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On the control of spin-boson systems

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***Abstract*— In this paper we study the so-called spin-boson system, namely a spin-$1/2$ particle in interaction with a distinguished mode of a quantized bosonic field. We control the system via an external field acting on the bosonic part.**

Applying geometric control techniques to the Galerkin approximation and using perturbation theory to guarantee non-resonance of the spectrum of the drift operator, we prove approximate controllability of the system, for almost every value of the interaction parameter.

I. INTRODUCTION

In this paper we study the so-called Rabi model, which describes the interaction between a bosonic mode and a two-level system. Mathematically, in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}, \mathbf{C}) \otimes \mathbf{C}^2$, we consider the Schrödinger equation

$$\mathrm{i}\,\partial_t \psi = H_{\mathrm{Rabi}} \psi, \tag{1}$$

where

$$H_{\mathrm{Rabi}} = \frac{\omega}{2}(-\partial_x^2 + x^2) \otimes \mathbb{1} + \frac{\Omega}{2} \mathbb{1} \otimes \sigma_3 + g\, x \otimes \sigma_1,$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

is the usual notation for the Pauli matrices.

The physical interpretation of the two factors in the tensor product varies according to the context. For instance, in cavity QED the Hamiltonian H_{Rabi} describes a two-level ion interacting with a single distinguished mode of the quantized electromagnetic field in the cavity. In this context, a simplified version of the operator H_{Rabi} is often called *Jaynes-Cummings Hamiltonian*, after the celebrated work [1] on maser theory (see [2] and references therein).

In a wide variety of experimental situations one can act on the system by an external field. The goal of the controller might be to lead the system from a given initial state to a prescribed final one. For spin-boson models, this amounts to study the control problem

$$\mathrm{i}\,\partial_t \psi(t) = H_{\mathrm{Rabi}} \psi(t) + H_c(u(t))\psi(t), \tag{2}$$

where H_c is a self-adjoint operator describing the coupling between the system and the controlled external field. In general, u takes values in $\mathbf{R}$ (or, more generally, in $\mathbf{R}^d$).

In most cases the external field can act on the bosonic mode only, while the spin mode is not directly accessible. This leads to a control Hamiltonian of the form

$$H_c(u(t)) = h_c(u(t)) \otimes \mathbb{1}. \tag{3}$$

where $h_c(u(t))$ is a self-adjoint operator acting in $L^2(\mathbf{R}, \mathbf{C})$.

One of the simplest form for the operator $h_c(u(t))$ is the following

$$h_c(u(t)) = u(t)x. \tag{4}$$

The linearity in u is a consequence of the dipole approximation which is valid in the limit of weak field. The linearity with respect to x of the multiplicative operator h_c represents the action of a force depending on time and constant in x.

The main result of the paper is the following.

Theorem 1: Assume that Ω is not an integer multiple of ω. Then system (2), with H_c taking the form (3)-(4), is approximately controllable for almost every $g \in \mathbf{R}$.

The precise definition of approximate controllability will be given in the next section. It basically means that for every choice of the initial and final state, there

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exists an admissible control law u depending on the time which steers the initial state arbitrarily close to the final one.

For control results on related spin-boson models, see [3], [4], [5], [6], [7].

A. Content of the paper

In Section II we recall an approximate controllability result obtained in [8], which is crucial for our study. In Section III we prove Theorem 1. To this purpose we study the applicability of the general approximate controllability result in dependence on the parameter g. In order to do so, we have to use perturbation theory in the parameter g up to order 4.

II. An approximate controllability result

We are going to recall a general controllability result for bilinear quantum systems in an abstract setting.

In a separable Hilbert space H, endowed with the Hermitian product $\langle\cdot,\cdot\rangle$, we consider the following control system

$$\frac{d}{dt}\psi = (A + u(t)B)\psi, \quad u(t) \in U, \qquad (5)$$

where (A, B, U) satisfies the following assumption.

Assumption 1: U is a subset of $\mathbf{R}$ and (A, B) is a pair of (possibly unbounded) linear operators in H such that

1) A is skew-adjoint on its domain $D(A)$;
2) there exists a Hilbert basis $(\phi_k)_{k\in\mathbf{N}}$ of H consisting of eigenvectors of A: for every k, $A\phi_k = \mathrm{i}\lambda_k\phi_k$ with λ_k in $\mathbf{R}$;
3) for every j in $\mathbf{N}$, ϕ_j is in the domain $D(B)$ of B;
4) $A + uB$ is essentially skew-adjoint for every $u \in U$;
5) $\langle B\phi_j, \phi_k\rangle = 0$ for every j, k in $\mathbf{N}$ such that $\lambda_j = \lambda_k$ and $j \neq k$.

If (A, B, U) satisfies Assumption 1, then $A + uB$ generates a unitary group $t \mapsto e^{t(A+uB)}$. By concatenation, one can define the solution of (5) for every piecewise constant function u taking values in U, for every initial condition ψ_0 given at time t_0. We denote this solution by $t \mapsto \Upsilon^u_{t,t_0}\psi_0$.

A pair (j, k) in $\mathbf{N}^2$ is a *non-resonant transition* of (A, B) if $b_{jk} \neq 0$ and, for every l, m, $|\lambda_j - \lambda_k| = |\lambda_l - \lambda_m|$ implies $\{j, k\} = \{l, m\}$ or $\{l, m\} \cap \{j, k\} = \emptyset$.

