2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

A Multiobjective Optimization Approach for Optimal Control Problems of Mechanical Systems with Uncertainties

Sina Ober-Blöbaum[1] and Albert Seifried[1]

Abstract— **This work deals with the optimal control of mechanical systems with uncertainties which are associated to e.g. unknown parameters such as friction coefficients or geometry parameters in the model. While in the deterministic setting we characterize the maximal performance of the control system by fulfilling a predefined task exactly, in presence of uncertainty a performance measure is introduced whose mean has to be minimized to guarantee maximal performance of the control system. Taking this notion of system performance as additional objective into account leads to a multiobjective optimal control problem. Based on techniques from multiobjective optimization we present a methodology to numerically approximate the set of Pareto optimal solutions. The approach is verified by means of a robot arm maneuver with uncertain model parameters.**

I. INTRODUCTION

The goal in classical optimal control theory is to determine control strategies that, if applied to the system, fulfill some predefined task optimally w.r.t. some given quantity. However, for many applications specific system parameters such as e.g. friction coefficients or geometry parameters are not known or cannot be measured exactly. Due to this uncertainty the execution of the task can no longer be guaranteed by the control system. Strategies in robust optimal control theory ([1], [2], [3]) are based on formulations that ensure that the task has to be fulfilled for all realizations of the uncertainty in a given set. If the distribution of the uncertain variables is known (or not exactly known), probabilistic guarantees for robust solutions are computable a priori dependent on the uncertainty set (see e.g. [1], [2] for an overview of tractability and probabilistic guarantees for special problem classes). In this work we choose a less conservative approach: Based on the introduction of a performance measure [4], which states how well the predefined task is fulfilled by the control system, we develop a methodology which takes this notion of system performance as additional objective into account. In particular, we want to design a control that, on the one hand, is optimal w.r.t. the prescribed objective functional and, on the other hand, maximizes the system performance. The resulting problem is a multiobjective optimal control problem, for which the optimization of several objectives at the same time is required. In our case, these objectives are conflicting such that no unique optimum can be determined. Rather, one is interested in determining a set of compromise solutions, the Pareto optimal solutions [5] such that a decision maker is able to select one Pareto optimal control strategy fitting his demands best. Based on existing numerical approaches for the solution of multiobjective optimization and optimal control problems we present a numerical method to approximate the Pareto set.

Common approaches to solve deterministic optimal control problems such as direct methods (cf. [6] for an overview) are based on a time-discretization of the control or state and control trajectories and the differential equation. In this way, the optimal control problem is transformed into a finite-dimensional nonlinear constrained optimization problem. Standard optimization techniques such as SQP [7] can be used to compute the optimal solution. The direct method DMOC [8] is particularly tailored to mechanical systems since it preserves specific geometric properties such as symmetries in the discrete solution. DMOC has also been extended to the case of optimal control problems with multiple objectives [9]. If system parameters are uncertain, an optimal control problem can be formulated by minimizing the mean of an objective functional w.r.t. a differential equation involving random parameters whose distribution is assumed to be known. Common solution methods are e.g. sample average approximation methods ([10], [11]) based on Monte Carlo simulations.

In the area of multiobjective optimization, numerical methods can be classified in global and local methods: Global methods stepwise iterate a complete set of candidate solutions to an approximation of the Pareto set while local methods use local information of single Pareto points to compute nearby solutions. Well known global methods are evolutionary algorithms (e.g. [12]) or subdivision methods (e.g. [13]). Typical local methods are the weighted sum method (e.g. [14], [12]), continuation methods (e.g. [15], [16]) or reference point methods (e.g. [14]). For the solution of optimal control problems with multiple objectives, typically direct methods for the solution of optimal control problems are combined with numerical multiobjective optimization methods (see e.g. [17], [9], [18]).

In Section II the optimal control problem in the presence of uncertainty is transformed into a multiobjective optimal control problem by introducing a performance measure. In Section III we recall basic definitions of multiobjective optimization problems and present the reference point technique, a local method for the approximation of Pareto sets. In Section IV the general form of a multiobjective optimal control problem with uncertainty is introduced and a solution

This contribution was partly developed and published in the course of the Collaborative Research Centre 614 “Self-Optimizing Concepts and Structures in Mechanical Engineering” funded by the German Research Foundation (DFG) under grant number SFB 614.

[1]S. Ober-Blöbaum and A. Seifried are with the Department of Mathematics, University of Paderborn, Warburger Str. 100, 33100 Paderborn, Germany {sinaob,seifried}@math.uni-paderborn.de

strategy is proposed. In Section V we investigate a control maneuver of a robot arm modeled as two-link manipulator, for which the arm lengths are assumed to be uncertain (due to measurement errors) with known probability distribution. For a successful operation a minimum arm tip positioning accuracy is required.

