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Stabilization of the Ball and Beam System by Dynamic Output Feedback using Incremental Measurements

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Abstract— **The present contribution deals with the development of a stabilizing dynamic output feedback controller for the ball and beam system. The measured states are composed of the relative rotation angle and the absolute ball position. The torque applied to the beam serves as the control input. Significant practical relevance of this contribution is due to the relative rotation angle measurement, provided by most commercially available encoders. The controller is designed by employing a published inverse Lyapunov stabilizing approach together with a novel nonlinear observer based on the I&I methodology. Simulation experiments illustrate the theory and show the effectiveness of the proposed design.**

I. Introduction

The ball and beam system (BBS) is an important benchmark system for modern nonlinear control engineering. The unstable dynamics of the system resemble some problems found in aerospace application but with the advantage of simplicity in its laboratory implementation. The BBS system consists of a beam, which is made to rotate in a vertical plane by applying a torque at the center of rotation, and a ball that is free to roll along the beam.

The exact input-output linearization approach cannot be directly applied to stabilize the system around the origin due to the fact that the system does not have a well-defined relative degree at the origin. The BBS is not feedback linearizable, although it is locally controllable around the origin. Several works can be found related to the control of the BBS: A control strategy based on an approximate feedback linearization is presented in [1]. The singularities are avoided neglecting certain terms what results in a good closed-loop performance although restricted to a small region. A control strategy based on switching between exact and approximate input-output linearization is developed in [2], and in [3] several switches between exact input-output linearization and fuzzy dynamic control are combined.

Among the more recent work is the publication of Aguilar-Ibañez et al. [4], which presents an inverse Lyapunov approach in conjunction with the energy shaping technique that demonstrates excellent behavior and whose closed-loop stability can be proven using the invariance theorem of LaSalle.

In the field of nonlinear observer design the Immersion and Invariance (I&I) methodology has been shown to be effective for several academic and practical examples (see, e.g. [5], [6] and the references therein). Considering the speed and rotation angle estimation of the BBS using as measurements the *relative* angle of the beam and the absolute position of the ball, a nonlinear observer using the I&I approach is proposed in [7].

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The present work focuses on a problem of practical relevance that combines a nonlinear stabilizing state feedback controller based on the Lyapunov approach with a nonlinear observer based on the I&I methodology. This results in a locally stabilizing dynamic output feedback controller. The practical importance of the present work is due to the measurement of the absolute position and the *relative* angle. Most of the angle measurement systems rely on encoders that work incrementally and can only provide relative measurements of the desired quantity. The habitual procedure of introducing a calibration phase at the beginning of the process can be avoided using the controller proposed in the contribution at hand.

The rest of the paper is organized as follows. Section II is devoted to the physical modelling. In Section III a nonlinear I&I observer is designed. The nonlinear Lyapunov control strategy is revisited in Section IV. There, the dynamic output feedback controller is stated and a sufficient condition for the local stability is given. In Section V, simulation experiments are provided which illustrate the theory and show the effectiveness of the proposed design. Finally, in Section VI the major contributions of the paper are summarized and possible avenues of future research are highlighted.

II. Dynamical Model of the Ball and Beam System

A. Experimental Setup

We consider the well known ball and beam system ([1], [8]), which is schematically depicted in Fig. 1.

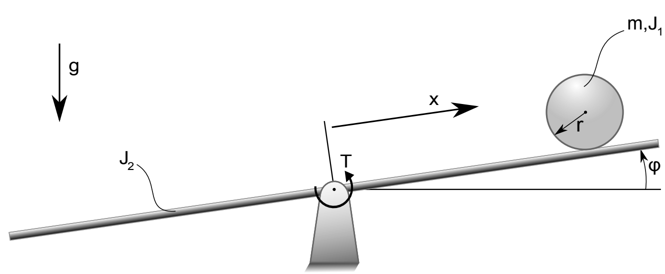

Fig. 1. Ball and beam system.

It consists of a ball that is rolling on a beam without slipping. The beam rotates around its pivot point where a torque T can be applied. The ball position x is measured with respect to the pivot point and the beam angle φ is defined relative to the horizontal plane. The values of the mass m and radius r of the ball, its moment of inertia J_1 as well as the moment of inertia J_2 of the beam are given in Table I together with the gravity constant g and are the same as those used in [8]. They will be used for the simulations in Section V.

TABLE I
SIMULATION PARAMETERS.

Symbol	Description	Value	Unit
m	ball mass	0.05	kg
r	ball radius	0.01	m
J_1	ball moment of inertia	2×10^{-6}	kg m^2
J_2	beam moment of inertia	0.02	kg m^2
g	gravity	9.81	m / s^2

A DC motor is used to generate the torque T on the beam. The ball position $x(t)$ is measured with an ultrasonic sound distance sensor. The rotation angle is measured using an *incremental encoder*. Due to this, we do not measure the absolute rotation angle $\varphi(t)$, but rather the relative angular difference with respect to the beginning of the measurement at $t = 0$, i.e. $\varphi(t)-\varphi_0$, where we have introduced the angular offset φ_0.