A subset S of $\mathbf{N}^2$ is a *chain of connectedness* of (A, B) if for every j, k in $\mathbf{N}$, there exists a finite sequence $p_1 = j, p_2, \ldots, p_r = k$ for which $(p_l, p_{l+1}) \in S$ for every l and $\langle\phi_{p_{l+1}}, B\phi_{p_l}\rangle \neq 0$ for every $l = 1, \ldots, r-1$. A chain of connectedness S of (A, B) is *non-resonant* if every (j, k) in S is a non-resonant transition of (A, B).

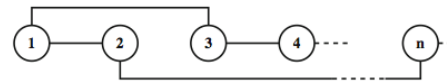

Fig. 1. Each vertex of the graph represents an eigenstate of A (when the spectrum is not simple, several nodes may be attached to the same eigenvalue). An edge links two vertices if and only if B connects the corresponding eigenstates. In this example, $\langle\phi_1, B\phi_2\rangle$ and $\langle\phi_1, B\phi_3\rangle$ are not zero, while $\langle\phi_1, B\phi_4\rangle = \langle\phi_2, B\phi_3\rangle = \langle\phi_2, B\phi_4\rangle = 0$.

Definition 1: Let (A, B, U) satisfy Assumption 1. We say that (5) is *approximately controllable* if for every $\varepsilon > 0$, for every $\psi_0, \psi_1 \in H$, there exists a piecewise constant function $u_\varepsilon : [0, T_\varepsilon] \to U$ such that $\|\Upsilon^u_{T_\varepsilon,0}\psi_0 - \psi_1\| < \varepsilon$.

Theorem 2 ([8]): Assume that $[0, \delta] \subset U$ for some $\delta > 0$ and let (A, B, U) satisfy Assumption 1 and admit a non-resonant chain of connectedness. Then system (5) is approximately controllable.

III. Proof of Theorem 1

We consider here the approximate controllability problem for a system of the form (2), where H_c takes the form (3)-(4).

The goal of this section is to prove Theorem 1. Assume then that Ω is not an integer multiple of ω. The proof of the theorem is based on a suitable application of Theorem 2.

The strategy of the proof is the following. We first show in Subsection III-A that, for almost every g in $\mathbf{R}$, some relevant pairs of eigenvalues of H_{Rabi} satisfy the non-resonance condition, see (7). This goal is reached by exploiting the analyticity of the eigenvalues and by using perturbation theory. Then, in Subsection III-B, we prove that these pairs of eigenvalues correspond to non-resonant transitions, according to the definition above.

Preliminarily, we introduce some additional notations. Denote by $H_{\mathrm{Rabi},0}$ the Hamiltonian H_{Rabi} where we set $g = 0$. Let $(\varphi_j)_{j\in\mathbf{N}}$ be the standard Hilbert basis of $L^2(\mathbf{R}, \mathbf{C})$ given by real eigenfunctions of $-\partial_x^2 + x^2$, so that $(-\partial_x^2 + x^2)\varphi_j = (2j+1)\varphi_j$ and $\int_{\mathbf{R}} x\varphi_j(x)\varphi_{j+1}(x)dx = \sqrt{(j+1)/2}$ for $j \geq 0$.

Based on $(\varphi_j)_{j\in\mathbf{N}}$, we obtain a Hilbert basis $(\Phi_{j,s})_{j\in\mathbf{N},\, s\in\{-1,1\}}$ of factorized eigenstates of $H_{\mathrm{Rabi},0}$

whose corresponding eigenvalues are

$$E_{j,s} = \omega\left(j + \frac{1}{2}\right) + \frac{s}{2}\Omega.$$

Since Ω is not an integer multiple of ω then each eigenvalue $E_{j,s}$ is simple. (See Figure 2.)

For $g \in \mathbf{R}$, denote by $E^g_{j,s}$, $j \in \mathbf{N}$, $s = \pm 1$, the eigenvalues of H_{Rabi} repeated according to their multiplicities, and by $\Phi^g_{j,s}$, $j \in \mathbf{N}$, $s = \pm 1$, an orthonormal basis of corresponding eigenstates. By a suitable global version of Rellich's theorem [9, Theorem XII.8], the ordering of the eigenvalues $E^g_{j,s}$'s and the eigenfunctions of H_{Rabi} inside the (possibly degenerate) eigenspaces can be chosen in such a way that $g \mapsto E^g_{j,s}$ and $g \mapsto \Phi^g_{j,s}$ are analytic functions, with values in $\mathbf{C}$ and $L^2(\mathbf{R}, \mathbf{C})$ respectively, for every $(j, s) \in \mathbf{N} \times \{-1, 1\}$. Without loss of generality we assume that $E^0_{j,s} = E_{j,s}$ and $\Phi^0_{j,s} = \Phi_{j,s}$ for every $(j, s) \in \mathbf{N} \times \{-1, 1\}$.

In the following, for ease of notations, we write in bold the elements of $\mathbf{N} \times \{-1, 1\}$, and for every $\mathbf{j} \in \mathbf{N} \times \{-1, 1\}$ we define $j(\mathbf{j}), s(\mathbf{j})$ in such a way that $\mathbf{j} = (j(\mathbf{j}), s(\mathbf{j}))$.