II. Optimal Control with Uncertainty

Consider a control system described by

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

with Lipschitz continuous f, with the state function $x : I \rightarrow \mathbb{R}^n$ and the control function $u : I \rightarrow \mathbb{R}^m$, $I = [t_0, t_f]$. Let $J(x,u) = \int_I C(x(t), u(t))\,dt$ be an objective functional which measures a quantity of interest and let $r : \mathbb{R}^n \rightarrow \mathbb{R}^{n_r}$ be a function describing a task that has to be fulfilled by the control system. An optimal control problem is given as:

Problem 2.1: Find a control u^* that solves

$$\min_u J(x,u) = \int_I C(x(t), u(t))\,dt \tag{2}$$

subject to the differential equation (1), to an initial value constraint of the form $x(t_0) = x^0$, and to a terminal constraint $r(x(t_f)) = 0$.

Each optimal solution of Problem 2.1 satisfies $r(x(t_f)) = 0$, i.e. the fulfillment of the task is guaranteed by the system. In this case, we say that the system performance is maximal. We now assume that some of the system parameters are uncertain and introduce random variables $\xi \in \mathbb{R}^s$ with known distributions associated with the uncertainty of the system parameters. The control system is now described by

$$\dot{x}(t) = f(x(t), u(t), \xi). \tag{3}$$

We assume that $x(t)$ can be expressed as function of ξ, i.e. $x(t, \xi)$. In the following we simply write $x(t)$ but keep in mind that x is also a function of ξ. In the uncertain case, a formulation of the optimal control problem has to be found for which the objective and the constraints have to remain meaningful for different realization of the random variables. The objective functional J is replaced by its expectation given by $\hat{J}(u) = \int J(x,u)\rho(\xi)\,d\xi$, where $\rho(\xi)$ is the probability density function (pdf) of ξ. Note that in the presence of uncertainty the terminal constraint $r(x(t_f, \xi))$ takes different values for different values of random variables ξ. Thus, the fulfillment of the task cannot be guaranteed anymore. Our aim is to design a control that, on the one hand, is still optimal w.r.t. to the prescribed objective functional and, on the other hand, optimizes the system performance. We introduce a single performance measure $Y(x, \xi) \geq 0$ that measures the system performance, i.e. how well the predefined task is fulfilled by the control system. For $r(x(t_f)) = 0$ we have $Y = 0$, which corresponds to maximal system performance as in the deterministic case. In the presence of uncertainty the performance of the system is defined to be optimal if the expectation value of the performance measure Y is minimized, i.e. we want to minimize

$$\hat{J}_2 = \int Y(x, \xi)\rho(\xi)\,d\xi. \tag{4}$$

For the performance measure chosen in (11) this means that maximal system performance corresponds to minimal mean violation of the terminal constraint.[1] This is in contrast to other approaches in robust optimal control, where e.g. the probability of a state constraint violation is required to be below some safety parameter, which is included as inequality constraint (denoted as chance or probabilistic envelope constraint [3]) or as new single objective function in the optimal control problem while the original optimality criterion is neglected (see e.g. [4]). Instead, we still take the original objective into account, treat (4) as an additional objective and are faced with a multiobjective optimal control problem for which optimal compromise solution are computed.

III. Multiobjective Optimization

We introduce the following

Problem 3.1: A *multiobjective optimization problem* is given by

$$\min_{y \in S} \{F(y)\}, \quad S := \{y \in \mathbb{R}^n \,|\, g(y) = 0, a(y) \leq 0\}, \tag{5}$$

with $F : \mathbb{R}^n \rightarrow \mathbb{R}^k, k > 1$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $a : \mathbb{R}^n \rightarrow \mathbb{R}^q$.
The set of Pareto optimal solutions of Problem 3.1 can be determined based on the concept of dominance.

Definition 3.1:
- (i) A vector $z \in S$ is *dominated* by a vector $y \in S$ (in short: $y \prec z$) with respect to Problem 3.1 if $F(y) \leq F(z)$ and $F(y) \neq F(z)$.
- (ii) A point $y \in S$ is called *Pareto optimal* or a *Pareto point* if there is no $z \in S$ which dominates y. ($F(y)$ is also called a Pareto point.)
- (iii) The *Pareto set* $\mathscr{P}$ is defined as the set of all Pareto points and the corresponding set in the image space $F(\mathscr{P})$ is called the *Pareto front*.

For the approximation of the Pareto set reference methods are appropriate in particular, if the dimension n is high. The basic idea is to generate unreachable reference points $T \in \mathbb{R}^k$ in the image space of F, where each of them is used for the following distance minimization:[2]

$$\min_{y \in S} \|F(y) - T\|, \quad S := \{y \in \mathbb{R}^n \,|\, g(y) = 0, a(y) \leq 0\}. \tag{6}$$

Standard minimization algorithms like SQP or Ipopt[3] can be applied to solve these single objective minimization problems. More precisely, for a given multiobjective optimization problem 3.1 and a Pareto optimal point $y_0 \in \mathscr{P}$ the reference point optimization algorithm works as follows:

$P := \{y_0\}$
$FP := \{F(y_0)\}$
for $i = 0, \ldots, M$ **do**
  for $j = 1, \ldots, k$ **do**
    Choose $T^i_j \in \mathbb{R}^k$ near $F(y_i)$ but outside of $F(S)$
    Solve $y^*_j := \arg\min_{y \in S} \|F(y) - T^i_j\|$
    if $\|F(y^*_j) - T^i_j\| > 0$ **then**

[1] Note that the expected value is only one choice of many measures and could be easily replaced.

