2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Linear reformulation of the Kuramoto model: asymptotic mapping and stability properties

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Abstract— **For the classical "all-to-all" Kuramoto model, we construct a family of auxiliary linear models that preserves information on the natural frequencies and interconnection gains of the original Kuramoto model and depends on its phase-locked solutions. Stability properties of the family of linear systems are analyzed, we show that there is only one system in this family which is stable and for almost all initial conditions its solutions exponentially converge to a stable periodic limit cycle. Finally, we show that asymptotically, in the time limit, this linear system maps on the original Kuramoto model.**

I. INTRODUCTION

During the last decade control and synchronization of complex networks gained a lot of interest in the control community, especially due to problems coming from various fields of science and engineering, such as physics, biology, social sciences, medicine, *etc*. In particular, behavior of complex networks, such as power grids [5], vehicle formations [14], social and neuronal networks [17], [7] has been at the center of attention.

The Kuramoto model of coupled phase oscillators [9] is probably the most simple model used to represent a network activity where activity of a single agent is seen just as an oscillator of fixed frequency at the price of losing all relevant information about its possibly complex dynamics. Even though the model is very simplistic, it captures the intrinstic charactersitics of the activity of many networks, in particular, it describes emergence of synchronization in a group of interacting agents and generation of large scale oscillations.

Different issues of synchronization for the Kuramoto model have been addressed by researches coming from physics, dynamical systems and control communities. For example, for the Kuramoto model with the all-to-all interconnections, existence and stability of the phase locked solutions were addressed in [2], [18], while ring and bipartite structure of the interconnections were considered in [3], [19]. Exponential frequency synchronization for the Kuramoto model was demonstrated in [4] for the all-to-all interconnections and in [8], [5] for the symmetric but otherwise arbitrary structure of interconnections. Detailed overview of various aspects of the Kuramoto model and conditions for its synchronization are given *e.g.* in [9], [15], [16], [6].

In general, the dynamic behavior of the Kuramoto model is very complex and difficult to analyse hence several attempts have been made to simplify the analysis. For example, in [13] a linear model was introduced that asymptotically can can be reduced to a Kuramoto model. However, the resulting reduced model obtained in [13] has a structure of interconnections that cannot be imposed beforehand and is different from the initial model.

Starting with the same idea we construct a family of auxiliary linear systems that preserve information on the natural frequencies and interconnection gains of the original Kuramoto model, analyse their stability properties and show that there is only one system in the family that is stable. Finally, we show that asymptotically, this linear system has a limit cycle and maps on the Kuramoto model with the same interconnection gains and natural frequencies.

Notations. Throughout the paper $\mathbf{1}_n$ denotes the n-dimensional vector of 1's, i.e., $\mathbf{1}_n^\top = [1, \dots, 1]$. Given a vector $a = col(a_1 \dots a_n) \in \mathbb{R}^n$, we will use the notations $diag\{a_1, \dots, a_n\}$ or $diag\{a\}$ for a diagonal matrix with elements a on the main diagonal. For the eigenvalues of a matrix $M \in \mathbb{C}^{n \times n}$ we will use the notation $\lambda(M)$, while $I_N = \{1, \dots, N\}$.

II. LINEAR REFORMULATION OF THE KURAMOTO MODEL : PROBLEM FORMULATION

Consider the classical all-to-all Kuramoto model for a network of N coupled oscillators [9]:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \qquad i \in I_N, \tag{1}$$

where $\omega_i \in \mathbb{R}$, $i \in I_N$ is the vector of natural frequencies and parameter $K > 0$ is the coupling strength.

In this paper we are interested in the following problem: find a linear system (of a complex variable) such that asymptotically, in the time limit, it can be reduced to the Kuramoto model (1) and compare its stability properties with those of Kuramoto model.

The underlying idea of our approach can be summarized as follows. Consider a parametrized (in terms of μ) linear system in the form

$$\dot{x}_k = (i\omega_k - \mu)x_k + \frac{K}{N}\sum_{j=1}^{N} x_j, \tag{2}$$

where $x_k \in \mathbb{C}$, $\mu_k \in \mathbb{R}$ and $k \in I_N$. Equivalently, this system can be written in the matrix form

$$\dot{\mathbf{x}} = \left(\frac{K}{N}\mathbf{1}\mathbf{1}^\top + i\Omega - \mathtt{M}\right)\mathbf{x} = A\mathbf{x}, \tag{3}$$

where $\Omega = diag\{\omega\}$ and $\mathtt{M} = diag\{\mu\} \in \mathbb{R}^{N\times N}$. Remark that except for the main diagonal, this matrix has the same structure of interconnections as the Kuramoto model (1).

