+(u_k) = F(x_k,u_k) = x_k^+(0) + \int_0^{u_k} G(x^+(v),v)\mathrm{d}v \quad (5)$$

with initial condition $x_k^+(0) = F_0(x_k)$. It is a matter of computation to verify that a given smooth map $F(x,u)$ can be described by equations of the form (2)-(3) provided $F_0 := F(.,0)$ is invertible.

The expansion in u of $G(.,u)$ defines a family of *control vector fields* on R^n which characterize the geometric structure of the flow associated with the differential equation (3) (see [11] for further details).

Given any C^1-function of the state x, $H(.) : R^n \to R$, the variation with respect to u of H around $H(x^+(0))$ admits the integral form

$$H(x^+(u)) - H(x^+(0)) = \int_0^u \mathrm{L}_{G(.,v)} H(x^+(v))\mathrm{d}v \quad (6)$$

so recovering $H(x_k^+(u_k)) = H(x_{k+1})$ and $H(x_k^+(0)) = H(F_0(x_k))$.

Remark - We note that (6) rewritten as

$$H(x^+(u)) - H(x^+(0)) = u\int_0^1 \mathrm{L}_{G(.,sv)} H(x^+(sv))\mathrm{d}s$$

is nothing else than the vectorial extension of the formula

$$\kappa(a+b) - \kappa(a) = b\int_0^1 \kappa'(a+bs)\mathrm{d}s$$

with κ a scalar C^1 function, $\kappa(0)=0$ and κ' denoting its derivative.

Definition A nonlinear strict-feedforward (F_0,G) representation is described by

$$\begin{aligned} x_1^+ &= x_1 + F_{01}(x_2,\ldots,x_n) \\ &\ldots \\ x_{n-1}^+ &= x_{n-1} + F_{0n-1}(x_n) \\ x_n^+ &= x_n \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{\partial x_1^+(u)}{\partial u} &= G_1(x_2^+(u),\ldots,x_n^+(u),u) \\ &\ldots \\ \frac{\partial x_{n-1}^+(u)}{\partial u} &= G_{n-1}(x_n^+(u),u) \\ \frac{\partial x_n^+(u)}{\partial u} &= G_n(u) \end{aligned} \quad (8)$$

with $F_0(x) = [x_i + F_{0i}(x_{i+1},\ldots,x_n)]_{i=1,n}, F_0(0) = 0$ and $G(.,u) := [G_1(.,u),\ldots,G_{n-1}(.,u),G_n(u)]$.

Proposition 2.1: Any nonlinear strict feedforward dynamics of the form (1) admits a strict-feedforward (F_0,G) representation with

$$F_{0i}(x) = x_i + f_i(x_{i+1},\ldots,x_n,0) \quad \text{for} \quad i=1,n$$

$$G_1(x_2^+(u),\ldots,x_n^+(u),u) = \frac{\partial f_1(x_2,\ldots,x_n,u)}{\partial u}\Big|_{x=F^{-1}(x^+(u),u)}$$

$$\ldots$$

$$G_{n-1}(x_n^+(u),u) = \frac{\partial f_{n-1}(x_n,u)}{\partial u}\Big|_{x_n=x_n^+(u)-u}$$

$$G_n(u) = \frac{\partial f_n(u)}{\partial u}.$$

Proof: For, it is sufficient to recall that $G(x,u)$ verifying by definition the condition $G(F(x,u),u) = \frac{\partial F(x,u)}{\partial u}$ is uniquely defined as

$$G(x,u) := \frac{\partial F(.,u)}{\partial u}\Big|_{x=F^{-1}(x,u)}$$

provided invertibility of $F(x,u)$ as verified by (1). More in detail, the expressions of the $(G_i)'$ are easily deduced from