[2] For the numerical computations we minimize the squared distance to ensure differentiability.

[3] https://projects.coin-or.org/Ipopt

$y_{|P|+1} := y_j^*$
$P := P \cup \{y_{|P|+1}\}$
$FP := FP \cup \{F(y_{|P|+1})\}$
end if
end for
end for

$M \in \mathbb{N}$ is a predefined number of steps, P is the resulting set that approximates the Pareto set $\mathscr{P}$ in the convex case and a connected part of it in the nonconvex case and FP is the corresponding front. The *if*-condition ensures that the chosen reference points are not reachable. For the reference point generation we choose a strategy suggested in [19] which is designed to approximate the Pareto front for a bi-criteria optimization problem with evenly spread points very fast.

IV. MULTIOBJECTIVE OPTIMAL CONTROL WITH UNCERTAINTY

The multiobjective optimal control problem with uncertainty introduced in Section II can be generally formulated as follows.

Problem 4.1: Find a control u^* that solves

$$\min_u \hat{J}(u) = \min_u \int J(x,u)\rho(\xi)\, d\xi \tag{7}$$

subject to (3) with vector valued objective functionals $\hat{J} = (\hat{J}_1, \dots, \hat{J}_k)$ and $J = (J_1, \dots, J_k)$ such that $\hat{J}_i(u) = \int J_i(x,u)\rho(\xi)\, d\xi$. Each $J_i(x,u)$ may consist of a Lagrange and a Mayer term $J_i(x,u) = \int_I C_i(x(t),u(t))\, dt + \psi_i(x(t_f))$, $i = 1, \dots, k$, with $C = (C_1, \dots, C_k)$ and $\psi = (\psi_1, \dots, \psi_k)$ being vector-valued, continuously differentiable functions.
The numerical solution of Problem 4.1 involves three key tasks: (i) the numerical solution of the state equation (3), (ii) a strategy to approximate the objective functional $\hat{J}$, and (iii) an optimization algorithm to approximate the Pareto set.

A. Numerical Approximation

For the first task a numerical integrator based on a prescribed time grid $\Delta t = \{t_0, \dots, t_N = t_f\}$ is used. Let $u_d = \{u_k\}_{k=0}^N$ be a discretization of the time-dependent function $u(t)$, where u_k is an approximation of $u(t_k)$. For a fixed control sequence u_d and fixed parameter ξ a variational integrator is applied to compute an approximation $x_d = \{x_k\}_{k=0}^N$ of the curve $x(t)$.[4] Variational integrators [20] are in particular efficient for Hamiltonian and Lagrangian systems since they preserve structural properties such as symmetries in the discrete approximation.

For the second task in general two approximations have to be performed to evaluate an objective functional of the form (7). We use the same approximation rules for each objective functional $\hat{J}_i$, $i = 1, \dots, k$ but omit the index i for simplicity in this section. The two approximations are based on a discretization in time (based on the discrete time grid $\{t_k\}_{k=0}^N$) and the space of uncertain parameters. We use a straight forward approximation technique that is based on a simple sampling of the uncertain parameters. We assume a bounded support of the pdf which is realized by cutting the pdf at appropriate bounds. For our numerical examples such a support is chosen by five times the standard deviation since it is unlikely that the parameters are outside this region. Note that also in most practical applications the region of uncertainty is bounded (system parameters can only change within a prescribed region). Thus, we introduce lower and upper bounds ξ_i^l and ξ_i^u on the random variables such that $\xi_i \in [\xi_i^l, \xi_i^u]$ for $i = 1, \dots, s$, and discretize ξ with $M_i + 1$ discretization points for each component ξ_i, i=1,…,s. We define equidistant discrete grids $\{\xi_{ik_i}\}_{k_i=0}^{M_i}$ with $\xi_{i0} = \xi_i^l$, $\xi_{ik_i} = \xi_{i0} + \Delta\xi_i k_i$ and $\xi_{iM_i} = \xi_{i0} + M_i \Delta\xi_i = \xi_i^u$ with grid sizes $\Delta\xi_i \in \mathbb{R}$, $i = 1, \dots, s$.[5] The discretized probability density function ρ_d is defined as

$$\rho_d(\xi) = \frac{\rho((\xi_{1k_1}, \cdots, \xi_{sk_s}))}{\Delta\xi_1 \cdots \Delta\xi_s \sum_{l_1=1}^{M_1} \cdots \sum_{l_s=1}^{M_s} \rho((\xi_{1l_1}, \cdots, \xi_{sl_s}))} \tag{8}$$

$$\text{for} \quad \xi_{ik_i-1} < \xi_i \leq \xi_{ik_i} \quad \text{for} \quad i = 1, \dots, s,$$