B. Equations of Motion

Following [1] and [9], the Lagrangian function

$$\mathcal{L} = \frac{1}{2}J_2\dot{\varphi}^2+\frac{1}{2}J_1\left(\frac{\dot{x}}{r}+\dot{\varphi}\right)^2+\frac{1}{2}m\left(\dot{x}^2+x^2\dot{\varphi}^2\right)-mgx\sin(\varphi)$$

defined as the difference between the kinetic energy and the potential energy is employed, yielding the following equations of motion

$$\left(J_2+J_1+mx^2\right)\ddot{\varphi}+2mx\dot{x}\dot{\varphi}+mgx\cos(\varphi)=T, \quad (1)$$

$$\left(\frac{J_1}{r^2}+m\right)\ddot{x}-mx\dot{\varphi}^2+mg\sin(\varphi)=0. \quad (2)$$

C. State-Space Representation

Introducing the state vector

$$\begin{aligned}\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5)^T &= (x, \dot{x}, \varphi, \dot{\varphi}, \varphi_0)^T \\ &\in \mathcal{D} \subset \mathbb{R}^5, \quad \text{with} \quad \varphi, \varphi_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\end{aligned} \quad (3)$$

and the input $u = T \in \mathbb{R}$ we obtain the state-space representation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \quad (4)$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \quad (5)$$

with

$$\boldsymbol{f}: \mathbb{R}^5 \times \mathbb{R} \to \mathbb{R}^5 : (\boldsymbol{x}, u) \mapsto \begin{pmatrix} x_2 \\ \frac{mx_1x_4^2 - mg\sin(x_3)}{A} \\ x_4 \\ \frac{u-2mx_1x_2x_4-mgx_1\cos(x_3)}{B+mx_1^2} \\ 0 \end{pmatrix} \quad (6)$$

and

$$\boldsymbol{h}: \mathbb{R}^5 \to \mathbb{R}^2 : \boldsymbol{x} \mapsto \begin{pmatrix} x_3 - x_5 \\ x_1 \end{pmatrix}, \quad (7)$$

where

$$A = \frac{J_1}{r^2} + m \quad \text{and}$$
$$B = J_2 + J_1$$

are employed for notational convenience.

III. OBSERVER DESIGN

This section is devoted to the observer design for the ball and beam system of Section II, where the measurements consist of the *absolute* ball position x and the *relative* rotation angle $\varphi - \varphi_0$.

Sticking tightly to the approach taken in [7], we first introduce the observer design based on the Immersion and Invariance methodology, as described in [9] and [10]. Thereafter, an observer for the ball and beam system is provided.

Opposed to [7], we consider the actuated system, i.e. $u \not\equiv 0$.

A. I&I Observer Design

Following [9], we consider the nonlinear, time-varying system

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \boldsymbol{f}_1(\boldsymbol{\eta}, \boldsymbol{y}, t), \\ \dot{\boldsymbol{y}} &= \boldsymbol{f}_2(\boldsymbol{\eta}, \boldsymbol{y}, t)\end{aligned} \quad (8)$$

where $\boldsymbol{\eta} \in \mathbb{R}^n$ is the *unmeasured part* of the state and $\boldsymbol{y} \in \mathbb{R}^m$ is the *measurable part* of the state. It is assumed that the vector fields $\boldsymbol{f}_1(\cdot)$ and $\boldsymbol{f}_2(\cdot)$ are forward complete, i.e. trajectories starting at time t_0 are defined for all times $t \geq t_0$.

To start with, we restate here the observer definition and (reduced-order) observer theorem as given in [9].

Definition 1 (Observer definition): The system

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\alpha}(\boldsymbol{\xi}, \boldsymbol{y}, t) \quad (9)$$

with $\boldsymbol{\xi} \in \mathbb{R}^p$, $p \geq n$, is called an observer for the system (8), if there exist mappings $\boldsymbol{\beta}(\boldsymbol{\xi}, \boldsymbol{y}, t) : \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}^p$ and $\boldsymbol{\phi}(\boldsymbol{\eta}, \boldsymbol{y}, t) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}^p$ that are left-invertible (w.r.t. their first argument) and such that the manifold

$$\begin{aligned}\mathcal{M} = \{(\eta, y, \xi, t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} : \\ \boldsymbol{\beta}(\xi, y, t) = \boldsymbol{\phi}(\eta, y, t)\}\end{aligned} \quad (10)$$

has the following properties.

(i) All trajectories of the extended sytem (8), (9) that start on the manifold $\mathcal{M}$ remain there for all future times, i.e. $\mathcal{M}$ is positively invariant.

(ii) All trajectories of the extended system (8), (9) that start in a neighborhood of $\mathcal{M}$ asymptotically converge to $\mathcal{M}$.