Using polar coordinates transformation $x_k = R_k e^{i\theta_k}$ the system (2) can be rewritten in the form

$$\dot{R}_k e^{i\theta_k} + iR_k e^{i\theta_k}\dot{\theta}_k = (i\omega_k - \mu)R_k e^{i\theta_k} + \frac{K}{N}\sum_{j=1}^{N} R_j e^{i\theta_j}. \tag{4}$$

Separating in the last equation real and imaginary parts and multiplying both sides by $\exp(-i\theta_k)$ we obtain the following equation corresponding to the imaginary part

$$R_k(t)\dot{\theta}_k = \omega_k R(t) + \frac{K}{N}\sum_{j=1}^{N} R_j(t)\sin(\theta_j - \theta_k).$$

From the last equation it is easy to see that existence of an $R > 0$ such that $\lim_{t\to\infty} R_j(t) = R \neq 0$ for all $j \in I_N$, would imply that asymptotically dynamics of θ is described by the Kuramoto model (1). To ensure asymptotic convergence of $R_j(t)$ to the same value it is enough to guarantee that the matrix A possesses the following properties:

P1: Matrix A has a purely imaginary eigenvalue and all elements of the corresponding right and left eigenvectors, taken in polar coordinates, have the same amplitude, i.e. there exists ω^* and $\phi \in \mathbb{R}^N$ such that $\lambda_1(A) = i\omega^*$ and

$$v_{1r}(A) = col(e^{i\phi_1}, \ldots, e^{i\phi_N}), \quad v_{1l}(A) = \bar{v}_{1l}(A),$$

satisfy the equalities

$$Av_{1r} = \lambda_1 v_{1r} \quad \text{and} \quad v_{1l}^* A = \lambda_1 v_{1r}^*.$$

P2: $N-1$ eigenvalues of the matrix A have negative real parts.

Intuition behind these requirements is as follows. Any matrix A that has properties **P1**, **P2** can be presented in the Jordan form as

$$A = SJS = S\left[\begin{array}{c|c}\lambda_\circ & \\ \hline & J_1\end{array}\right]S^{-1} = S\left[\begin{array}{c|c}i\omega_s & \\ \hline & J_1\end{array}\right]S^{-1}$$

where $J_1 \in \mathbb{R}^{(N-1)\times(N-1)}$ is the part of the Jordan matrix corresponding to $N-1$ eigenvalues with negative real parts, S is a $N\times N$ matrix whose columns are eigenvectors and generalized eigenvectors of A, in particular, the first column is the vector v_{1r}.

Using this representation of the matrix A, solutions of the system 3 can be written explicitly as

$$\mathbf{x}(t, \mathbf{x}_\circ) = Se^{Jt}S^{-1} = S\left[\begin{array}{c|c}e^{i\omega_s t} & 0 \\ \hline 0 & e^{J_1 t}\end{array}\right]S^{-1}\mathbf{x}_\circ.$$

Since the matrix J_1 corresponds to the eigenvalues with negative real part, asymptotically we have that $lim_{t\to\infty} e^{J_1 t} = 0$ and therefore behavior of $\mathbf{x}(t)$ asymptotically is defined only by the term $e^{i\omega_s t} v_{1r} v_{1r}^\top \mathbf{x}_\circ = c(\mathbf{x}_\circ)e^{i\omega_s t} v_{1r}$. Therefore asymptotically all the trajectories $\mathbf{x}_j(t)$ will be periodic and, in view of the property **P1**, will converge to the same limit cycle and hence the linear system (3) can be asymptotically mapped on the Kuramoto model.

In summary, in order to find such a linear system we need to find a vector $\mu \in \mathbb{R}^N$, or equivalently a matrix $\mathtt{M} = diag(\mu)$, such that the matrix $A = \frac{K}{N}\mathbf{1}\mathbf{1}^\top + i\Omega - \mathtt{M}$ satisfies the properties **P1, P2**. As a first step in this direction, we show next that there exist a finite number of vectors $\mu \in \mathbb{R}^N$ which define matrix A with the property **P1**.

We start by introducing the following assumption on the vector of the natural frequencies $\boldsymbol{\omega} \in \mathbb{R}^N$ and the interconnection gain K.

