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Necessary Conditions for Structural and Strong Structural Controllability of Linear Time-Varying Systems

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***Abstract*— In this paper, the notion of structural controllability of linear time-invariant systems is extended to the time-varying case $\dot{x}(t) = A(t) \cdot x(t) + B(t) \cdot u(t)$. We provide two Examples which show that neither the conditions for structural controllability of time-invariant systems are necessary, nor the conditions for strong structural controllability of time-invariant systems are sufficient for the controllability of time-varying systems. We present a necessary condition for structural controllability of linear time-varying systems and in our main result a necessary condition for strong structural controllability of linear time-varying systems is given. In a previous work, this necessary condition for strong structural controllability of linear time-invariant systems was shown to be sufficient, so that the strong structural controllability of linear time-invariant systems is now characterized. We want to emphasize, that our results cover the single and the multi-input case.**

I. Introduction

Consider the linear time-invariant control system

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t), \quad x(t_0) = x_0, \tag{1}$$

in which the state variables x and the input variables u take their values in $\mathbb{R}^n$ and $\mathbb{R}^r$, respectively. A basic question in the analysis of a system (1) is roughly speaking the ability to transfer the state from any x_0 to any other state $x(t_1) = x_1$ using the control function u. If this is possible, then the linear time-invariant system (1) is *controllable*.

There exists some well-known tests for the *controllability* of a certain system (1) [1, 2], but if the dimension n of the system becomes higher, the results of these numerical tests are more and more unreliable. Moreover, the entries of the matrices A and B are in many applications approximated and not exactly known, so that it is often advisable to consider a system (1) as an uncertain system with fixed zeros and free parameters.

One possibility to represent the parameter uncertainty is to make a distinction between zero and non-zero entries in the matrices A and B of (1) (see [3]–[12]), which was first proposed by Lin [3] in 1974. Then the matrices A and B are of a certain zero-nonzero pattern (or structure), and the system (1) is defined to be *structurally controllable* if there exists at least one controllable system with the same pattern of zero and nonzero elements in A and B. A few years later, Mayeda and Yamada [6] introduced the notion of *strong structural controllability*, in which all systems with the same zero-nonzero pattern are controllable. Both *structural properties* are characterized for the single and the multi-input ($r \geq 1$) case [3]–[7].

Hence, the structural controllability describes a necessary and the strong structural controllability a sufficient property for the controllability of a linear time-invariant system (1). This can be illustrated with the following matrices of the control system (1)

$$A = \begin{pmatrix} 0 & \rho_1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \rho_2 \\ \rho_3 \end{pmatrix}. \tag{2}$$

For controllability of (1) with (2), the determinant of the Kalman-Matrix $\det\big(B, AB\big) = \rho_1 \cdot \rho_3^2$ has to be nonzero. This is obviously true, if none of the two parameters ρ_1 and ρ_3 is zero, so that all systems (1) with the same zero-nonzero pattern as in (2) are controllable i.e. all systems (1) with the structure of (2) are structurally controllable as well as strongly structurally controllable. Notice that if $\rho_2 = 0$, then the matrix $A' = A|_{\rho_2=0}$ has a different structure than A, but it is obvious that all systems (1) with A' and B are strongly structurally controllable, too.

In this paper, we investigate the structural and the strong structural controllability of linear time-varying control systems of the form

$$\dot{x}(t) = A(t) \cdot x(t) + B(t) \cdot u(t), \quad x(t_0) = x_0, \tag{3}$$

in which the time-varying entries of $A\colon \mathbb{R} \to \mathbb{R}^{n\times n}$ and $B\colon \mathbb{R} \to \mathbb{R}^{n\times r}$ are analytic functions. For this purpose, we first analyze a control system (3) with the following matrices

$$A(t) = \begin{pmatrix} 0 & \rho_1 \\ 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} t+1 \\ 1 \end{pmatrix}. \tag{4}$$

Here, the parameter ρ_2 is replaced by $t+1$ and the parameter ρ_3 is set to 1 in the matrices (2), so that for all $t > 0$ the same elements in the matrices $A(t)$ and $B(t)$ of (4) and of (2) are zero and non-zero, respectively. The solution of (3) with (4), $\rho_1 = 1$, $x_0 = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$ and $t_0 = 0$ is

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} t+1 \\ 1 \end{pmatrix} \cdot \int_0^t u(\tau)\, d\tau. \tag{5}$$

The solutions x_1 and x_2 are dependent for all control functions u and it is not possible to control this system to an arbitrary state $x_1(t_e) \neq x_2(t_e) \cdot (t_e + 1)$ at the time $t_e > 0$. This time-varying system is not controllable, even if only one of the three parameters of the strongly structurally controllable, time-invariant system (2) is changing in time. The requirements for strong structural controllability of linear

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time-invariant systems (1) are not sufficient for linear time-varying control systems (3) (see [13] for another example).