In order to study the first and higher-order derivatives of $g \mapsto E^g_{j,s}$ at $g = 0$, it is useful to introduce the quantities

$$\begin{aligned} V_{\mathbf{i},\mathbf{j}} &= \langle \Phi_{\mathbf{i}}, (x \otimes \sigma_1)\Phi_{\mathbf{j}}\rangle \\ &= \left(\delta_{j(\mathbf{i}),j(\mathbf{j})-1}\sqrt{\frac{j(\mathbf{j})}{2}} + \delta_{j(\mathbf{i}),j(\mathbf{j})+1}\sqrt{\frac{j(\mathbf{j})+1}{2}}\right) \times \\ &\quad \times (1 - \delta_{s(\mathbf{i}),s(\mathbf{j})}). \end{aligned} \tag{6}$$

A. Step I: Relevant eigenvalue pairs are non-resonant.

Let us first prove that for almost every $g \in \mathbf{R}$ and every $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in \mathbf{N} \times \{-1, 1\}$, with $(\mathbf{i}, \mathbf{j}) \neq (\mathbf{k}, \mathbf{l})$ and $\mathbf{i} \neq \mathbf{j}$, one has $E^g_{\mathbf{i}} - E^g_{\mathbf{j}} \neq E^g_{\mathbf{k}} - E^g_{\mathbf{l}}$. In order to do so, we observe that it is enough to show that for fixed $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in \mathbf{N} \times \{-1, 1\}$ as before, the set

$$S_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} = \{g \mid E^g_{\mathbf{i}} - E^g_{\mathbf{j}} \neq E^g_{\mathbf{k}} - E^g_{\mathbf{l}}\} \tag{7}$$

is of full measure. By the analytic dependence on g of the eigenvalues of H_{Rabi}, this is equivalent to say that $g \mapsto E^g_{\mathbf{i}} - E^g_{\mathbf{j}}$ and $g \mapsto E^g_{\mathbf{k}} - E^g_{\mathbf{l}}$ have different Taylor expansions at $g = 0$.

Let us consider the Taylor expansion

$$E^g_{\mathbf{j}} = E_{\mathbf{j}} + \sum_{m=1}^{\infty} g^m E^{(m)}_{\mathbf{j}}.$$

The computation of the coefficients $E^{(m)}_{\mathbf{j}}$ carried on below is based on the Rayleigh–Schrödinger series (see, for instance, [9, Chapter XII]).

First of all we observe that $E_{\mathbf{i}} - E_{\mathbf{j}} = E_{\mathbf{k}} - E_{\mathbf{l}}$ is equivalent to $j(\mathbf{i}) - j(\mathbf{j}) = j(\mathbf{k}) - j(\mathbf{l})$ and $s(\mathbf{i}) - s(\mathbf{j}) = s(\mathbf{k}) - s(\mathbf{l})$, in the case in which Ω is not an integer multiple of $\omega/2$. If $\Omega = (2m+1)\omega/2$ for some non-negative integer m, then $E_{\mathbf{i}} - E_{\mathbf{j}} = E_{\mathbf{k}} - E_{\mathbf{l}}$ implies

$$\begin{aligned} j(\mathbf{i}) + j(\mathbf{l}) - j(\mathbf{j}) - j(\mathbf{k}) &= \\ \frac{2m+1}{4}\big(s(\mathbf{j}) + s(\mathbf{k}) &- s(\mathbf{i}) - s(\mathbf{l})\big), \end{aligned}$$

and thus, if the left-hand side is an integer number different from zero, it must be $|s(\mathbf{j}) + s(\mathbf{k}) - s(\mathbf{i}) - s(\mathbf{l})| = 4$, that is $s(\mathbf{j}) = s(\mathbf{k}) = -s(\mathbf{i}) = -s(\mathbf{l})$.

The term $E^{(1)}_{\mathbf{j}}$ coincides with $V_{\mathbf{j},\mathbf{j}} = \langle \Phi_{\mathbf{j}}, (X \otimes \sigma_1)\Phi_{\mathbf{j}}\rangle$, thus we deduce from (6) that $E^{(1)}_{\mathbf{j}} = 0$ for every $\mathbf{j}$.

Following [9] we have that

$$E^{(2)}_{\mathbf{j}} = -\sum_{\mathbf{m}\neq\mathbf{j}} (E_{\mathbf{m}} - E_{\mathbf{j}})^{-1} V_{\mathbf{j},\mathbf{m}} V_{\mathbf{m},\mathbf{j}}.$$

Thus

$$\begin{aligned} E^{(2)}_{\mathbf{j}} &= -\sum_{\mathbf{m}\neq\mathbf{j}} (E_{\mathbf{m}} - E_{\mathbf{j}})^{-1}(1 - \delta_{s(\mathbf{j}),s(\mathbf{m})})^2 \times \\ &\quad \times \left(\delta_{j(\mathbf{j}),j(\mathbf{m})-1}\sqrt{\frac{j(\mathbf{j})+1}{2}} + \right. \\ &\quad \left. + \delta_{j(\mathbf{j}),j(\mathbf{m})+1}\sqrt{\frac{j(\mathbf{j})}{2}}\right)^2 \\ &= -(E_{j(\mathbf{j})+1,-s(\mathbf{j})} - E_{\mathbf{j}})^{-1}\frac{j(\mathbf{j})+1}{2} + \\ &\quad - (E_{j(\mathbf{j})-1,-s(\mathbf{j})} - E_{\mathbf{j}})^{-1}\frac{j(\mathbf{j})}{2} \\ &= -(\omega - s(\mathbf{j})\Omega)^{-1}\frac{j(\mathbf{j})+1}{2} + \\ &\quad + (\omega + s(\mathbf{j})\Omega)^{-1}\frac{j(\mathbf{j})}{2} \\ &= \frac{\omega + s(\mathbf{j})\Omega(2j(\mathbf{j})+1)}{2(\Omega^2 - \omega^2)}. \end{aligned}$$