$k_i = 1, \dots, M_i$, $i = 1, \dots, s$ and $\rho_d(\xi) = 0$ if $\xi_i \notin [\xi_i^l, \xi_i^u]$ for at least one i. Note that this choice of ρ_d guarantees that $\int \rho_d(\xi)\, d\xi = 1$. Thus, $\hat{J}(u)$ can be approximated by

$$\begin{aligned} \hat{J}_d(u_d) &= \Delta\xi_1 \cdots \Delta\xi_s \cdot \\ &\sum_{l_1=1}^{M_1} \cdots \sum_{l_s=1}^{M_s} J_d(x_d(\xi_{1l_1}, \cdots, \xi_{sl_s}), u_d)\rho_d((\xi_{1l_1}, \cdots, \xi_{sl_s})) \\ &= \frac{\sum_{l_1=1}^{M_1} \cdots \sum_{l_s=1}^{M_s} J_d(x_d(\xi_{1l_1}, \cdots, \xi_{sl_s}), u_d)\rho((\xi_{1l_1}, \cdots, \xi_{sl_s}))}{\sum_{l_1=1}^{M_1} \cdots \sum_{l_s=1}^{M_s} \rho((\xi_{1l_1}, \cdots, \xi_{sl_s}))}. \end{aligned} \tag{9}$$

with J_d being the approximation of J based on time discretization and numerical quadrature rules (see e.g. [8]). Note that the probability P of ξ being in $[\xi_{1k_1-1}, \xi_{1k_1}] \times \cdots \times [\xi_{sk_s-1}, \xi_{sk_s}]$ can be approximated by

$$P_{k_1 \cdots k_s} = \frac{\rho((\xi_{1k_1}, \cdots, \xi_{sk_s}))}{\sum_{l_1=1}^{M_1} \cdots \sum_{l_s=1}^{M_s} \rho((\xi_{1l_1}, \cdots, \xi_{sl_s}))} \tag{10}$$

The multiobjective optimal control problem is thus transformed into a multiobjective optimization problem with objective function $\hat{J}_d$ and optimization parameters $y = u_d = \{u_0, \dots, u_N\}$, where $x_d(\xi)$ is computed with a variational integrator. The discretization of the time interval determines the problem dimension and is typically high to reach the desired accuracy requirements. Thus, for the computation of Pareto optimal control sequences $u_d = \{u_0, \dots, u_N\}$ and for the approximation of the Pareto set (the third task), the reference point optimization method described in Section III is applied. The discretization of the uncertain parameter space significantly influences the duration for the objective function evaluation. Note that for each evaluation of $\hat{J}_d(u_d)$ $(M_1 + 1) \cdots (M_s + 1)$ simulations of the differential equation (3) are required.

[4] Note that each x_k, $k = 0, \dots, N$, is a function of ξ.

[5] In the following the first index of ξ refers to the vector component where the second index represents the grid point.

B. Procedure of the Methodology

The proposed methodology can be summarized in the following steps:

1) Determine the optimal control sequence for the deterministic optimal control problem.
2) Formulate the multiobjective optimal control problem with uncertainty based on a performance measure.
3) Transform the problem into a multiobjective optimization problem with uncertain parameters.
4) Approximate the Pareto set.

In the first step we solve the optimal control problem 2.1 with objective functional J_1 for the deterministic system, i.e. $\xi = \hat{\xi}$, using the direct optimal control method DMOC. This enables us to compute a control sequence $u_d^*(\hat{\xi})$ that is an approximation of the optimal control function $u^*(\hat{\xi})$ such that the desired task defined by $r(x(t_f)) = 0$ is fulfilled by the deterministic control system. In the second step we introduce uncertain system model parameters. The objective functional becomes $\hat{J}_1 = \int J_1(x,u)\rho(\xi)\,d\xi$. Furthermore, we reformulate the terminal constraint $r(x(t_f)) = 0$ as an additional objective functional J_2 by defining a performance measure Y whose expectation value has to be minimized for optimal performance. We choose $Y = \|r(x(t_f))\|_2^2$ such that

$$\hat{J}_2 = \int \|r(x(t_f))\|_2^2 \rho(\xi)\,d\xi. \tag{11}$$

In the third step we discretize the multiobjective optimal control problem by approximating the solution of the differential equation as well as the objective functionals as described in the previous subsection. In the fourth step we apply the reference point method to solve the multiobjective optimization problem. Starting with the optimal control sequence $u_d^*(\hat{\xi})$ for the deterministic case as initial guess, we search for control sequences which are Pareto optimal w.r.t. $\hat{J}_{d,1}$ and $\hat{J}_{d,2}$. For the distance minimization (6) we choose the two-norm and minimize the squared distance as $\|\hat{J}_d(u_d) - T\|_2^2$. We use a SQP method implemented in the NAG library[6] to solve this single optimization problem. While the optimization of the deterministic problem can be solved quite efficiently (in the range of several seconds for the example described in Section V), the approximation of the Pareto set of the uncertain problem (step 4)) involves the highest computational effort (typically in the range of several hours depending on the problem size and the approximation accuracy given by the number of reference points).