Consider another class of linear time-invariant systems (1) with (2) and $\rho_1 = 0$. Then, none of the systems (1) with $A' = A|_{\rho_1=0}$ and B are controllable, because the determinant of the Kalman-Matrix is zero, so that all the linear time-invariant systems (1) with A' and B are not structurally controllable and of course not strongly structurally controllable, too. Take now the linear time-varying system (3) with (4) and $\rho_1 = 0$. This time-varying system has for $t > 0$ the same structure as the non controllable time-invariant system and the solution is

$$x(t) = \int_0^t \begin{pmatrix} \tau + 1 \\ 1 \end{pmatrix} \cdot u(\tau)\, d\tau \tag{6}$$

for $x_0 = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$ and $t_0 = 0$. Using a certain control function u, the system can be transferred to any arbitrary state $x(t_e) = \begin{pmatrix} x_{1e} & x_{2e} \end{pmatrix}^T$ with $t_e > 0$. One possible control function for $t_e = 1$ is $u(t) = 6(2t-1)x_{1e} + 2(5-9t)x_{2e}$.

The second example proves that the requirements for structural controllability of time-invariant systems (1) are not necessary for the controllability of linear time-varying control systems (3). Hence, neither the conditions for structural controllability of (1) are necessary, nor the conditions for strong structural controllability of (1) are sufficient for the controllability of linear time-varying system (3).

In [13] sufficient conditions for strong structural controllability of linear time-varying systems (3) are given. In this paper, we prove that these conditions are also necessary, so that the class of strongly structurally controllable, time-varying systems (3) is now characterized. One basic property of this class of systems is for example, that the time-varying coefficient matrix B has to have at least one column with only one nonzero element.

Moreover, we extend the notion of structural controllability of linear time-invariant systems (1) to the time-varying case (3) and we give a necessary condition for time-varying systems (3). We want to emphasize, that our results cover the single and the multi-input case.

The remaining paper is organized as follows. In the next section, the basic notation as well as well-known definitions and theorems to controllability and graphs are presented. In Section III, basic properties and definitions concerning structural matrices and systems are given and finally, our main result, Theorem 18, is presented in Section IV.

II. Preliminaries

A. Basic notation

The set $\mathbb{Z}$ and the set $\mathbb{R}$ denote the set of of all integers and all real numbers, respectively. For two sets N, M, $N \subseteq M$, $N \subset M$, $N \supseteq M$, $N \supset M$ means that N is a subset, a strict subset, a superset and a strict superset of M, respectively. The notation $[a, b]$, $]a, b[$, $[a, b[$, and $]a, b]$ stands for closed, open and half-open intervals in $\mathbb{R}$ with end points a and b and $[a; b]$, $]a; b[$, $[a; b[$, and $]a; b]$ for discrete intervals, e.g. $[a; b] = [a, b] \cap \mathbb{Z}$. A sufficient small interval around a point $t \in \mathbb{R}$, denoted by $\mathbb{I}_t$, is a continuous interval with end points $t - \alpha$ and $t + \beta$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha, \beta > 0$.

The set of real $n \times m$ matrices is denoted by $\mathbb{R}^{n \times m}$. For $X \in \mathbb{R}^{n \times m}$, $X_{i,j}$ denotes the entry at the ith row of the jth column of X, and X^T is the transpose of X. The rows of the time-varying coefficient matrix $X\colon \mathbb{R} \to \mathbb{R}^{n \times m}$ are linear dependent on the interval $\mathbb{T}$ if there exists a row vector $v \in \mathbb{R}^{1 \times n}$ such that for all $t \in \mathbb{T}$, $vX(t) = 0$.

The derivative of a function x at the time t is denoted by $\dot{x}(t)$. The $n \times n$ identity matrix is denoted by I_n.

B. Controllability

In this subsection we collect some well-known definitions and theorems concerning the solution and the controllability of linear systems (1) and (3) [1], [14]–[16].

The solution of the linear time-varying system (3) is

$$x(t) = \Psi_A(t)\Psi_A^{-1}(t_0)x_0 + \Psi_A(t) \int_{t_0}^t \Psi_A^{-1}(\tau)B(\tau)u(\tau)d\tau, \tag{7}$$

in which Ψ_A is the nonsingular fundamental matrix (or solution matrix) of the system satisfying

$$\dot{\Psi}_A(t) = A(t)\Psi_A(t) \quad \text{and} \quad \dot{\Psi}_A^{-1}(t) = -\Psi_A^{-1}(t)A(t).$$

In the case of linear time-invariant system (1) with a constant matrix A, the fundamental matrix is $\Psi_A(t) = e^{At}$.