Notice that the computation above is correct also for $j(\mathbf{j}) = 0$, even if in this case $E_{j(\mathbf{j})-1,-s(\mathbf{j})}$ is not a well defined eigenvalue of $H_{\mathrm{Rabi},0}$. Indeed, in this case the term $(E_{j(\mathbf{j})-1,-s(\mathbf{j})} - E_{\mathbf{j}})^{-1} j(\mathbf{j})/2$ counts as zero.

Let us identify the values $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}$ such that $E^{(2)}_{\mathbf{i}} - E^{(2)}_{\mathbf{j}} = E^{(2)}_{\mathbf{k}} - E^{(2)}_{\mathbf{l}}$ under the assumption that $E_{\mathbf{i}} - E_{\mathbf{j}} =$

$E_{\mathbf{k}} - E_{\mathbf{l}}$. Recall that we also assume that $(\mathbf{i}, \mathbf{j}) \neq (\mathbf{k}, \mathbf{l})$ and $\mathbf{i} \neq \mathbf{j}$. From the above expression of $E_{\mathbf{j}}^{(2)}$ we have

$$s(\mathbf{i})\,(2j(\mathbf{i})+1) - s(\mathbf{j})\,(2j(\mathbf{j})+1) = s(\mathbf{k})\,(2j(\mathbf{k})+1) - s(\mathbf{l})\,(2j(\mathbf{l})+1)\,. \quad (8)$$

If $\Omega = (2m+1)\omega/2$ for some non-negative integer m and $s(\mathbf{j}) = s(\mathbf{k}) = -s(\mathbf{i}) = -s(\mathbf{l})$ then (8) gives $j(\mathbf{i}) + j(\mathbf{j}) + j(\mathbf{k}) + j(\mathbf{l}) + 2 = 0$, which is impossible being the addends positive.

The remaining case is when $j(\mathbf{i}) - j(\mathbf{j}) = j(\mathbf{k}) - j(\mathbf{l})$ and $s(\mathbf{i}) - s(\mathbf{j}) = s(\mathbf{k}) - s(\mathbf{l})$, in which case

$$s(\mathbf{i})j(\mathbf{i}) - s(\mathbf{j})j(\mathbf{j}) = s(\mathbf{k})j(\mathbf{k}) - s(\mathbf{l})j(\mathbf{l}). \quad (9)$$

Then, either $s(\mathbf{i}) = s(\mathbf{j})$, which implies $s(\mathbf{k}) = s(\mathbf{l})$ and then, by (9), $s(\mathbf{i}) = s(\mathbf{j}) = s(\mathbf{k}) = s(\mathbf{l})$, or $s(\mathbf{i}) = -s(\mathbf{j})$, which implies $s(\mathbf{k}) = -s(\mathbf{l})$ and then, by (9), $j(\mathbf{i}) + j(\mathbf{j}) = j(\mathbf{k}) + j(\mathbf{l})$. In the latter case it must be $\mathbf{i} = \mathbf{k}$ and $\mathbf{j} = \mathbf{l}$, which is excluded by assumption. Therefore the nontrivial quadruples $\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}$ satisfying both the equalities $E_{\mathbf{i}} - E_{\mathbf{j}} = E_{\mathbf{k}} - E_{\mathbf{l}}$ and $E_{\mathbf{i}}^{(2)} - E_{\mathbf{j}}^{(2)} = E_{\mathbf{k}}^{(2)} - E_{\mathbf{l}}^{(2)}$ are those for which $s(\mathbf{i}) = s(\mathbf{j}) = s(\mathbf{k}) = s(\mathbf{l})$ and $j(\mathbf{i}) - j(\mathbf{j}) = j(\mathbf{k}) - j(\mathbf{l})$.

Let us now evaluate the terms $E_{\mathbf{j}}^{(3)}$ as in [9]. We have

$$E_{\mathbf{j}}^{(3)} = \sum_{\mathbf{m}\neq\mathbf{j},\mathbf{n}\neq\mathbf{j}} (E_{\mathbf{m}} - E_{\mathbf{j}})^{-1}(E_{\mathbf{n}} - E_{\mathbf{j}})^{-1} \times V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{n}}V_{\mathbf{n},\mathbf{j}} - \sum_{\mathbf{m}\neq\mathbf{j}} (E_{\mathbf{m}} - E_{\mathbf{j}})^{-2} V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{j}}V_{\mathbf{j},\mathbf{j}}.$$

Since $V_{\mathbf{a},\mathbf{b}} \neq 0$ only if $s(\mathbf{a}) = -s(\mathbf{b})$ it turns out that $V_{\mathbf{j},\mathbf{m}}$ and $V_{\mathbf{m},\mathbf{n}}$ are different from 0 only if $s(\mathbf{j}) = s(\mathbf{n}) = -s(\mathbf{m})$, but then $V_{\mathbf{n},\mathbf{j}} = 0$. Thus, recalling that $V_{\mathbf{j},\mathbf{j}} = 0$, we have $E_{\mathbf{j}}^{(3)} = 0$ for every $\mathbf{j} \in \mathbf{N} \times \{-1, 1\}$.