Assumption **A** *1:* Let parameter $K > 0$ and vector $\omega \in \mathbb{R}^N$ be given and denote $\omega_m = \frac{1}{N}\sum i = 1^N \omega_i$ and $\tilde{\omega}_i = \omega_i - \omega_m$ for all $i \in I_N$. We assume that among the following set of equations[1]

$$r_\infty = \frac{1}{N}\sum_{i=1}^{N} \pm\sqrt{1 - \left(\frac{\tilde{\omega}_i}{Kr_\infty}\right)^2}, \tag{5}$$

there is at least one equation that has a solution $r_\infty \in (0, 1]$ satisfying the following inequalities for all $i \in I_N$

$$-1 \leq \frac{\tilde{\omega}_i}{Kr_\infty} \leq 1. \tag{6}$$

Remark 1: Originally introduced in [9] with only positive signs, espression (5) is known in the literature on the Kuramoto model as the *consistency condition on* r_∞ of the phase locked solutions. Later, in [2] it was shown that Assumption 1 is a necessary and sufficient condition for existence of phased locked solutions, see also [2], [15], [18] for the detailed discussion on the subject.

As it was remarked in [2], the overal number of equations in (5) is 2^N, however, part of these equations does not have solutions in the interval $(0, 1]$, part of them would possibly have several solutions, that is why we denote by M the overall number of solutions r_∞ satifying the assumption 1 and by $\mathcal{R}_{K\omega} = \{r_\infty^j,\ \ j = 1, \ldots, M\}$ the set of all solutions of (5) satisfying the inequalities (6).

Given an $r_\infty^j \in \mathcal{R}_{K\omega}$, we define the corresponding vector $\mu_j \in \mathbb{R}^N$ as

$$\mu_i = \pm\sqrt{(Kr_\infty)^2 - (\tilde{\omega}_i)^2}, \tag{7}$$

where the choice of $\pm$ signs coincides with the corresponding choice of signs in (5). We will denote by $\mathcal{M}_{K\omega}$ the set of all thus defined vectors mu and by $\mathcal{A}_{K\omega}$ the corresponding set of matrices A, i.e. $\mathcal{A}_{K\omega} = \{A \in \mathbb{R}^{N\times N} :\quad A = \frac{K}{N}\mathbf{1}\mathbf{1}^\top + i\Omega - diag\{\mu^j\},\ \text{where}\ \ \mu^j \in \mathcal{M}_{K\omega}\}$.

[1]The $\pm$ sign indicates that each term in the sum can have either plus or minus sign and therefore (5) represents *a set of equations*.

We show next that assumption 1 is a necessary and sufficient conditions for the existence of a family of matrices A with the propoerty **P1**.

Theorem 1: Let $\omega \in \mathbb{R}^N$ and t $K > 0$ be given and satisfy the assumption 1. Then the matrix $= \frac{K}{N}\mathbf{1}\mathbf{1}^\top - \mathtt{M} + i\Omega$ satisfies the property **P1** if and only if $A \in \mathcal{A}_{K\omega}$. Moreover, $\lambda_1(A) = i\omega_m$ and elements of the eigenvector $\mathbf{v}_{1r}$ are defined as

$$\mathbf{v}_{1j} = e^{i\phi_j} = \frac{1}{Kr_\infty}(\mu_j + i\tilde{\omega}_j). \tag{8}$$

Proof. See the appendix.

Discussion. Comparing theorem 1 above and the theorem 1 of [2] it is easy to see that the eigenvector $\mathbf{v}_{1r}$ defined in (8) coincides with the expression for the phase locked solution in [2], see equations (2.19) in that article. Thus, as a result of the teorem 1 we obtained a set of matrices $\mathcal{A}_{K\omega}$ which are defined by the equilibrium points of the Kuramoto model. In the next section we'll analyse properties of the corresponding family of linear systems.

Let $\Phi = diag\{e^{i\phi}\}$ be a diagonal matrix defined by the eigenvector v_{1r} , then given a vector $\mu \in \mathcal{M}_{K\omega}$ and denoting $\tilde{\Omega} = diag\{\omega - \omega_m\}$, the matrix A can be rewritten in the following way

$$\begin{aligned} A &= \frac{K}{N}\mathbf{1}\mathbf{1}^\top - (\mathtt{M} - \tilde{\Omega}) + i\omega_m I_N \\ &= \frac{K}{N}\mathbf{1}\mathbf{1}^\top - diag\{\mu - i\tilde{\omega}\} + i\omega_m I_N = \frac{K}{N}\mathbf{1}\mathbf{1}^\top \\ &= \frac{K}{N}\mathbf{1}\mathbf{1}^\top - Kr_\infty \Phi^* + i\omega_m I_N, \end{aligned} \tag{9}$$

where we added and subtracted term $i\omega_m I$ in the first line and used (8) and the definition of Φ in the last line. For simplicity of future notations we introduce the matrix A_1 as follows

$$A_1 = \frac{K}{N}\mathbf{1}\mathbf{1}^\top - Kr_\infty \Phi^*, \tag{10}$$

with this notation $A = A_1 + i\omega_m I_N$.