(1) since the inverse mapping $F^{-1}(x,u)$ can be iteratively computed as follows

$$\begin{aligned} x_n &= x_n^+(u) - G_n(u) \\ x_{n-1} &= x_{n-1}^+(u) - f_{n-1}(x_n^+(u) - G_n(u), u) \\ x_{n-2} &= x_{n-2}^+(u) - f_{n-2}(x_{n-1}, x_n, u)\big|_{x=F^{-1}(x^+(u),u)} \\ &\cdots \\ x_1 &= x_1^+(u) - f_1(x_2, ..., x_n, u)\big|_{x=F^{-1}(x^+(u),u)}. \end{aligned}$$

■

We will refer in the sequel to systems of the form (7,8), equivalently (1), which are stabilizable in first approximation. It is easily verified that, due to the assumed triangular structure, this implies that such dynamics are controllable in first approximation.

III. Average passivity based controllers - AvPBC

The notion of average passivity of nonlinear discrete-time systems has been recently introduced by the authors to cope with the lack of direct input output link when dealing with passivity based control techniques. Let us denote by $\Sigma_d(H)$ any nonlinear discrete-time system composed with the dynamics in its (F_0, G) representation as in (2-3) and a real valued output function H; moreover 0 is assumed to be an equilibrium, $F_0(0) = 0$, satisfying $H(0) = 0$. The following results are recalled from [13].

Definition Given $\Sigma_d(H)$ then, for any pair $(x,u) \in X \times U$, $H_{av}^+(x,u)$ denotes the *u-average output mapping* defined by

$$H_{av}^+(x,u) := \frac{1}{u}\int_0^u H(x^+(v))\mathrm{d}v \qquad (9)$$

with $H_{av}^+(x,0) := H(x^+(0)) = H(F_0(x))$ and $x^+(0) = F_0(x)$.

Definition $\Sigma_d(H)$ is *u-average passive* (resp. *u-average lossless*) if there exists a nonnegative C^0-function S such that $S(0) = 0$ and for all $(x_k, u_k) \in X \times U$

$$S(x_{k+1}) - S(x_k) \le H_{av}^+(x_k, u_k)u_k \qquad (10)$$

$$\Big(\text{resp. } S(x_{k+1}) - S(x_k) = H_{av}^+(x_k, u_k)u_k\Big). \qquad (11)$$

Definition - Zero State Detectabiliby - ZSD - $\Sigma_d(H)$ is said zero state detectable when no trajectory of the uncontrolled dynamics ($u = 0$) can stay in $\{x \in X \quad \text{s.t.} \quad y = H(x) = 0\}$ other than those converging asymptotically to zero.

According to these definitions, assuming S positive definite with $S(0) = 0$ then, any feedback law $u = \gamma(x)$ satisfying the strict inequality $H_{av}^+(x, \gamma(x))\gamma(x) < 0$, achieves global asymptotic stabilization - GAS -. Let us recall the Average Passivity Based Controllers - AvPBC - proposed in [13].

Theorem 3.1: Negative u-average output feedback - Let $\Sigma_d(H)$ be *u-average passive* with positive C^1-storage function V satisfying $V(0) = 0$ and assume $\Sigma_d(H)$ zero state detectable, then any feedback $u = \gamma(x)$ solving the algebraic equation

$$u + KH_{av}^+(x,u) = 0 \quad \text{with positive gain} \quad K > 0 \qquad (12)$$

achieves asymptotic stabilization of the equilibrium $x = 0$.

With reference to a given Lyapunov stable dynamics, the following damping controller can be designed [13].

Theorem 3.2: Damping L_GV controller - Given a Lyapunov stable discrete-time dynamics (2-3), let $V : X \to R$ be a C^1-Lyapunov function for it, and assume zero state detectability with respect to the "fictitious" output mapping $H(.,0) := \mathrm{L}_{G(.,0)}V$, then any feedback $u = \gamma(x)$ solving the algebraic equation (12) with $H_{av}^+(x,u)$ given in (9) ensures global asymptotic stabilization.