We are going to complete the proof that the set $S_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}}$ defined as in (7) has full measure by showing that if $\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l} \in \mathbf{N} \times \{-1, 1\}$ are such that $(\mathbf{i}, \mathbf{j}) \neq (\mathbf{k}, \mathbf{l})$, $\mathbf{i} \neq \mathbf{j}$, $j(\mathbf{i}) - j(\mathbf{j}) = j(\mathbf{k}) - j(\mathbf{l})$ (which follows from $E_{\mathbf{i}} - E_{\mathbf{j}} = E_{\mathbf{k}} - E_{\mathbf{l}}$) and $s(\mathbf{i}) = s(\mathbf{j}) = s(\mathbf{k}) = s(\mathbf{l})$ (which follows from $E_{\mathbf{i}}^{(2)} - E_{\mathbf{j}}^{(2)} = E_{\mathbf{k}}^{(2)} - E_{\mathbf{l}}^{(2)}$), then $E_{\mathbf{i}}^{(4)} - E_{\mathbf{j}}^{(4)} \neq E_{\mathbf{k}}^{(4)} - E_{\mathbf{l}}^{(4)}$.

The general formula for $E_{\mathbf{j}}^{(4)}$ (see [9]) is

$$\begin{aligned} E_{\mathbf{j}}^{(4)} = &- \sum_{\mathbf{m}\neq\mathbf{j},\mathbf{n}\neq\mathbf{j},\mathbf{p}\neq\mathbf{j}} (E_{\mathbf{m}} - E_{\mathbf{j}})^{-1}(E_{\mathbf{n}} - E_{\mathbf{j}})^{-1} \times \\ &\times (E_{\mathbf{p}} - E_{\mathbf{j}})^{-1} V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{n}}V_{\mathbf{n},\mathbf{p}}V_{\mathbf{p},\mathbf{j}} \\ &+ \sum_{\mathbf{m}\neq\mathbf{j},\mathbf{n}\neq\mathbf{j}} V_{\mathbf{j},\mathbf{j}}V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{n}}V_{\mathbf{n},\mathbf{j}}[(E_{\mathbf{m}} - E_{\mathbf{j}})^{-1} \times \\ &\times (E_{\mathbf{n}} - E_{\mathbf{j}})^{-2} + (E_{\mathbf{m}} - E_{\mathbf{j}})^{-2}(E_{\mathbf{n}} - E_{\mathbf{j}})^{-1}] \\ &+ \sum_{\mathbf{m}\neq\mathbf{j},\mathbf{n}\neq\mathbf{j}} (E_{\mathbf{m}} - E_{\mathbf{j}})^{-2}(E_{\mathbf{n}} - E_{\mathbf{j}})^{-1} \times \\ &\times V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{j}}V_{\mathbf{j},\mathbf{n}}V_{\mathbf{n},\mathbf{j}} \\ &- \sum_{\mathbf{m}\neq\mathbf{j}} (E_{\mathbf{m}} - E_{\mathbf{j}})^{-3} V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{j}}V_{\mathbf{j},\mathbf{j}}^2. \end{aligned} \quad (10)$$

Since $V_{\mathbf{j},\mathbf{j}} = 0$, only the first and third term of the right-hand side must be evaluated.

Let us compute the first term in (10). In order to avoid null terms we must assume $s(\mathbf{j}) = -s(\mathbf{m}) = s(\mathbf{n}) = -s(\mathbf{p})$ and thus $j(\mathbf{j}) \neq j(\mathbf{n})$. Therefore the only nonzero terms in the sum are given by $j(\mathbf{j}) = j(\mathbf{m}) + 1 = j(\mathbf{n}) + 2 = j(\mathbf{p}) + 1$ (if $j(\mathbf{j}) > 1$) and $j(\mathbf{j}) = j(\mathbf{m}) - 1 = j(\mathbf{n}) - 2 = j(\mathbf{p}) - 1$. We have

$$\begin{aligned} &\sum_{\mathbf{k}\neq\mathbf{j},\mathbf{l}\neq\mathbf{j},\mathbf{i}\neq\mathbf{j}} (E_{\mathbf{k}} - E_{\mathbf{j}})^{-1}(E_{\mathbf{l}} - E_{\mathbf{j}})^{-1}(E_{\mathbf{i}} - E_{\mathbf{j}})^{-1} \times \\ &\times V_{\mathbf{j},\mathbf{k}}V_{\mathbf{k},\mathbf{l}}V_{\mathbf{l},\mathbf{i}}V_{\mathbf{i},\mathbf{j}} = (-\omega - s(\mathbf{j})\Omega)^{-1} \times \\ &\times (-2\omega)^{-1}(-\omega - s(\mathbf{j})\Omega)^{-1} \left(\frac{j(\mathbf{j})}{2}\right)\left(\frac{j(\mathbf{j})-1}{2}\right) \\ &+ (\omega - s(\mathbf{j})\Omega)^{-1}(2\omega)^{-1}(\omega - s(\mathbf{j})\Omega)^{-1} \times \\ &\times \left(\frac{j(\mathbf{j})+1}{2}\right)\left(\frac{j(\mathbf{j})+2}{2}\right). \end{aligned}$$

Notice that the formula is correct also in the case where $j(\mathbf{j}) = 0$ or $j(\mathbf{j}) = 1$.