III. Intrinsic properties of systems defined by set $\mathcal{A}_{K\omega}$

In the previous section we obtained a finite set of matrices $\mathcal{A}_{K\omega}$, each of them has an imaginary eigenvalue $\lambda_1(A) = i\omega_m$ and the corresponding eigenvector has the form $\mathbf{v}_1 = \mathrm{col}(e^{i\phi_1}, \dots, e^{i\phi_N})$. We consider now the family of linear systems defined by $\mathcal{A}_{K\omega}$

$$\dot{\mathbf{x}} = A\mathbf{x} = A_1\mathbf{x} + i\omega_m\mathbf{x}, \quad A \in \mathcal{A}_{K\omega} \tag{11}$$

and analyze its stability properties.

Similar to the notion of *phase order parameter* $re^{i\psi} \equiv \frac{1}{N}\sum_{j=1}^N e^{i\theta_j}$ which plays an important role in the analysis of the Kuramoto model [15], [2], [18], we introduce a notion of *a weighted order parameter* in the following way.

Definition 1: Let $A \in \mathcal{A}_{K\omega}$ and $\mathbf{v}_1 = \mathbf{e}^{\mathbf{i}\phi}$ be the corresponding eigenvector. Then weighted order parameter $z \in \mathbb{C}$ for the system $\dot{\mathbf{x}} = A\mathbf{x}$ is defined as

$$z = Re^{i\psi} = \sum_{j=1}^N e^{i\phi_j}\mathbf{x}_j = \mathbf{1}^\top \Phi \mathbf{x}, \tag{12}$$

where $\Phi = diag\{e^{i\phi_1}, \dots, e^{i\phi_N}\}$.

It is well known that in the coordinate frame rotating with the frequency ω_m, Kuramoto model has an invariant set defined by equality $\sum_{j=1}^N \theta_j(t, \theta_\circ) = \sum_{j=1}^N \theta_{\circ j}$. We show next that the amplitude R of the weighted order parameter z is invariant for the set of systems (11).

Proposition 1: Consider a linerar system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A \in \mathcal{A}_{K\omega}$, $\mathbf{x}(0) = \mathbf{x}_\circ$. Let $\mathbf{v} = \mathbf{e}^{i\phi}$ be the eigenvector corresponding to $\lambda(A) = i\omega_m$, then weighted order parameter $z = \mathbf{1}^\top \Phi \mathbf{x}$ is a periodic function of time with a period ω_m and $z(t, z_\circ) = z_\circ e^{i\omega_m t}$.

Scketch of the proof.

Differentiating z along trajectories of the system (11), we obtain

$$\begin{aligned} \dot{z} &= \frac{K}{N}\mathbf{1}^\top \Phi \mathbf{1}\mathbf{1}^\top \mathbf{x} - Kr_\infty \mathbf{1}^\top \mathbf{x} + i\omega_m \mathbf{1}^\top \Phi \mathbf{x} \\ &= i\omega_m \mathbf{1}^\top \Phi \mathbf{x} = i\omega_m z, \end{aligned} \tag{13}$$

where in the first line we used the definition of z and the property $\mathbf{1}^\top \Phi \mathbf{1} = \sum_{j=1}^N e^{i\phi_j} = Nr_\infty$ that make the first two terms in the sum cancel.

The rest of the proof is straightforward and is omitted due to space limitations.

Since weighted order parameter is a periodic function of time and its amplitude is constant, it is reasonable to expect that solutions of the system (11) don't converge to an equilibrium point. It is easy to show that solutions of the system (11) converge to an equilibrium point only for the set of initial conditions that has measure zero and is defined by the equality $\mathbf{1}^\top \Phi \mathbf{x}_\circ = 0$.

That is why in the sequel we concentrate on the stability analysis stability of the limit cycles defined by weighted order parameter z. In particular, we show that among all the systems (11) with the matrices from the set $\mathcal{A}_{K\omega}$, there is only one system for which the limit cycle is stable while for others it is unstable.