IV. Stabilization of nonlinear discrete-time dynamics in strict-feedforward form

With this in mind we describe in the sequel a stabilizing procedure for nonlinear discrete-time systems in strict-feedforward form (1). The design starts with the computation of an elementary feedback to stabilize the last equation and goes through the iterative application of a procedure which makes use at each step of a coordinates change for rendering an *augmented subsystem* stabilizable and computes the related *damping L_GV controller* according to Theorem 3.2. Due to the involved triangular structures, local exponential stability follows. In the sequel, starting from the (F_0, G) representation (7,8), we describe the two before mentioned tasks: the computation of the coordinates change on the augmented dynamics, the design of the *damping L_GV controller* to stabilize the augmented dynamics.

In the sequel the time index "$_k$" is omitted when obvious from the context.

A. The initial step: stabilization of the bottom subsystem

The procedure starts with the bottom subsystem in its DDR

$$\begin{aligned} x_n^+ &= x_n \\ \frac{\partial x_n^+(u)}{\partial u} &= G_n(u); \quad x_n^+(0) = x_n \end{aligned}$$

where $x_n^+(u) = x_n + G_n(u)$ and $G = G_n(u)\frac{\partial}{\partial x_n}$. Setting $V_n = \frac{1}{2}x_n^2$ and $G_n(u) = 1$ without loss of generality, one easily computes

$$H_n(x) := \mathrm{L}_{G_n}V_n(x) = \frac{\partial V_n}{\partial x_n} = x_n$$

with respect to which the dynamics is average lossless according to definition (11); i.e.

$$\Delta V_n(u_{nk}) \quad := \quad u_k H_{nav}^+(x_{nk}, u_{nk})$$

with

$$\begin{aligned} \Delta V_n(u_{nk}) &:= V_n(k+1) - V_n(k) = \frac{(x_{nk} + u_{nk})^2}{2} - \frac{x_{nk}^2}{2} \\ H_{nav}^+(x_{nk}, u_{nk}) &:= \frac{1}{u_{nk}}\int_0^{u_{nk}} H_n(x_{nk}^+(v))\mathrm{d}v = x_{nk} + \frac{u_{nk}}{2}. \end{aligned}$$

According to Theorem 3.2, the feedback $u := u_n$ satisfying $u_n + K_n H_{nav}^+(x, u_n) = 0$ with positive gain K_n achieves GAS due to the ZSD of the output $H_n(x) = x_n$. For $K_n = 1$, one sets

$$u_n = -H_{nav}^+(x_n, u_n)$$

to achieve in closed loop

$$\Delta V_n(u_n) \quad := \quad -(H^+_{nav}(x_n,u_n))^2 = -u_n^2 < 0$$

and GAS follows. In this elementary case, u_n is computed to solve $u_n + x_n + \frac{u_n}{2} = 0$ so getting

$$u_n = \gamma_n(x_n) = \frac{-2x_n}{3} \tag{13}$$

and $\Delta V_n(\gamma_n(x_n)) = \frac{-4x_n^2}{9}$. Accordingly one verifies local exponential stability.

B. The coordinates change and the augmented dynamics

Consider now the augmented two-dimensional dynamics

$$\begin{aligned} x^+_{n-1} &= x_{n-1} + F_{0n-1}(x_n) \\ x^+_n &= x_n \\ \frac{\partial x^+_{n-1}(u)}{\partial u} &= G_{n-1}(x^+_n(u),u) \\ \frac{\partial x^+_n(u)}{\partial u} &= 1 \end{aligned}$$

with

$$\begin{aligned} F_{0n-1}(x_n) &= f_{n-1}(x_n,0) \\ G_{n-1}(x^+_n(u),u) &= \frac{\partial f_{n-1}(x_n,u)}{\partial u}\Big|_{x_n = x^+_n(u)-u} \end{aligned}$$

