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Design of interval observers for LPV systems subject to exogenous disturbances

Rihab El Houda Thabet, Tarek Raïssi, Christophe Combastel, Ali Zolghadri

Abstract— **In this paper an interval observer for Linear-Parameter-Varying (LPV) Systems is proposed. Usually, the design of such observers is based on monotone systems theory and the observation error should have a cooperative dynamics. In many cases, such a property is hard to satisfy. In this paper, a time-varying change of coordinates is used to overcome this limitation. Two sufficient conditions, according to the center and the radius dynamics of the proposed interval observer and ensuring the stability, are given. The efficiency of the proposed observer is illustrated through computer simulations.**

Index Terms— **Interval observers, LPV systems, monotone systems, bounded errors.**

I. Introduction

The problem of unmeasurable system state estimation is challenging and its solution is required in many engineering applications. Luenberger observers [3], Kalman filter [4] or H_∞ based estimators are classical examples of widely used techniques to solve this problem. Interval observers, which provide two variables evaluating the lower and the upper bounds for state values of systems, are an interesting alternative approach [5]. Under some assumptions, interval observers allow the designer to cope with uncertainties and evaluate the set of admissible values of the state vector, at any time instant.

A number of approaches exist for designing interval observers [1], [2], [7], in linear setting [8]-[11], [19] or when the system exhibits non linear behaviour [6], [7]. The design of stable interval observers is mainly based on the monotone systems theory [1], [7]. This approach has been recently extended to some nonlinear systems using LPV representation with known minorant and majorant matrices [13]. One of the restrictive assumptions is the cooperativity [25] of the interval estimation error dynamics. In [8], [10] the assumption of cooperativity is relaxed for LTI systems by using a time-varying change of coordinates which transforms the LTI system into a Jordan canonical form. In [11], a similar transformation based on a diagonalization in the complex space is used and further extended to the Jordan form [19]. Furthermore, a time-invariant transformation is proposed in [6] to design a closed-loop observer for LTI systems where the transition matrix and the observer gain verify a Sylvester equation. This technique has also been extended to a class of nonlinear systems based on exact linearizations. Finally, time-varying systems have been investigated in [12] where the observer gain and the transition matrix are designed in order to obtain a cooperative observation error at each time. The main limitation of the technique proposed in [12] is that the matrix $A(t) - LC(t)$ should belong to a thin domain whose size is proportional to the inverse of the system dimension. The extension of this result to Linear-Parameter-Varying (LPV) systems remains a challenging task which is the motivation behind this work. It should be noted that there exist several other techniques for estimation of LPV systems such as the one based on the scheduled observer using an interpolation method to schedule the state and the gain matrices of the observer [21], the LPV observer for discrete-time LPV systems [22] and interval estimation applying High Order Sliding Mode (HOSM) techniques [23]. The goal of this paper is to design an interval observer for LPV systems with stable bounds dynamics. This result can be useful to deal with some nonlinear systems which can be transformed or approximated by LPV representations [14], [15], [24].

In this work, an interval observer for LPV systems with a known vector of scheduling parameters and additive disturbances is developed. It is based on a time varying change of coordinates. The LPV system is decomposed into a centered dynamic and a radius one. In addition, it is assumed that the system is subject to additive disturbances. According to these two dynamics (centered and radius dynamics), two sufficient conditions are given to ensure the stability of the bounds dynamics.

The paper is organized as follows. In section 2, the problem statement is formulated. Section 3 is devoted to some preliminaries about complex intervals and centered forms useful for the proposed techniques in section 4. Section 5 illustrates the efficiency of the interval observer through numerical simulations.

II. Problem Statement

Consider an LPV system described by:

$$\begin{cases} \dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) + E(t)d(t) \\ y(t) = Cx(t) + Fw(t) \\ x(0) \in [x^-(0), x^+(0)] \\ \forall t,\ d(t) \in [-1, +1]^p,\ w(t) \in [-1, +1]^r \\ \forall t,\ \rho(t) \in [-1, +1]^q \end{cases} \qquad (1)$$

The first author is a PhD student at the University of Bordeaux, IMS-CNRS, 351 cours de la libration, 33405 Talence, France, rihabelhouda.thabet@u-bordeaux1.fr. The second author is with CNAM, CEDRIC-LAB, 292 rue Saint Martin, 75141 Paris, tarek.raissi@cnam.fr. The third author is with ENSEA, ECS-Lab, 6 Avenue du Ponceau, 95014 Cergy, France, christophe.combastel@ensea.fr. The last author is with the University of Bordeaux, IMS-CNRS, France, ali.zolghadri@ims-bordeaux.fr. This work is done within the MAGIC-SPS project (Guaranteed Methods and Algorithms for Integrity Control and Preventive Monitoring of Systems) funded by the French National Research Agency (ANR) under the decision n ANR2011-INS-006.