IV. Stability analysis of linear systems defined by set $\mathcal{A}_{K\omega}$

We analyze next properties of the set of linear systems (11) and show that there is a strong link between stability properties of linear systems defined by $\mathcal{A}_{K\omega}$ and those of corresponding phased locked solutions of Kuramoto model.

For linear systems $\dot{\mathbf{x}} = L\mathbf{x}$, where L is a symmetric Laplacian matrix $L = L^\top \in \mathbb{R}^{n\times n}$, the consensus value, i.e. the quantity $\mathrm{Ave}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^n = \mathbf{x}_i \mathbf{1}^\top \mathbf{x}$ is invariant, that is $\mathbf{1}^\top \mathbf{x}(t, \mathbf{x}_\circ) = \mathbf{1}^\top \mathbf{x}_\circ$, see e.g. [12]. This property appears to be crucial in the consensus stability analysis, namely, the system dynamics is projected onto the subspace orthogonal to the eigenvalue $\mathbf{1}$ by introducing the *group disagreement vector* [2]

$$\delta = \mathbf{x} - \text{Ave}(\mathbf{x})\mathbf{1} \tag{14}$$

and further stability of the set $\delta = 0$ is analyzed.

Following this idea we project dynamics of the system (11) onto the subspace orthogonal to the vector $\mathbf{v}_{1r}$ and similar to (14) we indroduce the disagreement vector $\delta \in \mathbb{C}^N$ as

$$\delta = \mathbf{x} - cz(\mathbf{x})\mathbf{v}_{1r},$$

where $c = 1/\mathbf{1}^\top\Phi^2\mathbf{1}$ and $z = \mathbf{1}^\top\Phi\mathbf{x}$ is the weighted order parameter, its role here is similar to that of $\text{Ave}(\mathbf{x})$ in the consensus problems. Direct calculations show that thus defined vector δ satisfies the equalities $\mathbf{v}_{1l}^*\delta = \mathbf{v}_{1r}^\top\delta = 0$.

Recalling that $z = \mathbf{1}^\top\Phi\mathbf{x}$ and $\mathbf{v}_1 = \Phi\mathbf{1}$ we can rewrite the disagreement vector δ in the following way

$$\delta = \mathbf{x} - c\Phi\mathbf{1}\mathbf{1}^\top\Phi\mathbf{x} = (I - c\Phi\mathbf{1}\mathbf{1}^\top\Phi) = P\mathbf{x}, \tag{15}$$

where $P = I - c\Phi\mathbf{1}\mathbf{1}^\top\Phi$. Combining (11) and (15) we obtain the following equation for the dynamics of δ

$$\dot{\delta} = PA_1\mathbf{x} + i\omega_m P\mathbf{x} = A_1\mathbf{x} + i\omega_m P\mathbf{x} = A\delta. \tag{16}$$

To analyze stability properties of the system (16) we can introduce the following Lyapunov function

$$V(\delta) = \frac{1}{2}\delta^*\delta, \tag{17}$$

taking its derivative along the trajectories of (16) we obtain that

$$\dot{V}(\delta) = \frac{1}{2}\delta^*(A_1 + A_1^*)\delta = \delta^*\left(\frac{K}{N}\mathbf{1}\mathbf{1}^\top + Kr_\infty\text{Re}(\Phi)\right)\delta$$

and hence properties of the matrix $\frac{K}{N}\mathbf{1}\mathbf{1}^\top + Kr_\infty\text{Re}(\Phi)$ would define stability properties of the system. However the last task is not so easy, in order to simplify it and make one more link with the pproperties of the Kuramoto model, we introduce the following change of coordinates

$$\delta_\phi = \Phi^*\mathbf{x} \tag{18}$$

which represents a rotation of the δ coordinates.

Lyapunov function (17) can be rewritten in new coordinates as

$$V(\delta) = \frac{1}{2}\delta^*\delta = \frac{1}{2}\delta^*\Phi\Phi^*\delta_\phi = \frac{1}{2}\delta^*\delta_\phi,$$

the same way it's derivative can be rewritten as

$$\dot{V}(\delta) = \delta_\phi^*\text{Re}\left(\frac{K}{N}\Phi\mathbf{1}\mathbf{1}^\top\Phi^* + Kr_\infty(\Phi)\right)\delta_\phi.$$

[2]using terminology of [12]