At this step, one looks for a coordinates change $z = \Phi(x)$, of the form

$$z_{n-1} = x_{n-1} + \Phi_{n-1}(x_n), \quad z_n = x_n$$

in such a way to cancel the dynamics of z_{n-1} under the previously computed u_n; i.e. $z^+_{n-1}(u_n) = z_{n-1}$ or equivalently

$$x^+_{n-1}(u_n) + \Phi_{n-1}(x^+_n(u_n)) = x_{n-1} + \Phi_{n-1}(x_n)$$

with

$$\begin{aligned} x^+_{n-1}(u_n) &= x_{n-1} + f_{n-1}(x_n, \frac{-2x_n}{3}) \\ x^+_n(u_n) &= x_n - \frac{2x_n}{3} = \frac{x_n}{3}. \end{aligned}$$

Such an equality can be rewritten according to the DDR as

$$\begin{aligned} & x_{n-1} + f_{n-1}(x_n,u_n) + \Phi_{n-1}(x^+_n(0)) \\ + & \int_0^{u_n} \frac{\mathrm{d}\Phi_{n-1}(x^+_n(v))}{\mathrm{d}x_n}\mathrm{d}v = x_{n-1} + \Phi_{n-1}(x_n). \end{aligned}$$

After easy manipulations, because $x^+_n(0) = x_n$ and $u_n = \frac{-2x_n}{3}$, one gets the equality below to be satisfied by $\Phi_{n-1}(x_n)$

$$\int_0^{\frac{-2x_n}{3}} \frac{\mathrm{d}\Phi_{n-1}(x^+_n(v))}{\mathrm{d}x_n}\mathrm{d}v = -f_{n-1}(x_n, \frac{-2x_n}{3}). \tag{14}$$

Remark - In the polynomial case, a unique solution exists of the form $\Phi_{n-1}(x_n) = \sum_{i=1}^P \phi_i x_n^i$ with suitable coefficients ϕ_i and order P equal to the order in x_n of $f_{n-1}(x_n, \frac{-2x_n}{3})$. We note that in case of linear dynamics $f_{n-1}(x_n,u_n) = a_n x_n + b_n u$ so getting the explicit linear coordinates change $\Phi_{n-1}(x_n) = \frac{(a_n-2b_n)x_n}{2}$.

Under the so defined coordinates change, one transforms the *augmented dynamics* of dimension two into

$$\begin{aligned} z^+_{n-1} &= z_{n-1} \\ z^+_n &= \frac{z_n}{3} \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{\partial z^+_{n-1}(u)}{\partial u} &= \tilde{G}_{n-1}(z^+_n(u),u) \\ \frac{\partial z^+_n(u)}{\partial u} &= 1 \end{aligned} \tag{16}$$

where $\tilde{G}(.,u)$ denotes the transformed vector field $G(.,u)$ under $z = \Phi(x)$; i.e. (see [15])

$$\tilde{G}(.,u) := \frac{\partial \Phi(x)}{\partial x} G(.,u)\big|_{x=\Phi^{-1}(z)}. \tag{17}$$

Easy computations show that due to the triangular structure, one gets $\tilde{G}(.,u) = \tilde{G}_{n-1}(.,u)\frac{\partial}{\partial z_{n-1}} + \frac{\partial}{\partial z_n}$ with

$$\tilde{G}_{n-1}(z_n,u) \quad = \quad G_{n-1}(z_n,u) + \frac{\partial \Phi_{n-1}(z_n)}{\partial z_n}. \tag{18}$$

By construction, the augmented dynamics (15-16) is driftless in z_{n-1} and Lyapunov stable under u_n; i.e. setting

$$V_{n-1} = \frac{1}{2}z_{n-1}^2 + V_n \tag{19}$$

one verifies

$$\Delta V_{n-1}(u_n) = \Delta V_n(u_n)(= \frac{-4z_n^2}{9}) \leq 0. \tag{20}$$

C. The damping $L_G V$ controller

Setting now

$$u := u_{n-1} = u_n + v_{n-1}$$

one designs an AvPBC over v_{n-1} with Lyapunov function V_{n-1} as in (19). For, one computes

$$\begin{aligned} \Delta V_{n-1}(u_{n-1}) &:= V_{n-1}(z^+(u_{n-1})) - V_{n-1}(z) \\ &= V_{n-1}(u_{n-1}) - V_{n-1}(u_n) + \Delta V_{n-1}(u_n) \end{aligned}$$

so getting because of (20)

$$\Delta V_{n-1}(u_{n-1}) \quad \leq \quad \int_{u_n}^{u_n+v_{n-1}} \mathrm{L}_{\tilde{G}} V_{n-1}(z^+(v),v)\mathrm{d}v. \tag{21}$$