$x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^s$ and $\rho(t) \in \mathbb{R}^q$ are respectively the state vector, the known input, the measurements and a vector of scheduling variables. $d(t) \in \mathbb{R}^p$, $E(t) \in \mathbb{R}^{n\times p}$, $F \in \mathbb{R}^{s\times r}$ and $w(t) \in \mathbb{R}^r$ denote respectively the additive disturbances, a known time-varying matrix, a known matrix and output disturbances. The disturbances as well as the initial state are assumed to be unknown but bounded with known vector bounds. $u(t)$ and $E(t)$ are assumed to be known and bounded. The formulation of the interval observer proposed for the system (1) is based on a centered dynamics which is obtained by decomposing the matrices A and B in (1) as follows:

$$\begin{array}{l} A(\rho(t)) = A_0 + \Sigma_{i=1}^{q} A_i\, \rho_i(t) \ \in \mathbb{R}^{n\times n} \\ B(\rho(t)) = B_0 + \Sigma_{i=1}^{q} B_i\, \rho_i(t) \ \in \mathbb{R}^{n\times m} \end{array} \quad (2)$$

The matrices A_i and B_i $(i = 1, \ldots, q)$ are constant and assumed to be known. $\rho_i(t)$, $(i = 1, \ldots, q)$, are the components of the scheduling variables vector. $A_0 \in \mathbb{R}^{n\times n}$ can be stable or unstable.

In this work, it is assumed that (A_0, C) is detectable and the non observable part is $\mathbb{C}$-diagonalizable (i.e. diagonalizable in the field of complex numbers). An observer gain L which makes the matrix $(A_0 - LC)$ Hurwitz and $\mathbb{C}$-diagonalizable is introduced. Based on this assumption, an interval observer with guaranteed time-varying bounds is designed in the following. Note that the case of a zero gain $(L = 0)$ corresponds to the design of an interval predictor rather than an interval observer as no measurement is then taken as input.

III. Complex intervals and centered forms

Interval analysis was initially developed to take into account the quantification errors introduced in the representation of real numbers on computers [17]. In this section, some preliminaries about complex intervals are given. Complex intervals are usually defined by rectangles or disks in the complex plane [16], [18]. A polar representation has also been, proposed in [20]. In the following, centered forms are introduced as well as the main notations which are used in this paper. More details about complex intervals are given in [11],[16].

The complex intervals used in this work rely on a partial order defined over $\mathbb{C}$ with three statements: $\forall(a, b) \in \mathbb{C} \times \mathbb{C}, a \star b \Leftrightarrow (a^R \star b^R) \wedge (a^I \star b^I)$ where $\star \in \{=, <, >\}$. Similar statements also hold with $\star \in \{\leq, \geq\}$. $a^R \in \mathbb{R}$ (resp. $a^I \in \mathbb{R}$) denotes the real part (resp. the imaginary part) of $a \in \mathbb{C}$ (idem for b). In the following, the same superscript notation is used to refer to the real and imaginary parts of scalar, vector or matrix complex arguments. A complex interval $[a, b]$ is defined as $[a, b] = [a^R, b^R] + i[a^I, b^I]$ if $a \leq b$, where $[a^R, b^R] = [a, b]^R$ and $[a^I, b^I] = [a, b]^I$ are usual real intervals. In this work, a centered form is used and an operator $\pm$ making it possible to represent a complex interval by a centered form is defined as:

$$\begin{array}{rcl} \pm : \mathbb{C} \times \mathbb{C}^+ & \rightarrow & \mathbb{IC} \\ (c, r) & \mapsto & c \pm r = [c - r, c + r] \end{array} \quad (3)$$

where $\mathbb{C}^+ = \{r \in \mathbb{C}, r \geq 0\}$ is the *positive* complex numbers set and $\mathbb{IC}$ is the set of scalar complex intervals. c and r respectively denote the center and the radius of the complex interval $c \pm r$. The used partial order ensures that $r \geq 0 \Leftrightarrow c - r \leq c + r$. The restriction of $\pm$ to real numbers is defined by analogy to (3) with $\pm : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{IR}$. Hence, $(c \pm r)^R = c^R \pm r^R$ and $(c \pm r)^I = c^I \pm r^I$.

Two operators namely *cabs* and *ctimes*, defined in [11], are used in this paper to compute $a(c \pm r)$. Their definitions for scalar values are as follows: *cabs*:$|a| = |a^R| + i|a^I|$ and *ctimes*: $a \diamond b = |a||b| + 2|a^I||b^I|$ where $|.|$ is the absolute value operator. The element-by-element extension of these operators to vectors and matrices is introduced in [11]. Then, the linear image of a complex interval matrix can be obtained by applying *Theorem* 1.

Theorem 1: $\forall(M, C, R) \in \mathbb{C}^{n\times p} \times \mathbb{C}^{p\times q} \times \mathbb{C}^{p\times q}$,
$M(C \pm R) = (MC) \pm (M \diamond R) \in \mathbb{IC}^{n\times q}$,
where $M \diamond R = |M||R| + 2|M^I|R^I| \in (\mathbb{C}^+)^{p\times q}$ □

The proof of *Theorem* 1 is given in [11]. Two technical propositions which have been proven in [9] are also used in this work. They are recalled hereafter.