The above definition implies that an asymptotically converging estimate of the state η is given by the *estimation equation*

$$\hat{\boldsymbol{\eta}} = \boldsymbol{\phi}^L(\boldsymbol{\beta}(\boldsymbol{\xi}, \boldsymbol{y}, t), \boldsymbol{y}, t), \quad (11)$$

where ϕ^L denotes a left-inverse of ϕ. Note that the estimation error $\hat{\eta} - \eta$ is zero on the manifold $\mathcal{M}$. Moreover, if the property (ii) holds for any $(\eta(t_0), y(t_0), \xi(t_0), t_0) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}$ then (9) is a *global* observer for the system (8).

△

Theorem 1 (Observer theorem): Consider the system (8), (9)

$$\dot{\eta} = f_1(\eta, y, t),$$
$$\dot{y} = f_2(\eta, y, t),$$
$$\dot{\xi} = \alpha(\xi, y, t)$$

(which is re-stated here for convenience) and suppose that there exist $\mathcal{C}^1$ mappings $\beta(\xi, y, t) : \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}^p$ and $\phi(\eta, y, t) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}^p$ with a left-inverse $\phi^L : \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}^n$ such that the following hold

(A1) For all y, ξ and t, $\beta(\xi, y, t)$ is left-invertible w.r.t. ξ and

$$\det\left(\frac{\partial \beta}{\partial \xi}\right) \neq 0. \tag{12}$$

(A2) The system

$$\begin{aligned}\dot{z} = &-\frac{\partial \beta}{\partial y}\left(f_2(\hat{\eta}, y, t) - f_2(\eta, y, t)\right) \\ &+ \frac{\partial \phi}{\partial y}\Big|_{\eta=\hat{\eta}} f_2(\hat{\eta}, y, t) - \frac{\partial \phi}{\partial y} f_2(\eta, y, t) \\ &+ \frac{\partial \phi}{\partial \eta}\Big|_{\eta=\hat{\eta}} f_1(\hat{\eta}, y, t) - \frac{\partial \phi}{\partial \eta} f_1(\eta, y, t) \\ &+ \frac{\partial \phi}{\partial t}\Big|_{\eta=\hat{\eta}} - \frac{\partial \phi}{\partial t}\end{aligned} \tag{13}$$

with $\hat{\eta} = \phi^L(\phi(\eta, y, t) + z, y, t)$ has a (globally) asymptotically stable equilibrium at $z = \mathbf{0}$, uniformly in η, y and t.

Then the system (9) with

$$\begin{aligned}\alpha(\xi, y, t) = -\left(\frac{\partial \beta}{\partial \xi}\right)^{-1} &\left(\frac{\partial \beta}{\partial y} f_2(\hat{\eta}, y, t) + \frac{\partial \beta}{\partial t}\right. \\ &\left. -\frac{\partial \phi}{\partial y}\Big|_{\eta=\hat{\eta}} f_2(\hat{\eta}, y, t) - \frac{\partial \phi}{\partial \eta}\Big|_{\eta=\hat{\eta}} f_1(\hat{\eta}, y, t) - \frac{\partial \phi}{\partial t}\Big|_{\eta=\hat{\eta}}\right)\end{aligned} \tag{14}$$

where $\hat{\eta} = \phi^L(\beta(\xi, y, t), y, t)$, is a (global) observer for the system (8).

◊

Please note that $\alpha(\xi, y, t)$ can also be written as

$$\alpha(\xi, y, t) = -\left(\frac{\partial \beta}{\partial \xi}\right)^{-1}\left(\frac{\partial \beta}{\partial y} f_2(\hat{\eta}, y, t) + \frac{\partial \beta}{\partial t} - \frac{\mathrm{d}\phi}{\mathrm{d}t}\Big|_{\eta=\hat{\eta}}\right)$$

and that

$$z = \beta(\xi, y, t) - \phi(\eta, y, t), \tag{15}$$

represents the *off-the-manifold* coordinate.

Proof: The proof is given in [9]. ■

B. I&I Observer Design for the Ball and Beam System

First of all, the observer problem has to be stated in the form of (8). This can be achieved by defining

$$\eta = (\eta_1, \eta_2, \eta_3)^T = (\dot{\varphi}, \dot{x}, \varphi)^T \quad \text{and}$$
$$y = (y_1, y_2)^T (\varphi - \varphi_0, x)^T$$

which yields

$$\begin{aligned}\dot{\eta} &= f_1(\eta, y, u) \\ \dot{y} &= f_2(\eta, y)\end{aligned} \tag{16}$$

with

$$f_1(\eta, y, u) = \begin{pmatrix} \dfrac{u - 2my_2\eta_1\eta_2 - mgy_2\cos(\eta_3)}{B + mgy_2^2} \\ \dfrac{my_2\eta_1^2 - mg\sin(\eta_3)}{A} \\ \eta_1 \end{pmatrix} \quad \text{and}$$

$$f_2(\eta, y) = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

Please note that the dependence of the first entry of f_1 on u does not alter the z-dynamics in assumption (A2), due to the independence of the first column of $\frac{\partial \phi}{\partial \eta}$ on η. Based on the observer theorem, we are now able to provide an observer for the system (16).