Let us now compute the third term in (10). As before, to avoid null terms we assume $s(\mathbf{j}) = -s(\mathbf{m}) = -s(\mathbf{n})$. The nonzero terms in the sum are given by $j(\mathbf{m}) = j(\mathbf{j}) \pm 1$ and $j(\mathbf{n}) = j(\mathbf{j}) \pm 1$ thus we have to sum four terms. We have

$$\sum_{\mathbf{m}\neq\mathbf{j},\mathbf{n}\neq\mathbf{j}} (E_{\mathbf{m}} - E_{\mathbf{j}})^{-2}(E_{\mathbf{n}} - E_{\mathbf{j}})^{-1} V_{\mathbf{j},\mathbf{m}} V_{\mathbf{m},\mathbf{j}} V_{\mathbf{j},\mathbf{n}} V_{\mathbf{n},\mathbf{j}} =$$
$$(-\omega - s(\mathbf{j})\Omega)^{-2}(-\omega - s(\mathbf{j})\Omega)^{-1}\left(\frac{j(\mathbf{j})}{2}\right)^2$$
$$+ (-\omega - s(\mathbf{j})\Omega)^{-2}(\omega - s(\mathbf{j})\Omega)^{-1}\,\frac{j(\mathbf{j})}{2}\,\frac{j(\mathbf{j})+1}{2}$$
$$+ (\omega - s(\mathbf{j})\Omega)^{-2}(-\omega - s(\mathbf{j})\Omega)^{-1}\,\frac{j(\mathbf{j})}{2}\,\frac{j(\mathbf{j})+1}{2}$$
$$+ (\omega - s(\mathbf{j})\Omega)^{-2}(\omega - s(\mathbf{j})\Omega)^{-1}\left(\frac{j(\mathbf{j})+1}{2}\right)^2.$$

By summing up all the terms one sees that, for fixed $s = s(\mathbf{j})$, the term $E_{\mathbf{j}}^{(4)}$ depends quadratically on $j(\mathbf{j})$, i.e. $E_{\mathbf{j}}^{(4)} = C_0(s(\mathbf{j})) + C_1(s(\mathbf{j}))j(\mathbf{j}) + C_2(s(\mathbf{j}))j(\mathbf{j})^2$, where the coefficient $C_2(s(\mathbf{j}))$ is given by

$$C_2(s(\mathbf{j})) = s(\mathbf{j})\frac{\Omega(\omega^2 + 3\Omega^2)}{2(\omega^2 - \Omega^2)^3} \neq 0.$$

So, if $\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}$ are such that $s(\mathbf{i}) = s(\mathbf{j}) = s(\mathbf{k}) = s(\mathbf{l}) = s$ and $j(\mathbf{i}) - j(\mathbf{j}) = j(\mathbf{k}) - j(\mathbf{l})$, we have

$$\begin{aligned} & E_{\mathbf{i}}^{(4)} - E_{\mathbf{j}}^{(4)} = E_{\mathbf{k}}^{(4)} - E_{\mathbf{l}}^{(4)} \\ \Longleftrightarrow\ & C_1(s)(j(\mathbf{i}) - j(\mathbf{j})) + C_2(s)(j(\mathbf{i})^2 - j(\mathbf{j})^2) = \\ & C_1(s)(j(\mathbf{k}) - j(\mathbf{l})) + C_2(s)(j(\mathbf{k})^2 - j(\mathbf{l})^2) \\ \Longleftrightarrow\ & C_2(s)(j(\mathbf{i})^2 - j(\mathbf{j})^2) = C_2(s)(j(\mathbf{k})^2 - j(\mathbf{l})^2) \\ \Longleftrightarrow\ & C_2(s)(j(\mathbf{i}) + j(\mathbf{j})) = C_2(s)(j(\mathbf{k}) + j(\mathbf{l})) \\ \Longleftrightarrow\ & j(\mathbf{i}) = j(\mathbf{k}) \text{ and } j(\mathbf{j}) = j(\mathbf{l}). \end{aligned}$$

This concludes the proof that for almost every $g \in \mathbf{R}$ and every $\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l} \in \mathbf{N}\times\{-1,1\}$, with $(\mathbf{i},\mathbf{j}) \neq (\mathbf{k},\mathbf{l})$ and $\mathbf{i}\neq\mathbf{j}$, one has $E_{\mathbf{i}}^g - E_{\mathbf{j}}^g \neq E_{\mathbf{k}}^g - E_{\mathbf{l}}^g$.

B. Step 2: Coupling of the relevant energy levels.

The proof of Theorem 1 is then concluded, thanks to Theorem 2, if we show that the controlled Hamiltonian $x\otimes\mathbb{1}$ couples, directly or indirectly, all the energy levels for almost all $g \in \mathbf{R}$.