Proposition 1:

$$\begin{array}{c} A \in \mathbb{C}^{n\times p}, B \in \mathbb{C}^{n\times q} : |[A, B]|\mathbf{1} = |A|\mathbf{1} + |B|\mathbf{1} \\ A \in \mathbb{C}^{n\times p} : |A|\mathbf{1} \leq \|A\|\mathbf{1}(1 + i) \\ A \in \mathbb{C}^{n\times p}, \varsigma \in (\mathbb{R}^+)^n : \|A diag(\varsigma)\|\mathbf{1} = \|A\|\varsigma \end{array} \quad (4)$$

where $\|.\|$, $\mathbf{1}$, $[., .]$ and $\leq$ respectively denote the element-by-element modulus of a complex matrix argument, a column vector with all elements equal to 1, the usual horizontal concatenation operator and the inferior or equal element-by-element relation operator defined over complex vectors and matrices. □

Proposition 2: [9] Let z: $\mathbb{R}^+ \rightarrow \mathbb{C}^n$, $z^c : \mathbb{R}^+ \rightarrow \mathbb{C}^n$ and $z^r : \mathbb{R}^+ \rightarrow (\mathbb{C}^+)^n$ be three continuous functions (with respect to time t). If $\forall t \in \mathbb{R}^+, z(t) \in z^c(t) \pm z^r(t)$ with $z^r(t) > 0$, then a continuous function $\sigma : \mathbb{R}^+ \rightarrow [-1, +1]^{2n}$ returning bounded real vectors values exists and satisfies (5) where the operator $\triangle(.)$ is defined as in (6):

$$\forall t \in \mathbb{R}^+, z(t) = z^c(t) + \triangle(z^r(t))\sigma(t) \quad (5)$$

$$\forall \upsilon \in \mathbb{C}^n, \triangle(\upsilon) = [diag(\upsilon^R), i.diag(\upsilon^I)] \in \mathbb{C}^{n\times 2n} \quad (6)$$

□

IV. Interval observer

To design a stable interval observer for a LTI system, the choice of an observer gain L such that $(\tilde{A}_0 = A_0 - LC)$ is only Hurwitz stable is not sufficient. Moreover, it is required that $\tilde{A}_0$ should be a Metzler matrix which is difficult to satisfy without introducing a change of variables. In recent works, suitable time-varying changes of coordinates have been proposed to overcome this difficulty [10], [20], and to obtain a monotone error dynamic. Notice that the LTV case has also been investigated in [12] where the goal is to find a time-invariant change of coordinates and a gain L in order to obtain a cooperative observation error at each time. Nevertheless, the design of L remains a difficult task.

In the following, the case of LPV systems is investigated. Based on some transformation in the LPV model expression, the time-varying change of coordinates previously introduced for LTI systems is used to build an interval observer which ensures the positivity of the observation error. The condition for interval observer stability introduced with LTI models is no more valid in this case and new stability conditions are proposed.

This section is organized as follows: first, a transformation of the LPV model (1) is given in paragraph A using the decomposition (2) and introducing an observer gain L. Then, a diagonalization of the matrix A_0-LC is performed in order to design the structure of an interval observer (i.e. a framer expressed in centered form) which is given in *Theorem* 2. The stability analysis of the designed interval observer is the subject of paragraph B.

A. LPV model transformation and interval observer structure

Based on (1) and (2), the system (1) can be rewritten as:

$$\begin{cases} \dot{x}(t) = A_0x(t) + B_0u(t) + E(t)d(t) + \\ \qquad (\Sigma_{i=1}^{q} A_i x(t)\rho_i(t)) + (\Sigma_{i=1}^{q} B_i u(t)\rho_i(t)) \\ y(t) = Cx(t) + Fw(t) \end{cases} \tag{7}$$

As it is assumed that (A_0, C) is detectable, it is possible to compute a gain L such that $(A_0 - LC)$ is Hurwitz. The system (7) can then be rewritten as:

$$\begin{aligned} \dot{x}(t) = (A_0 - LC)x(t) + B_0u(t) + E(t)d(t) - \\ LFw(t) + Ly(t) + (\Sigma_{i=1}^{q} A_i x(t)\rho_i(t)) + \\ (\Sigma_{i=1}^{q} B_i u(t)\rho_i(t)) \end{aligned} \tag{8}$$

By introducing the notations $\tilde{A}_0 = (A_0 - LC)$, $\tilde{B}_0 = [B_0, L]$, $\tilde{U}(t) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$, $\breve{E}(t) = [E(t), -LF]$ and $\breve{d}(t) = \begin{bmatrix} d(t) \\ w(t) \end{bmatrix} \in [-1, 1]^{p+r}$, the system (8) becomes:

$$\dot{x}(t) = \tilde{A}_0 x(t) + \phi(t, x(t)) \tag{9}$$

$$\begin{aligned} \text{where } \phi(t, x(t)) = \tilde{B}_0\tilde{U}(t) + \breve{E}(t)\breve{d}(t) + \\ (\Sigma_{i=1}^{q} A_i x(t)\rho_i(t)) + (\Sigma_{i=1}^{q} B_i u(t)\rho_i(t)) \end{aligned} \tag{10}$$