Define now

$$H_{n-1}(z,v) := \mathrm{L}_{\tilde{G}(.,v)} V_{n-1}(z,v) = z_{n-1}\tilde{G}_{n-1}(z_n,v) + z_n \tag{22}$$

and conclude from (21) average passivity of the link $v_{n-1} \to H_{n-1}$ under the dynamics (15-16).

Theorem 3.2 provides a stabilizing controller. For, it is sufficient to compute v_{n-1} which solves the algebraic equality $v_{n-1} + K_{n-1}H^+_{(n-1)av}(z,v_{n-1}) = 0$ with $K_{n-1} = 1$. As a matter of facts, if

$$v_{n-1} = -H^+_{(n-1)av}(z,v_{n-1})$$

then one gets in closed loop

$$\Delta V_{n-1}(u_{n-1}) = -u_n^2 - v_{n-1}^2$$

and GAS is achieved. More in detail,

$$v_{n-1} = -\int_{u_n}^{u_n+v_{n-1}} (z_{n-1}^+(v)\tilde{G}_{n-1}(z_n^+(v),v) + z_n^+(v))\mathrm{d}v \qquad (23)$$

with $\tilde{G}_{n-1}(z,v)$ given in (18) and

$$\begin{aligned} z_{n-1}^+(v) &= z_{n-1} + \int_0^v \tilde{G}_{n-1}(z_n^+(w),w)\mathrm{d}w \\ z_n^+(v) &= \frac{z_n}{3} + v. \end{aligned}$$

As far as the ZSD of $H_{n-1}(z,0)$ defined in (22) is concerned, it is a matter of computations to verify that the only trajectories invariant under free evolution $z_{n-1}^+(0) = z_{n-1}, z_n^+(0) = \frac{z_n}{3}$ which give

$$z_{n-1}\tilde{G}_{n-1}(z_n) + z_n = z_{n-1}(G_{n-1}(z_n) + \frac{\partial \Phi_{n-1}(z_n)}{\partial z_n}) + z_n = 0$$

are those equal to 0 because of the controllability assumption of the linearized dynamics which guarantees $G_{n-1}(z_n) + \frac{\partial \Phi_{n-1}(z_n)}{\partial z_n} \neq 0$.

The procedure can be pursued along the same lines returning to the construction of an augmented dynamics of dimension 3. This is summarized in the main result below.

Theorem 4.1: Given the strict-feedforward polynomial discrete-time dynamics (1), equivalently (7,8), there exists a polynomial state feedback which achieves global stabilization of the equilibrium $x = 0$ and local exponential stability.

Proof: The proof is by induction. Starting from the stabilization of the bottom subsystem, the induction step is composed with successive coordinates change computation as in section *(IV.B)* and damping L_GV controller design as in section *(IV.C)*.

step $p > 1$. One starts with a dynamics of order p and a feedback $u_{n-p+1}(z_n, ...z_{n-p+1})$ which satisfies

$$\Delta V_{n-p+1}(u_{n-p+1}) = -u_n^2 - \sum_{i=1}^{p-1}(u_{n-p+i} - u_{n-p+i+1})^2 < 0$$

with $V_{n-p+1} = \frac{1}{2}\Sigma_{i=0}^{p-1} z_{n-i}^2$. At this $p-th$ step, one looks for $\Phi_{n-p}(z_n, ...z_{n-p+1}) = \Phi_{n-p}(z_{n/n-p+1})$ such that setting