Matrix $R_\phi = \text{Re}\big(\frac{K}{N}\Phi\mathbf{1}\mathbf{1}^\top\Phi^* + Kr_\infty(\Phi)\big)$ can be equally represented as

$$\begin{aligned} R_\phi &= \frac{K}{N}\text{Re}\left(\mathbf{v}_{1r}^{*}{}^\top\mathbf{v}_{1r}^\top\right) - Kr_\infty\text{Re}\,(\Phi) \\ &= \frac{K}{N}\left(\text{Re}(\mathbf{v}_{1r}^\top)\text{Re}(\mathbf{v}_{1r}) + \text{Im}(\mathbf{v}_{1r}^\top)\text{Im}(\mathbf{v}_{1r})\right) \\ &- Kr_\infty\text{Re}\,(\Phi^*) = \frac{K}{N}\left(\mathtt{M} + \mathbf{b}\mathbf{b}^\top + \mathbf{c}\mathbf{c}^\top\right), \end{aligned} \tag{19}$$

where $\mathbf{b} = Re(\mathbf{v}_1) = col(\frac{\mu_1}{Nr_\infty}, \dots, \frac{\mu_N}{Nr_\infty})$, $\mathbf{c} = Im(\mathbf{v}_1) = col(\frac{\tilde{\omega}_1}{Nr_\infty}, \dots, \frac{\tilde{\omega}_N}{Nr_\infty})$ and we recall that $\mathtt{M} = diag\{\mu\}$. The resulting expression for the matrix R_ϕ coincides with the Jacobian matrix of the linearized (around phase locked solutions) Kuramoto model, see equations (3.4)-(3.5) in [2], where properties of the set of matrices R_ϕ were analysed (see Lemmas 1-3 and Theorems 2-3 in this reference). In particular, it was shown that among the matrices R_ϕ there is only which is stable and correspondingly, among all phased locked solutions of Kuramoto model there is only one which is locally stable. Similar result holds for the linear systems, namely all, except one, systems (16) defined by $\mathcal{A}_{K\omega}$ are unstable as show the following two results.

Theorem 2: Let assumption **A1** be satisfied and consider a system(16) where matrix $A \in \mathcal{A}_{K\omega}$. If the matrix A defined in (9) corresponds to the equation (5) containing at least one minus sign, then the equilibruim set $\delta = 0$ is unstable for this system.

Next we consider the system $\dot{\mathbf{x}} = A\mathbf{x}$, where all the signs in the expressions for r_∞ in (5) and for μ in (6) are positive. We denote by A^+ and R_ϕ^+ matrices A and R_ϕ corresponding to this choice of signs.

Theorem 3: Let assumption **A1** be satisfied and consider the system

$$\dot{\mathbf{x}} = A^+\mathbf{x}. \tag{20}$$

Let the following condition be satisfied

$$\sum_{j=1}^{N} \frac{1 - 2\left(\frac{\tilde{\omega}_j}{Kr_\infty}\right)^2}{\sqrt{1 - \left(\frac{\tilde{\omega}_j}{Kr_\infty}\right)^2}} > 0. \tag{21}$$

Then the following statements are valid

1) The origin $\mathbf{x} = \mathbf{0}$ is stable for the system (20)
2) For all initial conditions $\mathbf{x}_\circ \in \mathbb{C}^N$ and all $t \geq 0$ the disagreement vector δ satisfies the following exponential bound

   $$\| \delta(t, \delta_\circ) \| \leq e^{-\alpha t} \| \delta_\circ \|,$$

   where $\alpha = \lambda_2(R_\phi^+)$ is the is the second smallest eigenvalue of the Laplacian matrix R_ϕ^+ and $\delta_\circ = (I_N - c\Phi\mathbf{1}\mathbf{1}^\top\Phi)\mathbf{x}_\circ$.
3) For almost all initial conditions $\mathbf{x}_\circ \in \mathbb{C}^N$ we have that coordinates of the vector $\mathbf{x}$ satisfy the following

equality

$$\lim_{t\to\infty} |\mathbf{x}_k(t,\mathbf{x}_\circ) - c z_\circ e^{i(\phi_k + i\omega_m t)}| = 0,$$

that is asymptotically all coordinates of the vector $\mathbf{x}(t,\mathbf{x}_\circ)$ rotate with the frequency ω_m and their positions are shifted on the angles ϕ_k relative to the weighted order parameter $z(t, z_\circ)$.