Assume that the gain L is chosen such that $\tilde{A}_0 \in \mathbb{R}^{n\times n}$ is $\mathbb{C}$-diagonalizable. A sufficient condition for $\tilde{A}_0$ to be $\mathbb{C}$-diagonalizable is that $\tilde{A}_0$ has no multiple poles, which can be achieved by a suitable choice of the gain L (pole placement) since the non observable part of (A_0, C) is assumed to be $\mathbb{C}$-diagonalizable. A necessary and sufficient condition to ensure this assumption is that each non observable mode of (A_0, C) has equal geometric and algebraic multiplicities. It should be noted that the eigenvalues of $\tilde{A}_0$ need not to be only real but can also be complex. Since $\tilde{A}_0 \in \mathbb{R}^{n\times n}$ is $\mathbb{C}$-diagonalizable, it can be expressed as:

$$\begin{aligned} \tilde{A}_0 = \nu^{-1}\, \theta\, \nu,\ \nu \in \mathbb{C}^{n\times n} \\ where\ \theta = diag(\xi + i\omega),\ \xi \in \mathbb{R}^n,\ \omega \in \mathbb{R}^n \end{aligned} \tag{11}$$

ξ and ω respectively denote the vector containing the real and the imaginary parts of the eigenvalues of $\tilde{A}_0$. The function $diag$ returns a diagonal matrix from its input vector. The Hurwitz stability of $\tilde{A}_0$ is equivalent to $\xi < 0$ (negative real part for the eigenvalues of $\tilde{A}_0$).

Note that the $\mathbb{C}$-diagonalization is not a restrictive assumption since a Jordan decomposition of $\tilde{A}_0$ can also be used without any additional investigation [10], [19]. Under some assumptions about (9) and using the diagonalization of $\tilde{A}_0$, an interval observer is proposed. Again, usually, the design of such observers assumes that $\tilde{A}_0$ is Metzler which is not usually the case. To overcome this difficulty, the time-varying change of coordinates:

$$z(t) = \Omega(t)x(t),\ \Omega(t) = diag(e^{-i\omega t})\nu \tag{12}$$

is introduced. The main result is now given by theorem 2.

Theorem 2: Given a system described by (9)-(10) with:
$A_1 : x(0) \in x^c(0) \pm x^r(0) \subset \mathbb{R}^n,\ z^c(0) = \nu x^c(0),\ z^r(0) = \nu \diamond x^r(0),\ x^r(0) \geq 0,\ z^r(0) \geq 0,$
$A_2 : \tilde{A}_0 \in \mathbb{R}^{n\times n}$ is a known $\mathbb{C}$-diagonalizable matrix,
$A_3 : u(t),\ E(t),\ d(t) \in [-1, +1]^p,\ w(t) \in [-1, +1]^r$ and $\rho(t) \in [-1, 1]^q$ are continuous w.r.t time t,
$A_4 : u(t)$ and $E(t)$ are known and bounded, $x^c(0)$ and $x^r(0)$ are known. The system

$$\begin{cases} \dot{z}^c(t) = (diag(\xi) + \Omega(t)(\Sigma_{i=1}^{q} A_i\rho_i(t))\Omega^{-1}(t)) \\ \qquad z^c(t) + \Omega(t)(\tilde{B}_0 + [\Sigma_{i=1}^{q} B_i\rho_i(t), 0])\tilde{U}(t) \\ \dot{z}^r(t) = diag(\xi)z^r(t) + |\Omega(t)\hat{E}(t, z^r(t))|\mathbf{1} \end{cases} \tag{13}$$

is an interval observer for (9)-(10) described by its center $(z^c(t))$ and radius $(z^r(t))$ state dynamics in the new base $(z(t))$ with

$$\begin{cases} \Omega(t) = diag(e^{-i\omega t})\nu,\ \Omega^{-1}(t) = \nu^{-1}diag(e^{i\omega t}) \\ \hat{E}(t, z^r(t)) = [\breve{E}(t)| \dots A_i\rho_i(t)\Xi(t) \dots] \\ \Xi(t) = \Omega^{-1}(t)\Delta(z^r(t)) \end{cases} \tag{14}$$

It satisfies the inclusion property given by (15) in the new base $(z(t))$ and by (16) in the original base $(x(t))$:

$$\forall t \in \mathbb{R}^+, z^r(t) \geq 0 \wedge z(t) \in z^c(t) \pm z^r(t) \subset \mathbb{C}^n \tag{15}$$

$$\forall t \in \mathbb{R}^+, x^r(t) \geq 0 \wedge x(t) \in x^c(t) \pm x^r(t) \subset \mathbb{C}^n \tag{16}$$

$$x^c(t) = \Omega^{-1}(t)z^c(t),\quad x^r(t) = \Omega^{-1}(t) \diamond z^r(t). \tag{17}$$

□

Based on the change of coordinates (12), theorem 2 gives the structure of an interval observer for the system (7), which makes use of the measurements $y(t)$ since $\tilde{U}(t) = [u(t);\ y(t)]$ in (13). The notation ";" refers to a vertical concatenation and $e^{(.)}$ refers to the elementwise exponential function, where each element of the input vector belongs to $\mathbb{C}$.