More precisely, we show below that $\langle\Phi_{\mathbf{j}}^g, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}^g\rangle \neq 0$ for almost every $g \in \mathbf{R}$ for all $\mathbf{j},\mathbf{k}$ such that $s(\mathbf{j}) = s(\mathbf{k})$ and $|j(\mathbf{j}) - j(\mathbf{k})| = 1$ or $s(\mathbf{j}) = -s(\mathbf{k})$ and $j(\mathbf{j}) = j(\mathbf{k})$. See Figure 2. As before, it is enough to show that the corresponding Taylor series in g is nonzero.

Set $\Phi_{\mathbf{j}}^g = \Phi_{\mathbf{j}} + \sum_{m=1}^{\infty} g^m\Phi_{\mathbf{j}}^{(m)}$. We have

$$\langle\Phi_{\mathbf{j}}, (x\otimes\mathbb{1})\Phi_{\mathbf{k}}\rangle =$$
$$\left(\delta_{j(\mathbf{j}),j(\mathbf{k})-1}\sqrt{\frac{j(\mathbf{j})+1}{2}} + \delta_{j(\mathbf{j}),j(\mathbf{k})+1}\sqrt{\frac{j(\mathbf{j})}{2}}\right)\delta_{s(\mathbf{j}),s(\mathbf{k})}.$$

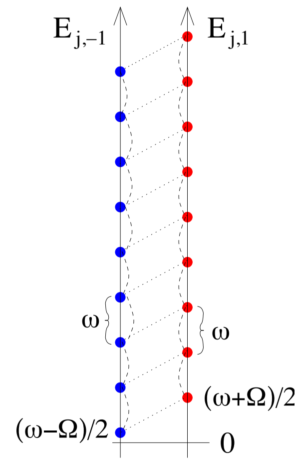

Fig. 2. The dashed lines connect eigenvalues of H_{Rabi} that are "coupled" by the controlled Hamiltonian when $g = 0$, while the dotted lines connect eigenvalues that are coupled by the controlled Hamiltonian for almost all $g \neq 0$.

This is enough to say that $\langle\Phi_{\mathbf{j}}^g, (x\otimes\mathbb{1})\Phi_{\mathbf{k}}^g\rangle \neq 0$ for almost every g for all $\mathbf{j},\mathbf{k}$ such that $s(\mathbf{j}) = s(\mathbf{k})$ and $|j(\mathbf{j}) - j(\mathbf{k})| = 1$.

The term $\Phi_{\mathbf{j}}^{(1)}$ can be characterized through the relation

$$\begin{aligned}(H_{\mathrm{Rabi},0} + gx\otimes\sigma_1)(\Phi_{\mathbf{j}} + g\Phi_{\mathbf{j}}^{(1)} + o(g)) = \\ (E_{\mathbf{j}} + gE_{\mathbf{j}}^{(1)} + o(g))(\Phi_{\mathbf{j}} + g\Phi_{\mathbf{j}}^{(1)} + o(g)).\end{aligned} \quad (11)$$

Regrouping the first-order terms in (11) we get

$$H_{\mathrm{Rabi},0}\Phi_{\mathbf{j}}^{(1)} + (x\otimes\sigma_1)\,\Phi_{\mathbf{j}} - E_{\mathbf{j}}\Phi_{\mathbf{j}}^{(1)} - E_{\mathbf{j}}^{(1)}\Phi_{\mathbf{j}} = 0. \quad (12)$$

Denote by Π the orthogonal projection on the orthogonal complement to $\Phi_{\mathbf{j}}$. Applying Π to (12), we get

$$(H_{\mathrm{Rabi},0} - E_{\mathbf{j}}\mathbb{1})\Phi_{\mathbf{j}}^{(1)} + \Pi(x\otimes\sigma_1)\Phi_{\mathbf{j}} = 0.$$

Notice that the orthogonal complement to $\Phi_{\mathbf{j}}$ is an invariant space for the operator $H_{\mathrm{Rabi},0} - E_{\mathbf{j}}\mathbb{1}$, which is invertible when restricted to it. We write $(H_{\mathrm{Rabi},0} - E_{\mathbf{j}}\mathbb{1})^{-1}$ to denote its inverse (whose values are in the orthogonal complement to $\Phi_{\mathbf{j}}$). Thus,

$$\begin{aligned}\Phi_{\mathbf{j}}^{(1)} &= -(H_{\mathrm{Rabi},0} - E_{\mathbf{j}}\mathbb{1})^{-1}\Pi(x\otimes\sigma_1)\Phi_{\mathbf{j}} \\ &= \sum_{\mathbf{l}\neq\mathbf{j}}(E_{\mathbf{j}} - E_{\mathbf{l}})^{-1}\langle\Phi_{\mathbf{l}}, (x\otimes\sigma_1)\Phi_{\mathbf{j}}\rangle\Phi_{\mathbf{l}}.\end{aligned}$$

The linear term in the Taylor expansion of $\langle\Phi_{\mathbf{j}}^g, (x\otimes$