Remark 2: Associating coordinates of the vector $\mathbf{x}$ with the individual oscillators and using terminology from the literature on dynamical systems we can reformulate last statement of the theorem as follows: for almost all initial conditions $\mathbf{x}_\circ \in \mathbb{C}^N$ a network of linear oscillators defined by equation (20) frequency synchronizes with the average frequency ω_m and moreover, asymptotically the oscillators are *phase locked.*

V. Mapping of the linear system on the Kuramoto model

Using polar coordinates representation $x_k = \rho_k e^{i\theta_k}$ and recalling that $A^+ = \frac{K}{N}\mathbf{1}\mathbf{1}^\top - K r_\infty \Phi^* + i\omega_m I_N$, we can rewrite equation (20) as

$$\dot\rho_k e^{i\theta_k} + i\rho_k e^{i\theta_k}\dot\theta_k = \frac{K}{N}\sum_{j=1}^{N}\rho_j e^{i\theta_j} + K r_\infty e^{\theta_k} + i\omega_m \rho_k e^{\theta_k} \tag{22}$$

and separating real and imaginary parts we obtain the following equation for the imaginary parts

$$\rho_k\dot\theta_k = \omega_k + \frac{K}{N}\sum_{j=1}^{N}\rho_j \sin(\theta_j - \theta_k) \tag{23}$$

In the previous section we proved that asymptotically the difference $\mathbf{x}_k(t,\mathbf{x}_\circ) - z_\circ e^{i(\phi_k + i\omega_m t)}$ converges to zero for almost all initial conditions, reformulated in polar coordinates this result implies that $lim_{t\to\infty}\rho_k(t,\mathbf{x}_\circ) = z_\circ$. Therefore equation (23) reduces in the time limit to the Kuramoto model

$$\dot\theta_k = \omega_k + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_k)$$

and for almost all initial conditions $\mathbf{x} \in \mathbb{C}^N$ we can consider Kuramoto model as an asymptotic projection of the linear system (20).

VI. Simulation results

We consider the Kuramoto model (1) and the corresponding linear system given by (3) with $N = 4$, $\boldsymbol{\omega} = col(-1.3, 4, -7.2, 10.8)^\top$ and 2 different gains $K = \{20, 50\}$. Invariance of the limit cycle for weighted order parameter $z(t)$ is illustrated in the Fig. 1.a, while convergence of the solutions of the linear system to the limit cycle is depicted on the Fig. 1.b

Using polar coordinates representation for the linear system $x_k = \rho_k e^{i\xi_k}$ we compare next behavior of the linear system (3) and Kuramoto model. Phase differences for both systems are presented in the Fig.2.

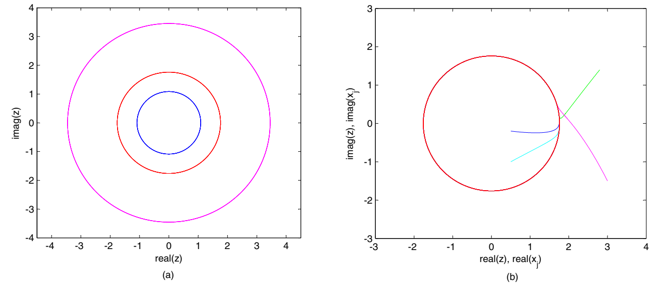

Fig. 1. (a) Weighted order parameter z for the system (3) with different initial conditions. (b) Convergence of the system trajectories to the invariant set defined by z.

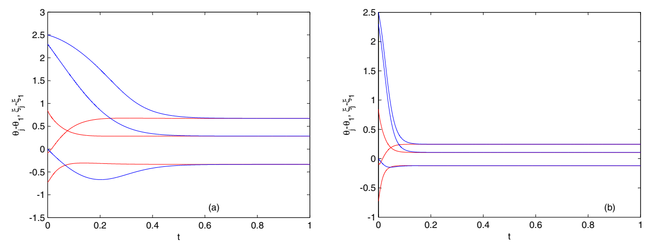

Fig. 2. Phase differences $\theta_j - \theta_1$ for the system (1) (in blue) versus phase differences $\xi_j - \xi_1$ for the system (3) (in red), for $K = 20$ (a) and for $K = 50$ (b).

VII. Conclusions

Starting with the idea of [13] to find a linear system that maps on the Kuramoto model, we formulated conditions this linear system has to satisfy and found the corresponding family of linear systems. Stability analysis of this family showed that among all these systems there is only one that is stable. Finally, we proved that asymptotically linear system has a limit cycle and moreover it maps on the Kuramoto model with the same gains of interconnections and natural frequencies. Simulation results, illustrate our theorems.