V. NUMERICAL EXAMPLE

We demonstrate our approach by a robot arm modeled as a two-link manipulator. The lengths L_1 and L_2 of the two links are assumed to be not known or exactly measurable. The goal is to determine control sequences $u(t)$ that are Pareto optimal w.r.t. control effort and performance. The performance of the system is determined by the arm tip state error at the final time t_f compared to a prescribed goal state.

A. The Deterministic Optimal Control Problem

The two-link manipulator (see Fig. 1 on the left) consists of two coupled planar rigid bodies with mass m_i, length l_i and moment of inertia J_i, $i = 1,2$, respectively. For $i \in 1,2$, we let θ_i denote the orientation of the ith link measured counterclockwise from the positive horizontal axis. If we assume one end of the first link to be fixed in an inertial reference frame, the configuration of the system is specified by $(\theta_1(t), \theta_2(t))$. Control torques $u(t) = (\tau_1(t), \tau_2(t))$ are applied at the base of the first link and at the joint between the two links. For a detailed model description we refer to [8]. The goal is to determine a control sequence $u(t)$ which steers the robot arm tip (modeled as end point of the two-link mechanism) from a prescribed initial state x_0 to a prescribed final state x_{ref}, where $x = (q, v)$ consists of the arm tip's position $q = (L_1\cos\theta_1 + L_2\cos\theta_2, L_1\sin\theta_1 + L_2\sin\theta_2)^T$ and its velocity $v = \dot{q} = (-L_1\dot{\theta}_1\sin\theta_1 - L_2\dot{\theta}_2\sin\theta_2, L_1\dot{\theta}_1\cos\theta_1 + L_2\dot{\theta}_2\cos\theta_2)^T$.

In the first step, the deterministic optimal control problem is solved, for which the lengths are fixed to some reference value $\xi = \hat{\xi} = (\hat{L}_1, \hat{L}_2)$. The endpoint condition $x(t_f) - x_{\text{ref}} = 0$ is incorporated as terminal constraint in the optimal control problem such that only one objective functional, the control effort given as $J_1(u) = \int_{t_0}^{t_f} \frac{1}{2}\|u(t)\|_2^2\,dt$, is considered. To determine the optimal control sequence $u_d^*(\hat{\xi})$ the numerical method DMOC [8] is employed which bases on the discretization of the underlying optimal control problem. We choose the link lengths to be $\hat{L}_i = 1.0$, $i = 1,2$. The prescribed goal state is given by $x_{\text{ref}} = (q_{\text{ref}}, \dot{q}_{\text{ref}}) = (0, 1.5, 0, 0)$ which corresponds to a not fully but almost stretched final up-up position of the manipulator[7] with zero velocity at time $t_f = 1.0$. For the discretization a discrete time grid with $N+1 = 20$ discretization points is chosen leading to 78 optimization variables in DMOC (two times 20 discrete configuration and two times 19 discrete control parameters).

B. Performance in Presence of Uncertainty

In the second step, the control problem is reconsidered including uncertainty. It is assumed that both unknown lengths L_1 and L_2 can vary randomly in a range of 0.5% around the given value $\hat{L}_i = 1.0$, $i = 1,2$, i.e. the region of uncertainty is $[0.995, 1.005] \times [0.995, 1.005]$. Furthermore, the random parameters are assumed to be independent and normally distributed around the reference value $\hat{L}_i = 1.0$, $i = 1,2$. The probability density function is given by $\rho(\xi) = \rho_1(\xi_1)\rho_2(\xi_2)$, where $\rho_i(\xi_i) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\xi_i - \mu}{\sigma}\right)^2\right)$ is the pdf of $\xi_i = L_i$, $i = 1,2$, with $\sigma = 0.001$ and $\mu = 1.0$.

For the numerical treatment the space of uncertain parameters $[0.995, 1.005] \times [0.995, 1.005]$ is discretized by two equidistant grids $\{\xi_{i0}, \xi_{i2}, \ldots, \xi_{iM}\}$, $M_i = 33$, $i = 1,2$, and we approximate the pdf by the discretized probability density function $\rho_d(\xi)$ using formula (8). In Fig. 1 on the right the probability P of ξ being in $[\xi_{1k-1}, \xi_{1k}] \times [\xi_{2l-1}, \xi_{2l}]$ is depicted as given by P_{kl} in equation (10).