Proof. To prove Theorem 2, a transformation of (9) in the new base $(z(t))$ as $\dot{z}(t) = diag(\xi)z(t) + \psi(t)$ is first detailed and the expressions of $\psi^c(t)$ and $\psi^r(t)$ such that $\psi(t) \in \psi^c(t) \pm \psi^r(t)$ are given. Finally, using the new expression of (9), the obtained expressions $(\psi^c(t),\ \psi^r(t))$ and based on Theorem 3 in [11], the structure of the interval observer as well as the inclusion property are justified. It should be noted that the inclusion property does not depend on the observer stability.

• *The system dynamics in the new base:*

Using the change of coordinates $z(t) = \Omega(t)x(t)$ and based on theorem 1 in [11], the state equation (9) becomes:

$$\dot{z}(t) = diag(\xi)z(t) + \Omega(t)\phi(t, z(t)) \quad (18)$$

where $\phi(t, z(t))$ is defined as $\phi(t, z(t)) = \tilde{B}_0\tilde{U}(t) + \breve{E}(t)\breve{d}(t) + (\Sigma_{i=1}^{q} A_i\Omega^{-1}(t)z(t)\rho_i(t)) + (\Sigma_{i=1}^{q} B_iu(t)\rho_i(t))$. Theorem 1 in [11] justifies the equivalence between (9) and (18). Denote by β_i the influence of the i^{th} bounded scalar scheduling variable $\rho_i(t)$, with $i = 1, \cdots, q$, i.e. $\beta_i(t) = A_ix(t)\rho_i(t)$ in the original base. $\beta_i(t)$ is given by $\beta_i(t) = \Gamma_i(t)z(t)\rho_i(t)$ in the new base $(z(t))$, where $\Gamma_i(t) = A_i\Omega^{-1}(t)$. Using Proposition 2, the expression of $\beta_i(t)$ can be rewritten as in (19) where $\sigma(t) \in [-1, 1]^{2n}$.

$$\beta_i(t) = \Gamma_i(t)z^c(t)\rho_i(t) + \Gamma_i(t)\Delta(z^r(t))\sigma(t)\rho_i(t) \quad (19)$$

Then, (18) becomes:

$$\dot{z}(t) = diag(\xi)z(t) + \Omega(t)(\tilde{B}_0\tilde{U}(t) + \tilde{E}(t)\tilde{d}(t)) \quad (20)$$

$$\text{where } \tilde{E}(t)\tilde{d}(t) = \breve{E}(t)\breve{d}(t) + \Sigma_{i=1}^{q}(\Gamma_i(t)z^c(t) + B_iu(t))\rho_i(t) + \Gamma_i(t)\Delta(z^r(t))\sigma(t)\rho_i(t) \quad (21)$$

Based on the time-varying change of coordinates, we have $z^c(t) = \Omega(t)x^c(t)$ (the proof is given in [11]) and $\Gamma_i(t)z^c(t) = A_ix^c(t)$. Then, by replacing $\Gamma_i(t)z^c(t)$ by $A_ix^c(t)$ in (21), (20) becomes:

$$\dot{z}(t) = diag(\xi)z(t) + \Omega(t)(\tilde{B}_0\tilde{U}(t) + \Sigma_{i=1}^{q}(A_ix^c(t) + B_iu(t))\rho_i(t) + \hat{E}(t)\hat{d}(t)) \quad (22)$$

$$\text{with } \hat{E}(t)\hat{d}(t) = \breve{E}(t)\breve{d}(t) + A_i\Xi(t)\sigma(t)\rho_i(t) \; and \; \Xi(t) = \Omega^{-1}(t)\Delta(z^r(t)) \quad (23)$$

(22) can be rewritten as:

$$\dot{z}(t) = diag(\xi)z(t) + \Omega(t)(\Sigma_{i=1}^{q} A_i\rho_i(t))\Omega^{-1}(t) z^c(t) + \Omega(t)(\tilde{B}_0 + [\Sigma_{i=1}^{q} B_i\rho_i(t), 0])\tilde{U}(t) + \Omega(t)\hat{E}(t, z^r(t))\hat{d}(t) \quad (24)$$

$$\text{where } \hat{E}(t, z^r(t)) = [\breve{E}(t)| \ldots A_i\rho_i(t)\Xi(t) \ldots] \quad (25)$$

$$\hat{d}(t) = [\breve{d}(t); \; \sigma(t)] \; \in [-1, 1]^{p+r+2n} \quad (26)$$

Finally, the expression (24) can be rewritten as follows:

$$\dot{z}(t) = diag(\xi)z(t) + \psi(t) \quad (27)$$

with $\psi(t) = \Omega(t)(\Sigma_{i=1}^{q} A_i\rho_i(t))\Omega^{-1}(t)z^c(t) + \Omega(t)(\tilde{B}_0 + [\Sigma_{i=1}^{q} B_i\rho_i(t), 0])\tilde{U}(t) + \Omega(t)\hat{E}(t, z^r(t))\hat{d}(t)$.