Proposition 1: Given the mappings

$$\phi(\eta, y) = \begin{pmatrix} \sqrt{B + my_2^2}\,\eta_1 \\ \sqrt{A}\,\eta_2 \\ \frac{\sin(\eta_3)}{k_3} + \sqrt{A}\,\eta_2 \end{pmatrix}, \tag{17}$$

$$\beta(\xi, y) = \begin{pmatrix} \sqrt{B + my_2^2}\left(k_1 y_1 (1 + \xi_2^2) + \xi_1\right) \\ k_2 a_2(y_2) + \xi_2 \\ \xi_3 \end{pmatrix}, \tag{18}$$

the function $a_2(y_2)$ for which the properties

$$\frac{\mathrm{d}a_2(y_2)}{\mathrm{d}y_2} = a_2'(y_2) > k_3 mg > 0 \quad \text{and} \tag{19}$$

$$my_2 a_2(y_2) \geq 0 \tag{20}$$

hold true, and the observer gains

$$k = (k_1, k_2, k_3)^T \tag{21}$$

such that

$$k_1 > \frac{c^*}{2} \quad \text{with} \quad c^* = \left|\frac{my_2}{(B + my_2^2)\sqrt{A}}\right| \tag{22}$$

$$k_2 > 1 \tag{23}$$

$$k_3 > 0 \tag{24}$$

holds true, the system

$$\begin{aligned}\dot{\xi} = \alpha(\xi, y) = -\left(\frac{\partial \beta}{\partial \xi}\right)^{-1} \times \\ \left(\frac{\partial \beta}{\partial y} f_2(\hat{\eta}, y) - \frac{\partial \phi}{\partial y}\Big|_{\eta=\hat{\eta}} f_2(\hat{\eta}, y) - \frac{\partial \phi}{\partial \eta}\Big|_{\eta=\hat{\eta}} f_1(\hat{\eta}, y, u)\right)\end{aligned} \tag{25}$$

with

$$\left(\frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\xi}}\right)^{-1} = \begin{pmatrix} \frac{1}{\sqrt{B+my_2^2}} & -2k_1y_1\xi_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{y}} = \begin{pmatrix} \sqrt{B+my_2^2}k_1(1+\xi_2^2) & \frac{(k_1y_1(1+\xi_2^2)+\xi_1)my_2}{\sqrt{B+my_2^2}} \\ 0 & k_2a_2'(y_2) \\ 0 & 0 \end{pmatrix},$$

$$\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{y}} = \begin{pmatrix} 0 & \frac{my_2}{\sqrt{B+my_2^2}}\eta_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and}$$

$$\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\eta}} = \begin{pmatrix} \sqrt{B+my_2^2} & 0 & 0 \\ 0 & \sqrt{A} & 0 \\ 0 & \sqrt{A} & \frac{\cos(\eta_3)}{k_3} \end{pmatrix}$$

is an observer for the system (16) and the estimation equation is given by

$$\hat{\boldsymbol{\eta}} = \begin{pmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \\ \hat{\eta}_3 \end{pmatrix} = \begin{pmatrix} \frac{\beta_1}{\sqrt{B+my_2^2}} \\ \frac{\beta_2}{\sqrt{A}} \\ \arcsin(k_3(\beta_3-\beta_2)) \end{pmatrix}. \tag{26}$$

Proof: It has to be shown that the assumptions (A1) and (A2) of the observer theorem are fulfilled.

Solving (18) for $\boldsymbol{\xi}$ yields

$$\xi_1 = \frac{\beta_1 - \sqrt{B+my_2^2}\left(k_1y_1(1+\xi_2^2)\right)}{\sqrt{B+my_2^2}}$$
$$\xi_2 = \beta_2 - k_2a_2(y_2)$$
$$\xi_3 = \beta_3,$$

showing that $\beta(\boldsymbol{\xi}, \boldsymbol{y})$ is left-invertible w.r.t. $\boldsymbol{\xi}$. Together with

$$\det\left(\frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\xi}}\right) = \det \begin{pmatrix} \sqrt{B+my_2^2} & 2\sqrt{B+my_2^2}k_1y_1\xi_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \sqrt{B+my_2^2} > 0,$$

we see that assumption (A1) is fulfilled.