[6] http://www.nag.co.uk

[7] Note that the up-up position corresponds to $q = (0, 2.0)$.

First, we investigate the influence of uncertainty on the system performance if the optimal control $u_d^*(\hat{\xi})$, that was computed for the deterministic case, is applied to the system. We integrate the differential equation (3) with fixed optimal control sequence $u_d = u_d^*(\hat{\xi})$ and different grid values $\xi_{ij} = (\xi_{1i}, \xi_{2j})$, $i, j = 0, \ldots, M$. In Fig. 2 it can be seen that the different final states $x(t_f, \xi_{ij})$ of the arm tip strongly deviate from the reference value for the final position $q_{\text{ref}} = (0, 1.5)$ (Fig. 2 on the left) and the final velocity $v_{\text{ref}} = (0, 0)$ (Fig. 2 on the right). The colors indicate the probability values as given in Fig. 1. Thus, the performance of the system given by (11) is relatively bad in presence of uncertain lengths and new control sequences have to be determined to improve the system performance.

C. Multiobjective Optimal Control with Uncertainty

To reduce the deviation from the reference goal state, the mean of the performance measure Y (11) is treated as additional objective function. The same maneuver should be performed in such a way that on the one hand, the required control effort and on the other hand, the mean deviation from the prescribed goal state x_{ref} is minimized in presence of the uncertain length parameters $\xi = (L_1, L_2)$. The reference point optimization method is employed to determine Pareto optimal control sequences using 36 reference points for the approximation of the Pareto front. For the minimization of (6) a SQP method implemented in NAG is used. Each minimization involves 38 optimization parameters (two times $N = 19$ discrete controls) and each function evaluation requires $M_1 \cdot M_2 = 1156$ simulations.[8]

Figure 3 (left) shows the approximation of the Pareto front. A first interesting observation is that the deterministic solution $u_d(\hat{\xi})$ computed by DMOC (blue cross) is not part of the Pareto front. It is dominated by a less expensive control belonging to the closest point on the Pareto front indicated as red circle in Fig. 3 (left). The final state distribution for this Pareto optimal point is shown in Fig. 4 (a)-(b). The distribution's center in configuration space does not lie in the reference configuration q_{ref} leading to a worse performance in configuration space. However the distribution in velocity space is concentrated more closely around the reference velocity v_{ref} compared to the distribution in the deterministic case (cf. Fig. 2 (right)), which results in a better performance in velocity space. Thus, a similar performance in state space can be reached by using less control effort.x

Generally, it can be observed in Fig. 3 (left) that for increasing control effort the mean of the deviation from the final goal state decreases. In Fig. 3 (right) the configuration of the arm tip for three selected Pareto points PP1-PP3 with $L_i = 1.0$, $i = 1, 2$, is shown. The red trajectory belongs to a solution with low control effort but high mean deviation, whereas the magenta trajectory requires a higher control effort and presents a solution with smaller mean deviation. This is also illustrated in Fig. 4 showing the reached final

[8]To speed up the computations we used less grid points M_1 and M_2 and interpolated the trajectory's final state at the grid points in between.

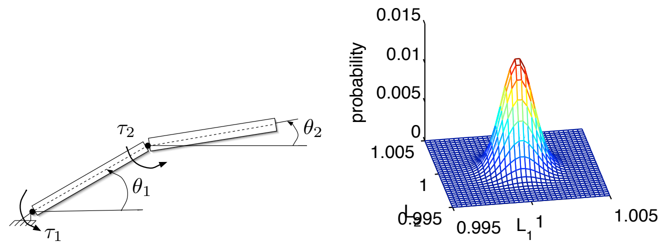


Fig. 1. Left: Model of the two-link manipulator. Right: Discretization of uncertain parameter space and approximated probability.

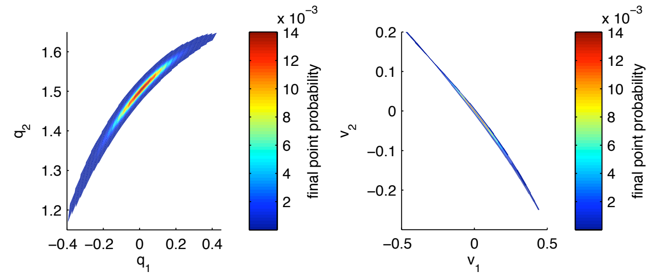


Fig. 2. Final positions $q(t_f, \xi)$ (left) and velocity $v(t_f, \xi)$ (right) for different values of lengths parameters ξ and optimal control sequence $u_d^*(\hat{\xi})$.