• *Expressions of $\psi^c(t)$ and $\psi^r(t)$:*

Using (27) and $\hat{d}(t) \in [-1, +1]^{p+r+2n}$, then $\psi(t) \in \Omega(t)(\Sigma_{i=1}^{q} A_i\rho_i(t))\Omega^{-1}(t)z^c(t) + \Omega(t)(\tilde{B}_0 + [\Sigma_{i=1}^{q} B_i\rho_i(t), 0])\tilde{U}(t) + \Omega(t)\hat{E}(t, z^r(t))(\mathbf{0} \pm \mathbf{1})$. Based on Theorem 1, the previous expression can be expressed as: $\psi(t) \in \psi^c(t) \pm \psi^r(t)$ with $\psi^c(t) = \Omega(t)(\Sigma_{i=1}^{q} A_i\rho_i(t))\Omega^{-1}(t)z^c(t) + \Omega(t)(\tilde{B}_0 + [\Sigma_{i=1}^{q} B_i\rho_i(t), 0])\tilde{U}(t)$ and $\psi^r(t) = |\Omega(t)\hat{E}(t, z^r(t))|\mathbf{1}$.

• Verifying the following assumptions

$A_1 : \dot{z}(t) = diag(\xi)z(t) + \psi(t),$
$A_2 : z(t) \in \mathbb{C}^n, z^r(0) \geq 0, z(0) \in z^c(0) \pm z^r(0),$
$A_3 : \psi^c(t) \in \mathbb{C}^n \; and \; \psi^r(t) \in \mathbb{C}^n \; are \; continuous \; w.r.t \; t,$
$A_4 : \psi^r(t) \geq 0 \; and \; \psi(t) \in \psi^c(t) \pm \psi^r(t),$
$A_5 : obs^c \; and \; obs^r \; are \; two \; continuous \; time \; complex \; dynamical \; systems \; respectively \; defined \; as \; in \; (28) \; and \; (29) :$

$$\dot{z}^c(t) = diag(\xi)z^c(t) + \psi^c(t) \quad (28)$$

$$\dot{z}^r(t) = diag(\xi)z^r(t) + \psi^r(t) \quad (29)$$

theorem 3 in [11] shows that $obs = (obs^c, obs^r)$ is an interval observer for the system defined in A_1 verifying the inclusion property (15). Thus, Theorem 2 is proven. □

B. Stability of the interval observer

Even if the inclusion property (15) ensures that the states trajectories stay within the calculated bounds, the stability of the resulting interval observer cannot be directly deduced from the stability of $\tilde{A}_0$. On the one hand, the center dynamics $(z^c(t))$ depends on $\Sigma_{i=1}^{q} A_i\rho_i(t)$, where $\rho_i(t)$ are time-varying variables. On the other hand, the radius dynamics $z^r(t)$ depends on $\hat{E}(t)$ i.e. $\hat{E}(t) = \hat{E}(t, z^r(t))$ depends on the state $z^r(t)$ of the interval observer. The stability of the interval observer derives from the stability of the dynamics described by (13). That is why, a design condition ensuring the stability of the interval observer center dynamics $z^c(t)$ as well as a sufficient condition ensuring the non-divergence of the interval observer radius dynamics $z^r(t)$ are given in the following. The design condition ensuring the stability of the center dynamics $z^c(t)$ is established in the original base as shown in the following. Thus, a backward transformation of the first equation of (13) in the original base $x(t)$ is given by corollary 1.

Corollary 1: The center dynamic of the interval observer, in the original base $(x(t))$ is described by (30):

$$\dot{x}^c(t) = (A_0 - LC + \Sigma_{i=1}^{q} A_i\rho_i(t))x^c(t) + (\tilde{B}_0 + [\Sigma_{i=1}^{q} B_i\rho_i(t), 0])\tilde{U}(t) \quad (30)$$

□

Proof. The expression (30) is justified by using the time-varying change of coordinates $z(t) = \Omega(t)x(t)$. In fact, we have $z^c(t) = \Omega(t)x^c(t)$ which implies $\dot{z}^c(t) = diag(-i\omega)\Omega(t)x^c(t) + \Omega(t)\dot{x}^c(t)$. Replacing $\dot{z}^c(t)$ by its expression in the first equation of (13) we obtain $diag(-i\omega)\Omega(t)x^c(t) + \Omega(t)\dot{x}^c(t) = (diag(\xi) + \Omega(t)(\Sigma_{i=1}^{q} A_i\rho_i(t))\Omega^{-1}(t))\Omega(t)x^c(t) + \Omega(t)(\tilde{B}_0 + [\Sigma_{i=1}^{q} B_i\rho_i(t), 0]))\tilde{U}(t)$. Then, left multiplying each side of this equality by $\Omega^{-1}(t)$ and replacing $diag(i\omega) + diag(\xi)$ by θ we obtain $\dot{x}^c(t) = \Omega^{-1}(t)\theta\Omega(t)x^c(t) + (\Sigma_{i=1}^{q} A_i\rho_i(t))x^c(t) + (\tilde{B}_0 + [\Sigma_{i=1}^{q} B_i\rho_i(t), 0]))\tilde{U}(t)$. Finally, by using (11), the last expression becomes $\dot{x}^c(t) = \tilde{A}_0x^c(t) + (\Sigma_{i=1}^{q} A_i\rho_i(t))x^c(t) + (\tilde{B}_0 + [\Sigma_{i=1}^{q} B_i\rho_i(t), 0]))\tilde{U}(t)$ and (30) is obtained. □