Using (16), (17), and (18), the error dynamics stated in assumption (A2) evaluate to

$$\dot{\boldsymbol{z}} = -\boldsymbol{\Gamma}\boldsymbol{z} + \boldsymbol{S}_1\boldsymbol{z} + \boldsymbol{w} \tag{27}$$

with

$$\boldsymbol{\Gamma} = \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}, \quad \text{where}$$

$$\gamma_{11} = \frac{1}{\sqrt{B+my_2^2}}\frac{\partial \beta_1}{\partial y_1} + \frac{my_2}{(B+my_2^2)\sqrt{A}}\beta_2,$$
$$\gamma_{12} = l_{22}\frac{\partial \beta_1}{\partial y_2},$$
$$\gamma_{21} = \frac{1}{\sqrt{B+my_2^2}}\frac{\partial \beta_2}{\partial y_1} - \frac{my_2}{(B+my_2^2)\sqrt{A}}\beta_1,$$
$$\gamma_{22} = \frac{1}{\sqrt{A}}\frac{\partial \beta_2}{\partial y_2} - \frac{k_3mg}{\sqrt{A}}, \quad \gamma_{23} = \gamma_{33} = \frac{k_3mg}{\sqrt{A}},$$
$$\gamma_{31} = \frac{1}{\sqrt{B+my_2^2}}\frac{\partial \beta_3}{\partial y_1}, \quad \gamma_{32} = \frac{1}{\sqrt{A}}\frac{\partial \beta_3}{\partial y_2} - \frac{k_3mg}{\sqrt{A}}$$

and

$$\boldsymbol{S}_1 = \frac{my_2\eta_1}{\sqrt{B+my_2^2}\sqrt{A}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$\boldsymbol{w} = \begin{pmatrix} -\frac{mgy_2}{\sqrt{B+my_2^2}}\left(\cos(\hat{\eta}_3) - \cos(\eta_3)\right) \\ 0 \\ \frac{\cos(\hat{\eta}_3)}{k_3}\hat{\eta}_1 - \frac{\cos(\eta_3)}{k_3}\eta_1 + \frac{my_2}{\sqrt{A}}(\hat{\eta}_1^2 - \eta_1^2) \end{pmatrix}.$$

The asymptotic stability of the error dynamics of the nominal system $\dot{\boldsymbol{z}} = -\boldsymbol{\Gamma}\boldsymbol{z} + \boldsymbol{S}_1\boldsymbol{z}$ is established with the Lyapunov function

$$V_z(\boldsymbol{z}) = \frac{1}{2}\boldsymbol{z}^T\boldsymbol{z}, \tag{28}$$

whose time-derivative along the trajectories of (27) evaluates to

$$\dot{V}_z = -\gamma_{11}z_1^2 - \gamma_{22}z_2^2 - \gamma_{33}z_3^2 - (\gamma_{12}+\gamma_{21})z_1z_2 - (\gamma_{23}+\gamma_{32})z_2z_3 - \gamma_{31}z_1z_3$$

Noting that

$$\gamma_{12} + \gamma_{21} = \gamma_{23} + \gamma_{32} = \gamma_{31} = 0$$

and (using (19)-(20) and (22)-(24))

$$\gamma_{11} = k_1(1+\xi_2^2) + \frac{my_2(k_2a_2(y_2)+\xi_2)}{(B+y_2^2)\sqrt{A}} > 0$$
$$\gamma_{22} = \frac{k_2a_2'(y_2) - k_3mg}{\sqrt{A}} > 0$$
$$\gamma_{33} = \frac{k_3mg}{\sqrt{A}} > 0$$

results in

$$\dot{V}_z < 0.$$

The disturbance term $\boldsymbol{w}$ is dealt with in a perturbed systems setting. Plugging in the estimation equation (11), the ϕ-mapping (17), and the z-definition (15) in $\boldsymbol{w}$ shows that $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{z})$ vanishes at the origin $\boldsymbol{z} = \boldsymbol{0}$. Following the reasoning provided in [11] for vanishing perturbations, we argue that $\boldsymbol{k}$ can be chosen large enough so as to dominate the perturbation term[1]. Therefore, assumption (A2) of the observer theorem is also fulfilled, which proves the claim. ■

[1]Precise bounds on $\boldsymbol{k}$ are dependent on physical constraints of the system.

IV. Controller Design

This section is devoted to the controller design for the ball and beam system of Section II.

Fist of all, the state feedback controller developed by Aguilar-Ibañez et al. [4] is revisited. This controller is combined with the observer of Section III-B in order to yield a dynamic output feedback controller. After that, we outline a sufficient condition for the local stability of the closed loop system under dynamic output feedback control.

Obviously, the state $x_5 = \varphi_0$ with $\dot{x}_5 = 0$ is not controllable. However, because x_5 is merely a parameter which is converted into a state for the sake of observer design and due to the independence of the vector field $\boldsymbol{f}(\boldsymbol{x}, u)$ on x_5, it can be omitted in the controller design.