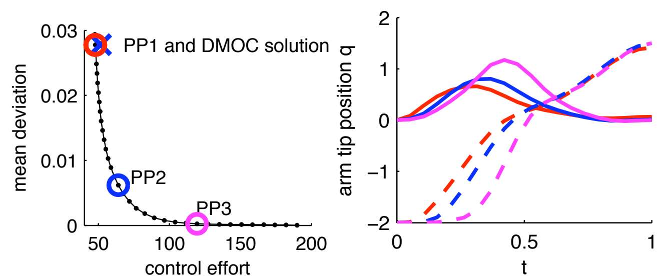
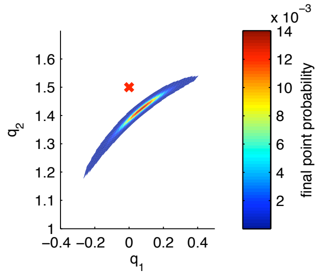


Fig. 3. Left: Approximation of the Pareto front (black curve) with three selected Pareto points PP1-PP3. The optimal solution computed with DMOC for the deterministic case (blue cross) is not Pareto optimal (dominated by PP1). Right: Configuration of arm tip for PP1-PP3 with $L_i = 1.0$, $i = 1, 2$ (q_1 solid, q_2 dashed).

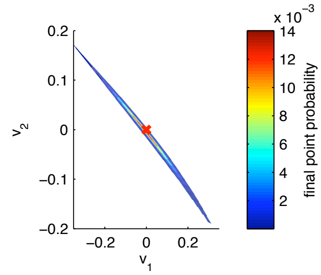
configurations and velocities for random lengths. For example, for PP1 less control effort is required to control the manipulator to a lower position compared to the goal position (note that the distribution's center in configuration space in Fig. 4 (a) lies below the goal state). Along the Pareto front the area of reached final states contracts and thus the system performance increases as desired.

VI. CONCLUSIONS

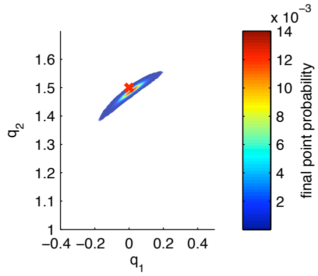
In this contribution we presented a numerical framework for the treatment of optimal control problems in the presence of uncertain parameters. In addition to the first objective prescribed for the deterministic optimal control problem, the maximal system performance was introduced as second objective based on a prescribed performance measure. The problem can be transformed into a finite-dimensional multiobjective optimization problem by numerical approximation techniques. The Pareto set of this problem was approximated with a reference point technique that iteratively searches for



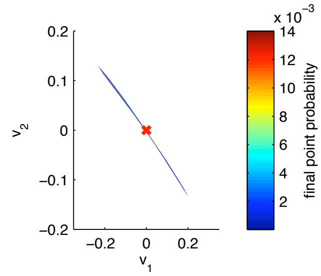
(a) Final configuration (PP1).



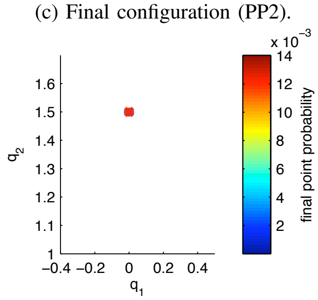
(b) Final velocity (PP1).



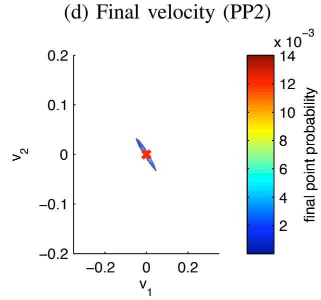
(c) Final configuration (PP2).



(d) Final velocity (PP2)



(e) Final configuration (PP3).



(f) Final velocity (PP3).

Fig. 4. Behavior of final points in configuration and velocity space for three selected points of the approximated Pareto front.

new Pareto optimal solutions. For the robot arm maneuver it could be observed that the objectives minimal control effort and maximal performance are indeed contrary, i.e. a higher system performance leads to higher control effort. Based on the information of the whole Pareto set, the decision maker is able to select one Pareto optimal control strategy dependent on the actual demands concerning system performance and control efficiency. For a further validation of the proposed methodology different applications have to be considered. In particular, different formulations have to be investigated, which are more closely related to the robust optimization literature, e.g. using probabilistic envelope constraints [3] to reformulate the terminal constraint or minimizing the probability of failure as second objective [4]. First tests already indicate a similar conflicting behavior between minimal control effort and minimal probability of failure. From a numerical point of view we will investigate strategies to decrease the computational expense (for the described example the approximation of the Pareto set took several hours), e.g. by using concepts from polynomial chaos theory [21]. Furthermore, a comparison concerning numerical effort and accuracy with different numerical methods for the different steps would be interesting, e.g. using evolutionary algorithms for the approximation of the Pareto set.

References

[1] A. Ben-Tal and A. Nemirovski, "Robust optimization – methodology and applications," *Mathematical Programming*, vol. 92, no. 3, pp. 453–480, 2002.