• *Stability of center dynamic $x^c(t)$:*

The error dynamic related to (30) is stable if $\dot{e}(t) = (A_0 - LC + \Sigma_{i=1}^{q} A_i \rho_i(t))e(t)$ is stable where $e(t) = x(t) - x^c(t)$ refers to an observation error term. In this work, $\rho_i(t)$ is assumed to be bounded and the stability of the interval observer center dynamic results from the stability of:

$$\dot{x}(t) = (A_0 - LC + \Sigma_{i=1}^{q} A_i \rho_i(t))x(t) \qquad (31)$$

Let a Lyapunov function be defined by $V(x(t)) = x^T(t)Px(t)$. This quadratic function guarantees the stability of (31) if there exists a matrix $P = P^T \succ 0$ such that:

$$\begin{array}{c}(A_0 + \Sigma_{i=1}^{q} A_i \rho_i(t) - LC)^T P + P(A_0 + \\ \Sigma_{i=1}^{q} A_i \rho_i(t) - LC) \prec 0\end{array} \qquad (32)$$

where $\prec$ indicates a negative definite matrix. By introducing the notation $X = PL$ (with $X^T = L^T P$), the LMI relaxation of equation (32) becomes:

$$A^T(t)P + PA(t) - C^T X^T - XC \prec 0 \qquad (33)$$

where $A(t) = A_0 + \Sigma_{i=1}^{q} A_i \rho_i(t)$. The centered affine matrix form $(A_0 + \Sigma_{i=1}^{q} A_i \rho_i(t))$ can be expressed in the polytopic form using Lemma 1 given as follows.

Lemma 1: The following statements are equivalent:
$S_1 : \exists \rho(t) \in [-1, 1]^q,\ A(t) = A_0 + \Sigma_{i=1}^{q} A_i \rho_i(t)$
$S_2 : \exists \alpha(t) \in (\mathbb{R}^+)^N,\ A(t) = \Sigma_{j=1}^{N} \alpha_j(t) \hat{A}_j,\ \Sigma_{i=1}^{N} \alpha_j(t) = 1$
where $\hat{A} = \{A_0 + \Sigma_{i=1}^{q} A_i \eta_i,\ \eta \in \{-1, 1\}^q\}$, $N = 2^q$ and $\hat{A}_j$ denotes the j^{th} elements of $\hat{A}$, $j = 1, \ldots, N$. □
Lemma 1 states an equivalence result between the affine representation of interval matrix uncertainties used in this paper (S_1) and polytopic uncertainties (S_2). A proof of this lemma directly follows from the link between $\rho(t)$ and $\alpha(t)$ that results from a convex polytopic transformation obtained by taking the scalar elements of $\rho(t)$ as premise variables [20]. Using Lemma 1, the inequality (33) is rewritten as

$$\Sigma_{j=1}^{N} \alpha_j(t)(\hat{A}_j^T P + P\hat{A}_j - C^T X^T - XC) \prec 0 \qquad (34)$$

$$\hat{A}_j^T P + P\hat{A}_j - C^T X^T - XC \prec 0,\ \forall j = 1, \ldots, N \qquad (35)$$

As $\alpha_j(t) > 0$ for $j = 1, \ldots, N$, if (35) is verified then (34) is also true. As a result, the design condition ensuring the stability of the interval observer center dynamic in the original base $x(t)$ is given by the following proposition.

Proposition 3: A system described by the first equation in (13) or, equivalently, by (30) is stable if there exists two matrices $P = P^T \succ 0$ and X such that $\hat{A}_j^T P + P\hat{A}_j - C^T X^T - XC \prec 0,\ \forall j = 1, \ldots, N$ where $\hat{A}_j$ is defined in Lemma 1 and $L = P^{-1}X$. □

• *Non divergence of radius dynamic $z^r(t)$:*
A sufficient condition ensuring the non divergence of the interval observer radius dynamics $z^r(t)$ is given by *Proposition* 4.

Proposition 4: Let $S = \Sigma_{i=1}^{q} \|\upsilon A_i \upsilon^{-1}\| \in (\mathbb{R}^+)^{n\times n}$ with $\rho(t) \in [-1, 1]^q$ and $\|.\|$ denotes the elementwise modules of a complex matrix argument. If $\xi < 0$ ($\tilde{A}_0$ is Hurwitz stable) and if the Metzler matrix $[(diag(\xi)+S), S; S, (diag(\xi)+S)]$ is Hurwitz stable, then $\forall t, 0 \le z^r(t) \le \bar{z}^r(t)$ and $\bar{z}^r(t)$ follows a stable dynamic. □
The proof of this proposition is similar to the proof of *Theorem* 8 in [9]. To conclude, the interval observer given in *Theorem* 2 is stable if the *Propositions* 3 and 4 are satisfied.

Remark. A_0 being stable, (13) reduces to a simple predictor related to (9) when $L = 0$ (null observer gain).