A. State Feedback Controller

Following [4], the *partial feedback linearization* stage

$$u = (B + mx_1^2)v + 2mx_1x_2x_4 + mgx_1\cos(x_3) \tag{29}$$

is applied. Together with the normalization

$$d = \frac{m}{A} \quad \text{and} \quad n = \frac{mg}{A}$$

and introducing the state vector

$$\begin{aligned}\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)^T = (x, \dot{x}, \varphi, \dot{\varphi})^T \\ \in \mathcal{D} \subset \mathbb{R}^4, \quad \text{with} \quad \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\end{aligned} \tag{30}$$

we obtain the state-space representation

$$\dot{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{f}}(\bar{\boldsymbol{x}}, v), \quad \bar{\boldsymbol{x}}(0) = \bar{\boldsymbol{x}}_0 \tag{31}$$

with[2]

$$\bar{\boldsymbol{f}} : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4 : (\bar{\boldsymbol{x}}, u) \mapsto \begin{pmatrix} \bar{x}_2 \\ d\bar{x}_1\bar{x}_4^2 - n\sin(\bar{x}_3) \\ \bar{x}_4 \\ v \end{pmatrix} \tag{32}$$

The control law is given as

$$\begin{aligned} v = &- k_d(-\bar{x}_2 + \bar{x}_4) - \left(n\sin(\bar{x}_3) + \frac{\partial V_p}{\partial \bar{x}_3}\right) \\ &- \left(-d\bar{x}_1\bar{x}_4^2 + d\bar{k}_1\bar{x}_1\bar{x}_2(\bar{x}_2 + \bar{x}_4)\mathrm{e}^{-d\bar{x}_1^2}\right)\end{aligned} \tag{33}$$

with

$$\begin{aligned}\frac{\partial V_p}{\partial \bar{x}_3} = n\bar{k}_1\left(\cos(\bar{x}_3 - \bar{x}_1)I_{\cos\mathrm{r}} - \sin(\bar{x}_3 - \bar{x}_1)I_{\sin\mathrm{r}}\right) \\ - k_p(\bar{x}_1 - \bar{x}_3)\end{aligned} \tag{34}$$

[2]Note that we model the ball as rolling without slipping. Therefore, there is no damping coefficient β or b, respectively, as in [4].

and

$$\begin{aligned} I_{\cos\mathrm{r}} &= \alpha_1\phi_s(\bar{x}_1) \\ I_{\sin\mathrm{r}} &= \alpha_1\left(\alpha_0 + \phi_c(\bar{x}_1)\right) \\ \alpha_0 &= 2\,\mathrm{Im}\left(\mathrm{erf}\left(\frac{\jmath}{2\sqrt{d}}\right)\right) \\ \alpha_1 &= \frac{\sqrt{\pi}\exp(-1/(4d))}{4\sqrt{d}} \\ \phi_s(\bar{x}_1) &= 2\,\mathrm{Re}\left(\mathrm{erf}\left(\frac{\jmath + 2d\bar{x}_1}{2\sqrt{d}}\right)\right) \\ \phi_c(\bar{x}_1) &= -2\,\mathrm{Im}\left(\mathrm{erf}\left(\frac{\jmath + 2d\bar{x}_1}{2\sqrt{d}}\right)\right)\end{aligned}$$

The controller parameters must satisfy the conditions

$$\bar{k}_1 > 0, \quad k_d > 0, \quad k_p > n\bar{k}_1. \tag{35}$$

For the stability analysis, the Lyapunov function

$$V_x(\bar{\boldsymbol{x}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T K_c(\bar{x}_1)\dot{\boldsymbol{q}} + V_p(\boldsymbol{q}) \tag{36}$$

is employed with

$$K_c(\bar{x}_1) = \begin{pmatrix} 1 + \bar{k}_1\mathrm{e}^{-d\bar{x}_1^2} & -1 \\ -1 & 1 \end{pmatrix} \tag{37}$$

and

$$\begin{aligned}V_p(\boldsymbol{q}) = n\bar{k}_1\int_0^{\bar{x}_1}\sin(\bar{x}_3 - \bar{x}_1 + s)\mathrm{e}^{-ds^2}\,\mathrm{d}s \\ + \frac{k_p}{2}(\bar{x}_1 - \bar{x}_3)^2\end{aligned} \tag{38}$$

where $\boldsymbol{q}^T = (\bar{x}_1, \bar{x}_3)$ and $\dot{\boldsymbol{q}}^T = (\bar{x}_2, \bar{x}_4)$.

In [4] it is shown that the system (31) in closed loop with the control law (33) is locally asymptotically stable with the domain of attraction

$$\Omega_{\tilde{c}} = \left\{\bar{\boldsymbol{x}} \middle| (\bar{x}_1, \bar{x}_3) \in Q \wedge V_x(\bar{\boldsymbol{x}}) < \tilde{c}\right\} \tag{39}$$

with

$$Q = \left\{(\bar{x}_1, \bar{x}_3) \middle| |\bar{x}_1| \leq L \wedge |\bar{x}_3| < \frac{\pi}{2}\right\} \tag{40}$$

where L denotes the beam length.