$$z_{n-p} = x_{n-p} + \Phi_{n-p}(z_{n/n-p+1}) \quad z_n = x_n$$

and keeping unchanged the other z_i-components, one satisfies

$$z_{(n-p)(k+1)} = z_{(n-p)k}^+(0) = z_{(n-p)k}$$

under u_{n-p+1}. An augmented dynamics of dimension $(p+1)$ is so defined. Setting

$$V_{n-p} = \frac{1}{2}z_{n-p}^2 + V_{n-p+1}$$

one immediately notes that under u_{n-p+1}

$$\Delta V_{n-p}(u_{n-p+1}) = \Delta V_{n-p+1}(u_{n-p+1}) < 0.$$

Assuming now

$$u_{n-p} = u_{n-p+1} + v_{n-p}$$

and computing $\Delta V_{n-p}(u_{n-p+1} + v_{n-p})$, one has

$$\begin{aligned} \Delta V_{n-p}(u_{n-p+1} + v_{n-p}) &= \Delta V_{n-p+1}(u_{n-p+1}) \\ + \int_{u_{n-p+1}}^{u_{n-p+1}+v_{n-p}} & \mathrm{L}_{\tilde{G}} V_{n-p}(z^+(v),v)\mathrm{d}v \end{aligned}$$

with $\tilde{G}$ described below. Average passivity of the input/output link $v_{n-p} \rightarrow \mathrm{L}_{\tilde{G}} V_{n-p}(z,v)$ follows and a GAS stabilizing feedback can be computed as described in section *III.C*.

At this step p, the vector field (say $G(z_{n/n-p+1}, u)$) defining the augmented DDR of dimension $p+1$ is transformed under the coordinates change

$$\begin{aligned} z_{n-p} &= x_{n-p} + \Phi_{n-p}(z_{n/n-p+1}) \\ z_{n-p+1} &= z_{n-p+1} \\ &... \\ z_n &= z_n \end{aligned}$$

into $\tilde{G}(.,u)$ according to the rule (17) so generalizing (18) into

$$\tilde{G}_{n-p}(z_{n/n-p+1}, u) = G_{n-p}(z_{n/n-p+1}, u) + \sum_{i=0}^{p-1} \frac{\partial z_{n-p}}{\partial z_{n-i}}. \quad (24)$$

It is a matter of computations to verify that by construction, at each step, the computed coordinates changes are diffeomorphisms and that with respect to the functions $H_{n-p}(z)$, ZSD holds true because of controllability in first approximation.

At the last step, $p = (n-1)$, the procedure ends with a coordinates change

$$z_i = x_i + \Phi_i(z_{n/i+1}); i \in [1, n-1]; z_n = x_n$$

a control Lyapunov function

$$V_1(z) = \Sigma_{i=1}^n \frac{z_i^2}{2} = \frac{z_1^2}{2} + V_2(z)$$

and a feedback control law of the form

$$u(z) = u_1(z) = u_n(z_n) + \sum_{i=1}^{n-1} v_i$$

with

$$v_i = \frac{1}{v_i}\int_{u_{i+1}}^{u_{i+1}+v_i} \mathrm{L}_{\tilde{G}} V_i(z^+(v), v_i)\mathrm{d}v_i$$

which achieves GAS and LES since $\frac{\partial v_i}{\partial z}(0) \neq 0$ holds too. ∎

We note that the proof is constructive for the Lyapunov function.