[2] D. Bertsimas, D. Brown, and C. Caramanis, "Theory and applications of robust optimization," *SIAM Review*, vol. 53, no. 3, pp. 464–501, 2011.

[3] H. H. Xu, C. Caramanis, and S. Mannor, "Optimization under probabilistic envelope constraints," *Oper. Res.*, vol. 60, no. 3, pp. 682–699, 2012.

[4] S. Leyendecker, L. J. Lucas, H. Owhadi, and M. Ortiz, "Optimal control strategies for robust certification," *Journal of Computational and Nonlinear Dynamics*, vol. 5, no. 3, pp. 031 008–031 008, 2010.

[5] J. Figueira, S. Greco, and M. Ehrgott, *Multiple Criteria Decision Analysis: State of the Art Surveys*. Boston, Dordrecht, London: Springer Verlag, 2005.

[6] T. Binder, L. Blank, H. G. Bock, R. Bulirsch, W. Dahmen, M. Diehl, T. Kronseder, W. Marquardt, J. Schlöder, and O. von Stryk, "Introduction to model based optimization of chemical processes on moving horizons," in *Online Optimization of Large Scale Systems: State of the Art*, M. Grötschel, S. O. Krumke, and J. Rambau, Eds. Springer, 2001, pp. 295–340.

[7] P. E. Gill, L. O. Jay, M. W. Leonard, L. R. Petzold, and V. Sharma, "An SQP method for the optimal control of large-scale dynamical systems," *Journal of Computational and Applied Mathematics*, vol. 120, pp. 197–213, 2000.

[8] S. Ober-Blöbaum, O. Junge, and J. E. Marsden, "Discrete mechanics and optimal control: an analysis," *Control, Optimisation and Calculus of Variations*, vol. 17, no. 2, pp. 322–352, 2011.

[9] S. Ober-Blöbaum, M. Ringkamp, and G. zum Felde, "Solving multiobjective optimal control problems in space mission design using discrete mechanics and reference point techniques," in *51st IEEE International Conference on Decision and Control*, Maui, HI, USA, 10-13 December 2012, pp. 5711–5716.

[10] Y. Cao, M. Hussaini, and T. Zang, "An efficient Monte Carlo method for optimal control problems with uncertainty," *Computational Optimization and Applications*, vol. 26, pp. 219–230, 2003.

[11] A. J. Kleywegt, A. Shapiro, and T. Homem-de Mello, "The sample average approximation method for stochastic discrete optimization," *SIAM J. on Optimization*, vol. 12, no. 2, pp. 479–502, Feb. 2002.

[12] K. Deb, *Multi-Objective Optimization Using Evolutionary Algorithms*. Wiley, 2001.

[13] M. Dellnitz, O. Schütze, and T. Hestermeyer, "Covering Pareto sets by multilevel subdivision techniques," *Journal of Optimization Theory and Applications*, vol. 124, pp. 113–155, 2005.

[14] K. Miettinen, *Nonlinear Multiobjective Optimization*. Kluwer Academic Publishers, 1999.

[15] C. Hillermeier, *Nonlinear Multiobjective Optimization - A Generalized Homotopy Approach*. Berlin: Birkhäuser, 2001.

[16] M. Ringkamp, S. Ober-Blöbaum, M. Dellnitz, and O. Schütze, "Handling high dimensional problems with multi-objective continuation methods via successive approximation of the tangent space," *Engineering Optimization*, vol. 44, no. 6, pp. 1117–1146, 2012.

[17] F. Logist, B. Houska, M. Diehl, and J. V. Impe, "Fast Pareto set generation for nonlinear optimal control problems with multiple objectives," *Structural and Multidisciplinary Optimization*, vol. 42, pp. 591–603, 2010.

[18] O. Schütze, K. Witting, S. Ober-Blöbaum, and M. Dellnitz, "Set oriented methods for the numerical treatment of multi-objective optimization problems," in *EVOLVE – A Bridge Between Probability, Set Oriented Numerics, and Evolutionary Computation*, ser. Studies in Computational Intelligence, E. T. et al., Ed. Springer Berlin Heidelberg, 2013, vol. 447, pp. 187–219.

[19] C. Romaus, J. Böcker, K. Witting, A. Seifried, and O. Znamenshchykov, "Optimal energy management for a hybrid energy storage system combining batteries and double layer capacitors," in *IEEE Energy Conversion Congress and Expo (ECCE)*, San Jose, CA, USA, 2009.

[20] J. E. Marsden and M. West, "Discrete mechanics and variational integrators," *Acta Numerica*, vol. 10, pp. 357–514, 2001.

[21] M. S. Eldred, "Recent Advances in Non-Intrusive Polynomial Chaos and Stochastic Collocation Methods for Uncertainty Analysis and Design," in *Structures, Structural Dynamics, and Materials Conference*. Sandia National Laboratories, Albuquerque, NM 87185, 2009.