B. Dynamic Output Feedback Controller

The dynamic output feedback controller $\hat{u}$ is obtained by using the control signal (33) in combination with the partial feedback linearization stage (29). Because the true state $\boldsymbol{x}$ or $\bar{\boldsymbol{x}}$, respectively, is not available, we plug in the estimated state

$$\begin{aligned}\hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5)^T = (y_2, \hat{\eta}_2, \hat{\eta}_3, \hat{\eta}_1, \hat{\eta}_3 - y_1)^T \\ \in \mathcal{D} \subset \mathbb{R}^5\end{aligned} \tag{41}$$

which is obtained using (5), (7) and (26).

C. Local Stability of the Closed-Loop System

A possible Lyapunov function for the overall system being composed of the observer and the plant is given by

$$V(\bar{\boldsymbol{x}}, \boldsymbol{z}) = V_x(\bar{\boldsymbol{x}}) + V_z(\boldsymbol{z}) \tag{42}$$

using (36) and (28).

The trajectories of the overall system are described by (4) (using $\bar{\boldsymbol{x}}$ instead of $\boldsymbol{x}$ and omitting x_5) and (27) and using the control law $\hat{u}$ of Section IV-B.

A sufficient condition for local asymptotic stability with the domain of attraction

$$\Omega_{\hat{c}} = \Big\{ (\boldsymbol{x}, \boldsymbol{z}) \Big| (x_1, x_3) \in Q \wedge V(\bar{\boldsymbol{x}}, \boldsymbol{z}) < \hat{c} \Big\} \tag{43}$$

is given by

$$\dot{V}(\bar{\boldsymbol{x}}, \boldsymbol{z}) < 0 \tag{44}$$

where the theorem of LaSalle needs to be employed for the same reason as in [4].

V. Controller Performance

In this section, we provide simulation results in order to show the performance of the developed dynamic output feedback controller.

Two different scenarios are considered which differ in their observer and controller parameters and consequently result in different plant behavior. The first scenario is an example for a parameter set that results in a finite escape time. In the second example, the parameters are chosen so that they are in the region of local stability.

In all simulations, the rotation angle offset is set to

$$\varphi_0 = -\frac{\pi}{6} \approx -0.523599 \,(\doteq -30\,^\circ)\,.$$

The initial conditions of the plant are set to

$$\boldsymbol{x}_0 = (0.5, 0.0, 0.4, 0.0, -0.523599)^T \tag{45}$$

and those for the observer are set to

$$\hat{\boldsymbol{\eta}}_0 = (0.0, 0.0, 0.0)^T\,. \tag{46}$$

A. Simulation Results for Scenario 1

The controller parameters and observer parameters are set to

$$\boldsymbol{k}_{\text{ctrl}} = \left(k_p, \bar{k}_1, k_d\right)^T = (3, 0.1427, 2)^T\,,$$
$$\boldsymbol{k}_{\text{obsv}} = (k_1, k_2, k_3)^T = (10, 10, 1)^T\,.$$

Figs. 2 and 3 show the trajectories x_1 and x_3 of the closed-loop system. The corresponding control signal is depicted in Fig. 4.

B. Simulation Results for Scenario 2

The controller parameters and observer parameters are set to

$$\boldsymbol{k}_{\text{ctrl}} = \left(k_p, \bar{k}_1, k_d\right)^T = (10, 0.1427, 1)^T\,,$$
$$\boldsymbol{k}_{\text{obsv}} = (k_1, k_2, k_3)^T = (10, 10, 1)^T\,.$$

Figs. 5 and 6 show the trajectories x_1 and x_3 of the closed-loop system. The corresponding control signal is depicted in Fig. 7.

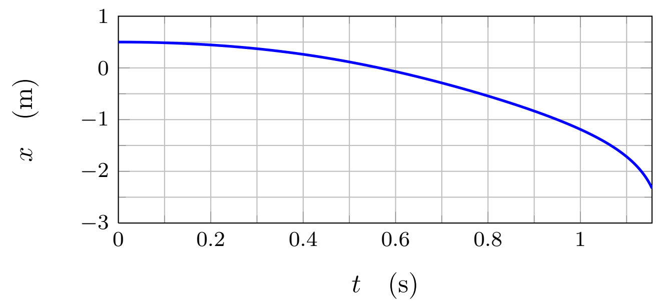

Fig. 2. Trajectory x_1 (scenario 1).

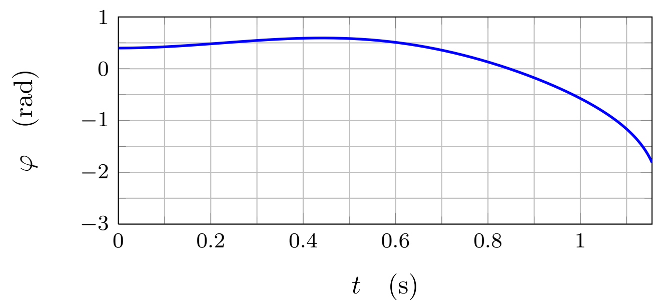

Fig. 3. Trajectory x_3 (scenario 1).