$\mathbb{1})\Phi^g_{\mathbf{k}}\rangle$ with respect to g is given by

$$\begin{aligned}&\langle\Phi_{\mathbf{j}},(x\otimes\mathbb{1})\Phi^{(1)}_{\mathbf{k}}\rangle+\langle\Phi^{(1)}_{\mathbf{j}},(x\otimes\mathbb{1})\Phi_{\mathbf{k}}\rangle=\\&\sum_{\mathbf{l}\neq\mathbf{k}}(E_{\mathbf{k}}-E_{\mathbf{l}})^{-1}\langle\Phi_{\mathbf{l}},(x\otimes\sigma_1)\Phi_{\mathbf{k}}\rangle\langle\Phi_{\mathbf{j}},(x\otimes\mathbb{1})\Phi_{\mathbf{l}}\rangle+\\&\sum_{\mathbf{l}\neq\mathbf{j}}(E_{\mathbf{j}}-E_{\mathbf{l}})^{-1}\langle\Phi_{\mathbf{l}},(x\otimes\sigma_1)\Phi_{\mathbf{j}}\rangle\langle\Phi_{\mathbf{l}},(x\otimes\mathbb{1})\Phi_{\mathbf{k}}\rangle.\end{aligned}$$

Taking into account only the nonzero terms in the first sum gives $s(\mathbf{j})=s(\mathbf{l})=-s(\mathbf{k})$, $|j(\mathbf{j})-j(\mathbf{l})|=1$ and $|j(\mathbf{k})-j(\mathbf{l})|=1$, so for fixed $\mathbf{j},\mathbf{k}$ we only have (at most) two terms. We assume $j(\mathbf{j})=j(\mathbf{k})$, thus we have

$$\begin{aligned}&\langle\Phi_{\mathbf{j}},(x\otimes\mathbb{1})\Phi^{(1)}_{\mathbf{k}}\rangle=\\&=\left[(-\omega+s(\mathbf{k})\Omega)^{-1}\frac{j(\mathbf{k})+1}{2}+(\omega+s(\mathbf{k})\Omega)^{-1}\frac{j(\mathbf{k})}{2}\right]\\&=\frac{\omega+s(\mathbf{k})\Omega(1+2j(\mathbf{k}))}{2(\Omega^2-\omega^2)}.\end{aligned}$$

Similarly,

$$\langle\Phi^{(1)}_{\mathbf{j}},(x\otimes\mathbb{1})\Phi_{\mathbf{k}}\rangle=\frac{\omega-s(\mathbf{k})\Omega(1+2j(\mathbf{k}))}{2(\Omega^2-\omega^2)}.$$

Hence,

$$\langle\Phi_{\mathbf{j}},(x\otimes\mathbb{1})\Phi^{(1)}_{\mathbf{k}}\rangle=\frac{\omega}{\Omega^2-\omega^2},$$

which is different from zero for every $\mathbf{k}$. Thus the Taylor series of $\langle\Phi^g_{\mathbf{j}},(x\otimes\mathbb{1})\Phi^g_{\mathbf{k}}\rangle$ is nonzero, which concludes the proof of the theorem.

Conclusion

We analyzed the controllability properties of the Rabi model, describing the interaction between a bosonic mode and a two-level system, subject to an external field acting on the bosonic mode only. Namely, approximate controllability has been proved under a non-resonance assumption between the bosonic term and the spin term in the Hamiltonian, and generically with respect to the strength of the interaction term between the two modes. The method relies on perturbation arguments for the spectrum of the Hamiltonian, which allow to apply a general controllability result for the bilinear Schrödinger equation. Future work will address the issue of extending the result to a general class of control terms, possibly removing the non-resonance condition.

References

[1] E. T. Jaynes and F. W. Cummings, “Comparison of quantum and semiclassical radiation theories with application to the beam maser,” *Proceedings of the IEEE*, vol. 51, no. 1, pp. 89–109, 1963.

[2] D. Braak, “Integrability of the Rabi model,” *Physical Review Letters*, vol. 107, no. 10, p. 100401, 2011.

[3] C. K. Law and J. H. Eberly, “Arbitrary control of a quantum electro-magnetic field,” *Phys. Rev. Lett.*, vol. 76, no. 7, pp. 1055–1058, 1996.

[4] A. M. Bloch, R. W. Brockett, and C. Rangan, “Finite controllability of infinite-dimensional quantum systems,” *IEEE Trans. Automat. Control*, vol. 55, no. 8, pp. 1797–1805, 2010.

[5] S. Ervedoza and J.-P. Puel, “Approximate controllability for a system of Schrödinger equations modeling a single trapped ion,” *Ann. Inst. H. Poincaré Anal. Non Linéaire*, vol. 26, no. 6, pp. 2111–2136, 2009.

[6] P. Rouchon, “Quantum systems and control,” *ARIMA Rev. Afr. Rech. Inform. Math. Appl.*, vol. 9, pp. 325–357, 2008.

[7] R. Adami and U. Boscain, “Controllability of the Schrödinger equation via intersection of eigenvalues,” in *Proceedings of the 44th IEEE Conference on Decision and Control, December 12-15*, 2005, pp. 1080–1085.

[8] U. Boscain, M. Caponigro, T. Chambrion, and M. Sigalotti, “A weak spectral condition for the controllability of the bilinear Schrödinger equation with application to the control of a rotating planar molecule,” *Comm. Math. Phys.*, vol. 311, no. 2, pp. 423–455, 2012.

[9] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*. New York: Academic Press [Harcourt Brace Jovanovich Publishers], 1978.