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APPENDIX

A. Proof of the Theorem 1.

Sufficiency. Let the matrix A satisfy properties **P1, P2**, i.e there exist ω^*, μ and $\phi \in \mathbb{R}^N$ such that $\lambda_1(A) = i\omega^*$ and $v_1 = e^{i\phi}$ is the corresponding eigenvector, $A\mathbf{v}_1 = i\omega^*\mathbf{v}_1$ or, equivalently

$$i\omega^*\mathbf{v}_1 = A\mathbf{v}_1 = \frac{K}{N}\mathbf{1}\mathbf{1}^\top\mathbf{v}_1 - diag(\mu - i\bar{\omega})\mathbf{v}_1 + i\omega^*\mathbf{v}_1, \quad (24)$$

where $\bar{\omega}_j = \omega_j - \omega^*$, $j = 1, \dots, N$. Last equation can be rewritten as

$$\frac{K}{N}\mathbf{1}\mathbf{1}^\top\mathbf{v}_1 = diag(\mu - i\bar{\omega})\mathbf{v}_1 = diag(\nu)\mathbf{v}_1, \quad (25)$$

where $\nu \in \mathbb{C}^N$ and $\nu_j = \mu_j - i\bar{\omega}_j$.

Next we introduce the quantity

$$\gamma = \mathbf{1}^\top\mathbf{v}_1 = \sum_{j=1}^{N} e^{i\phi_j}. \quad (26)$$

If $\mathbf{v}_1$ is an eigenvector of A, then any vector $\mathbf{v} = \eta\mathbf{v}_1$ (where $\eta \in \mathbb{C}$) is also an eigenvector of A and therefore, without loss of generality we can assume that $\gamma \in \mathbb{R}$, i.e. $\gamma = \sum_{j=1}^{N} \alpha_j \; e^{i\phi_j} = \sum_{j=1}^{N} \alpha_j \; cos(\phi_j)$.

Then, using (26) we have $\frac{K}{N}\mathbf{1}\mathbf{1}^\top\mathbf{v}_1 = \frac{K}{N}\gamma\mathbf{1}$ and can rewrite (25) in the following form

$$\frac{K}{N}\gamma\mathbf{1} = diag(\nu)\mathbf{v}_1. \quad (27)$$

Since $\gamma \in \mathbb{R}$, we have that the vector $\mathbf{v}_1$ has to satisfy the following equations

$$\nu_j = \gamma\frac{K}{N}\bar{\mathbf{v}}_{1j} = \gamma\frac{K}{N}e^{-\phi_j}. \quad (28)$$

Next, recalling that $\nu_j = \mu_j - i\bar{\omega}_j$, from the last equation we obtain that

$$\mu_j - i\bar{\omega}_j = \gamma\frac{K}{N}e^{-i\phi_j} = \gamma\frac{K}{N}\left(\cos(\phi_j) - i\sin(\phi_j)\right) \quad (29)$$

and therefore

$$\sin(\phi_j) = \frac{N}{K\gamma}\bar{\omega}_j \quad \text{and} \quad \cos(\phi_j) = \pm\sqrt{1 - \frac{N^2\bar{\omega}_j^2}{(K\gamma)^2}}. \quad (30)$$

In order to define value of ω^* we use again the fact that γ is real and from the last equation we have that $\sum_{j=1}^{N} sin(\phi_j) = \frac{N}{K\gamma}\sum_{j=1}^{N}\bar{\omega}_j = 0$ and therefore $\omega^* = \omega_m$.

Combining equalities (26) and (30) we obtain that

$$\gamma = \sum_{j=1}^{N}\cos(\phi_j) = \sum_{j=1}^{N} \pm\sqrt{1 - \frac{N^2}{(K\gamma)^2}\tilde{\omega}_j^2}. \quad (31)$$

Finally, denoting $r_\infty = \gamma/N$ we get that equation (31) coincides with the assumption **A1** and therefore matrix A has an eigenvalue $i\omega_m$. From (30) it also follows that the diagonal elements of the matrix $\mathcal{M}$ are defined by

$$\mu_j = \pm\frac{K\gamma}{N}\sqrt{1 - \frac{N^2}{(K\gamma)^2}\tilde{\omega}_j^2} = \pm\sqrt{(Kr_\infty)^2 - \tilde{\omega}_j^2}.$$

Necessity. To prove necessity it is enough to show that $\mathbf{v}_1$ defined by (8) is an eigenvector of the matrix A defined by (24) with coefficients μ_i defined as in the theorem and that it corresponds to the eigenvalue $i\omega_m$. Direct calculations show that this is indeed the case.

□