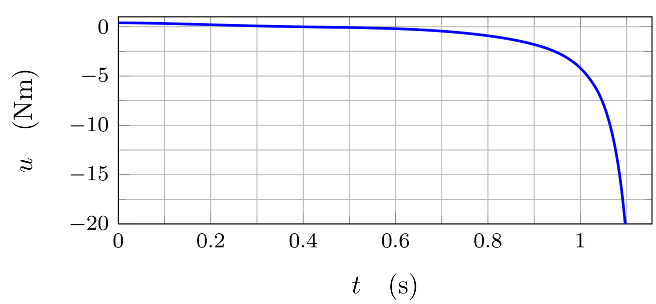

Fig. 4. Control signal u (scenario 1).

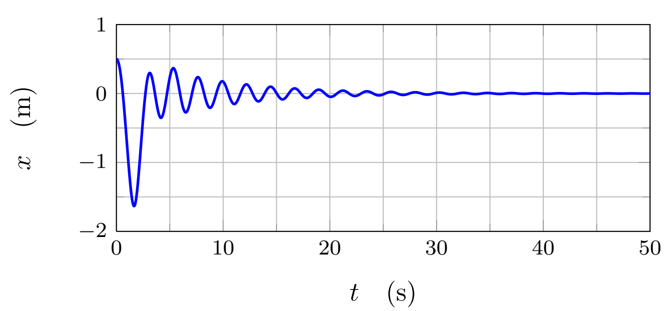

Fig. 5. Trajectory x_1 (scenario 2).

C. Discussion

The simulation results confirm the theoretic analysis. First of all, the scenario 1 is an example for the well known

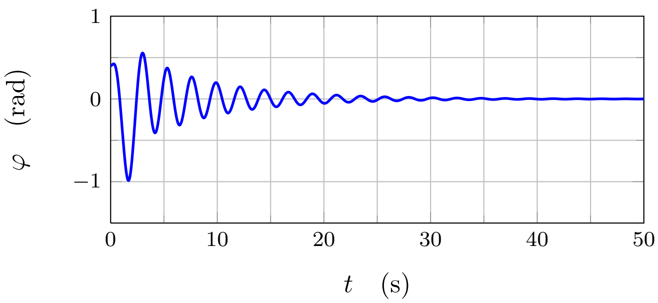

Fig. 6. Trajectory x_3 (scenario 2).

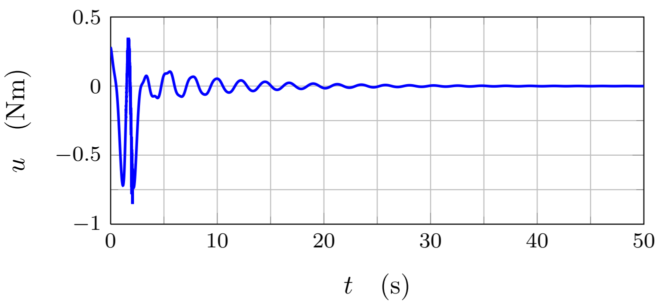

Fig. 7. Control signal u (scenario 2).

fact that the combination of an asymptotically stable state feedback controller with a converging observer might lead to a closed-loop system with finite escape time when dealing with nonlinear systems, where the separation principle does not hold true.

However, as scenario 2 shows, for adequately chosen parameter sets, the dynamic output feedback control shows excellent performance for the stabilization of the BBS with incremental rotation angle measurements and, therefore, unknown angular offset φ_0, which is an experimental setup of great practical interest.

VI. Conclusion and Further Work

The present contribution combines a nonlinear stabilizing state feedback controller based on the Lyapunov approach with a nonlinear observer based on the I&I methodology which is applied to a problem of practical relevance, i.e. the ball and beam system, which represents a mechanical system with unstable dynamics and which is not suitable to be input-output linearized around the origin due to the fact that the system does not have a well-defined relative degree at the origin. The practical importance of the present work is due to the measurement of the absolute position and the *relative* angle. Angle measurement systems usually rely on encoders which can only provide relative measurements. Commonly this problem is addressed by a calibration phase at the beginning of the process. With the proposed controller we offer a possibility to avoid this laborious routine. The result of the presented work is a locally stabilizing dynamic output feedback controller. To assess the performance and effectiveness of the proposed control strategy, some numerical simulations are carried out, which – for adequately chosen parameter sets – show excellent behavior for real-life conditions.

Future research is dedicated to the estimation of the respective domains of attraction and will include a detailed parametric study to assess the validity regions of the parameter space as well as noise distributions provided by the sensors in the design procedure. These studies are relevant because we will extend the results to a comprehensive class of nonlinear mechanical systems.

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