V. An example

Let the discrete-time strict feedforward dynamics

$$\begin{aligned} x_{1k+1} &= x_{1k} + x_{2k} + x_{2k}^2 \\ x_{2k+1} &= x_{2k} + u_k. \end{aligned}$$

A GAS state feedback can be computed according to the forwarding procedure.

induction step - It is performed in section IVA so setting $u_{2k} = -\frac{2x_{2k}}{3}$ with Lyapunov function $V_2 := \frac{1}{2}x_2^2$ and closed loop performances $\Delta V_2(u_{2k}) = -\frac{2x_{2k}^2}{9} \leq 0$. Exponential stabilization is achieved too. Then, according to section IVC, one sets $z_1 = \phi_1 x_2 + \phi_2 x_2^2$ to solve the equality (14); i.e.

$$\int_0^{-\frac{2x_2}{3}} (\phi_1 + 2\phi_2(x_2+v))\mathrm{d}v = -x_2 - x_2^2$$

so getting $z_1 = x_1 + \frac{3}{2}x_2 + \frac{9}{8}x_2^2$.
By construction, the z-dynamics under $v_{1k}+u_{2k}$ is driftless in z_1 and described by

$$\begin{aligned} z_{1k+1} &= z_{1k} + \frac{3}{2}v_{1k} + \frac{9}{8}v_{1k}^2 + \frac{3}{4}z_{2k}v_{1k} \\ z_{2k+1} &= \frac{z_{2k}}{3} + v_{1k} \end{aligned}$$

or equivalent DDR

$$\begin{aligned} z_1^+ &= z_1 \\ z_2^+ &= \frac{z_2}{3} \end{aligned}$$

$$\begin{aligned} \frac{\partial z_1^+(v)}{\partial v} &= \frac{3}{2}(1+\frac{3}{3}z_2^+(v)) \\ \frac{\partial z_2^+(v)}{\partial v} &= 1. \end{aligned}$$

It is Lyapunov stable with $V_1 := \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$ and average passive for the output mapping

$$H_1(z,v) = \mathrm{L}_{\tilde{G}(.,v)}V_1(z) := \frac{3}{2}z_1(1+\frac{3}{2}z_2) + z_2$$

with

$$\tilde{G}(z,v) = \frac{3}{2}(1+\frac{3}{2}z_2)\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}.$$

Computing

$$\begin{aligned} H_{1av}^+(z,v) &= \frac{1}{v}\int_0^v \mathrm{L}_{\tilde{G}(.,w)}V_1(z^+(w))\mathrm{d}w \\ &= \frac{1}{v}\int_0^v (\frac{3}{2}z_1^+(w) + \frac{9}{4}z_1^+(w)z_2^+(w) + z_2^+(w))\mathrm{d}w \end{aligned}$$

it follows that the feedback $u_1 = v_1 - \frac{2z_2}{3}$ with v_1 satisfying the implicit equality $v_1 = -H_{1av}^+(z,v_1)$; i.e.

$$\begin{aligned} v_1 &= -(\frac{3}{2}z_1 + \frac{1}{3}z_2 + \frac{3}{4}z_1 z_2) \\ &- \frac{v_1}{2^3}(13 + 3^2 z_1 + 3^2 z_2 + \frac{3^2}{2^2}z_2^2) - v_1^2\frac{3^3}{2^4}(1+\frac{z_2}{2}) - v_1^3\frac{3^4}{2^7} \end{aligned}$$

achieves GAS of 0. It is a matter of computations to verify that

$$H_1(z,0) = \frac{3}{2}z_1(1+\frac{3}{2}z_2) + z_2$$

is ZSD.

VI. CONCLUDING REMARKS

It has been shown that the concept of average-passivity, recently introduced by the authors, can be profitably used to provide a complete procedure achieving global asymptotic stabilization of strict-feedforward discrete-time dynamics. Even though the computations of discrete-time controllers remains a difficult task, they are characterized as the solutions of algebraic equations. Applying the procedure to dynamics with polynomial generating functions make feasible the computations. Sampled-data dynamics are of peculiar interest as they represent realistic case studies. In that case, it is possible to compute the control through executable algorithms taking advantage of its dependency on the sampling period. This is a promising perspective of this work in relation with other approaches developed to precisely handle the sampled-data aspect [3]. The approach here proposed could turn out profitable in the analysis and design of hybrid systems, a context where the interest to habe similar control procedures for continuous-time and discrete-time dynamics is well known.

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