V. NUMERICAL EXAMPLE

To illustrate the proposed methodology, let us consider the LPV system with additive disturbances described by (36).

$$\left\{\begin{array}{l}\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) + E(t)d(t) \\ y(t) = Cx(t)\end{array}\right. \qquad (36)$$

where $x(t) \in \mathbb{R}^2$ is the state, $u(t)$ is the known input equal to one ($u(t) = 1$), $d(t) \in [-1, +1]^2$ is the disturbance, $E(t) = [0.2, 0;\ 0, 0.5] \in \mathbb{R}^{2\times 2}$, $B(\rho(t)) = B_0 = [0.1; 0.1] \in \mathbb{R}^{2\times 1}$, $C = [0, 1] \in \mathbb{R}^{1\times 2}$. For simulation, $A(\rho(t))$ is chosen as:

$$\begin{array}{rcl}A(\rho(t)) & = & \left[\begin{array}{cc}-0.632 - 0.16sin(t) & 0.1cos(3t) \\ -0.14cos(2t) & 0.06sin(t)\end{array}\right] \\ & = & A_0 + \Sigma_{i=1}^{3} A_i \rho_i(t)\end{array}$$

where $\rho(t) = [\rho_1(t);\ \rho_2(t);\ \rho_3(t)] = [sin(t);\ cos(3t);\ cos(2t)]$, $A_0 = [-0.632, 0;\ 0, 0]$, $A_1 = [-0.16, 0;\ 0, 0.06]$, $A_2 = [0, 0.1;\ 0, 0]$ and $A_3 = [0, 0;\ -0.14, 0]$. Since A_0 is not Hurwitz, an observer gain $L = [0;\ 4.368]$ satisfying the propositions 3 and 4 is introduced. The system (36) is similar to (9) where $\tilde{A}_0 = [-0.632, 0;\ 0, -4.368]$, $\tilde{B}_0 = [0.1, 0;\ 0.1, 4.368]$, $\tilde{U}(t) = [1;\ y(t)]$, $\breve{E}(t) = E(t)$ ($F = 0$) and $\breve{d}(t) = d(t)$. The initial observer state bounds, in the original base ($x(t)$) are $x^-(0) = x^c(0) - x^r(0)$ and $x^+(0) = x^c(0) + x^r(0)$ where $x^c(0) = [1.4;\ 0.05]$ and $x^r(0) = [1.4;\ 1.45]$.

Even if the system (36) is unstable, by introducing a gain L verifying the conditions in *Proposition* 3, the obtained $\tilde{A}_0$ is stable and diagonalizable. All assumptions of theorem 2 are valid and the results of simulation between $t = 0$ and $t = 50$ (with a step size h=0.001) of the designed interval observer, in the original base ($x(t)$), are reported in Fig.1 for both states $x_1(t)$ and $x_2(t)$.

The interval observer bounds $(\underline{x}_1, \bar{x}_1)$ and $(\underline{x}_2, \bar{x}_2)$ are consistent with the stability properties obtained through the use of propositions 3 and 4. Considering $x^c(t)$ as a point estimate for the state $x(t)$, the observation error dynamics is stable even the studied system is not. In addition, $x^r(t)$ is also positive and bounded by $\bar{x}^r(t)$ which follows a stable dynamic. Moreover, as shown in Fig.1, the inclusion property is verified: $x_1(t) \in [\underline{x}_1, \bar{x}_1]$ and $x_2(t) \in [\underline{x}_2, \bar{x}_2]$, as it is expected from an interval observer.

VI. CONCLUSION

This work deals interval estimation for LPV systems subject to additive disturbances. The structure of such observer is based on a time-varying change of coordinates under some mild assumptions. A design condition of an observer gain L ensuring the positivity and the stability of the estimation error dynamics is given. This methodology can be useful to take nonlinear systems through LPV transformations.

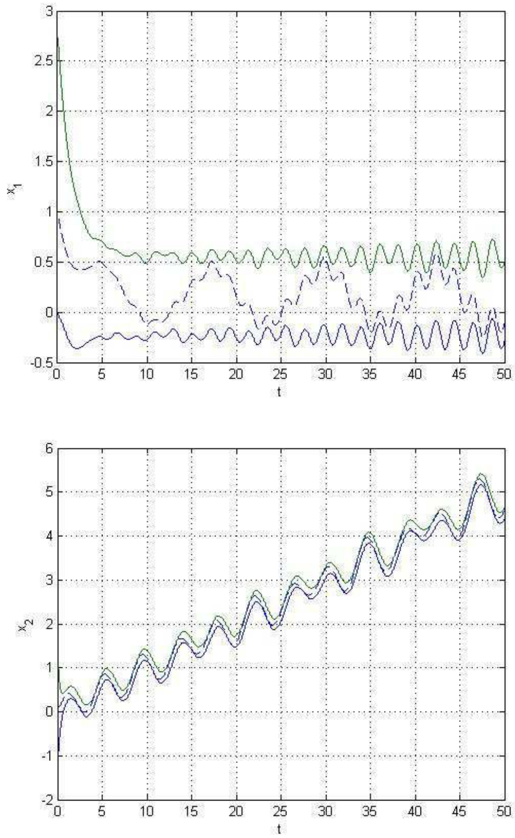

Fig. 1. Simulation results: LPV system state and time-varying interval enclosing the set of possible state values

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