Definition 1 : The linear time-invariant system (1) is *controllable* if for any initial state x_0 at t_0 and any state x_1 at $t_1 \geq t_0$, there exists an input $u\colon [t_0, t_1] \to \mathbb{R}^r$ such that $x(t_1) = x_1$.

Theorem 2 : Let matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times r}$ be given, then the following are equivalent.

(i) The linear time-invariant system (1) with A and B is controllable.

(ii) rank $\hat{Q}_{n,A,B} = n$ with

$$\hat{Q}_{k,A,B} = \left(A^{k-1}B, \ldots, AB, B\right) \in \mathbb{R}^{n \times k \cdot r}. \tag{8}$$

(iii) rank $(\lambda \cdot I_n - A, B) = n$ for all $\lambda \in \mathbb{C}$.

In the case of linear time-varying systems (3), we need to distinguish between different kinds of controllability.

Definition 3 : The linear time-varying system (3) is *(completely) controllable on the interval* $\mathbb{T} = [t_0, t_1]$ if for any initial state x_0 at t_0 and any state x_1 at t_1, there exists an input $u\colon \mathbb{T} \to \mathbb{R}^r$ such that $x(t_1) = x_1$.

Definition 4 : The linear time-varying system (3) is *totally controllable on the interval* $\mathbb{T}$ if it is completely controllable on every subinterval of $\mathbb{T}$.

Theorem 5 : For any interval $\mathbb{T}$ and any time-varying coefficient matrices $A\colon \mathbb{T} \to \mathbb{R}^{n \times n}$ and $B\colon \mathbb{T} \to \mathbb{R}^{n \times r}$, in which the entries of A and B are analytic on $\mathbb{T}$, the following are equivalent.

(i) The linear time-varying system (3) with A and B is totally controllable on $\mathbb{T}$.
(ii) The rows of $\Theta_{A,B} = \Psi_A^{-1}B$ are not linear dependent on every subinterval of $\mathbb{T}$.
(iii) $Q_{n,A,B}(t)$ does not have rank less than n on a set of points t everywhere dense in $\mathbb{T}$, in which

$$Q_{k,A,B}(t) = \left(P_{k-1,A,B}(t), \quad \ldots, \quad P_{0,A,B}(t)\right), \quad (9)$$

$$P_{i+1,A,B}(t) = -A(t) \cdot P_{i,A,B}(t) + \dot{P}_{i,A,B}(t) \quad (10)$$

and $P_{0,A,B}(t) = B(t)$.

We remark that if the matrices A and B of system (3) are constant on the interval $\mathbb{T}$, then complete and total controllability are equivalent and equal to controllability of linear time-invariant systems (1).

C. Graphs

For any matrix $X \in \mathbb{R}^{n\times m}$, the *graph of* X, denoted $\mathcal{G}(X)$, is the pair $(\mathcal{V},\mathcal{E})$ of vertices and directed edges. If $m > n$, the set of vertices is $\mathcal{V} = \{1, 2, \ldots, m\}$ and the set of directed edges $\mathcal{E}$ is given by

$$\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} | X_{j,i} \neq 0\}.$$

See [8]. The vertices $\{1, \ldots, n\}$ in $\mathcal{V}$ are the *state vertices* and the vertices $\{n+1, \ldots, m\}$ in $\mathcal{V}$ are the *input vertices* of $\mathcal{G}(X)$. For any directed edge $(i, j) \in \mathcal{E}$ of $\mathcal{G}(X)$, the vertex j is a *successor* of i, and the vertex i is a *predecessor* of j. A *path from the vertex* i_1 *to the vertex* i_k of $\mathcal{G}(X)$ is a sequence of $k \geq 2$ distinct vertices $\{i_1, i_2, \ldots, i_k\}$ if for all $l \in [2; k], (i_{l-1}, i_l) \in \mathcal{E}$. For any set N of vertices of $\mathcal{G}(X)$, $|N|$ denotes the number of distinct vertices in N and $T(N)$ denotes the set of predecessors of vertices in N,

$$T(N) = \{i | (i, j) \in \mathcal{E} \text{ and } j \in N\}.$$

A state vertex i of $\mathcal{G}(X)$ is said to be *non-accessible* if there exists no path from any input vertex j to i in $\mathcal{G}(X)$ and the graph $\mathcal{G}(X)$ is said to contain a *dilation* if there exists a set N of state vertices such that $|T(N)| < |N|$.

III. Structural Matrices, Properties and Systems

A. Structural matrices

Two matrices $X, Y \in \mathbb{R}^{n\times m}$ are *structural equivalent*, denoted by $X \sim Y$ if X and Y are of the same zero-nonzero pattern,

$$X \sim Y :\Leftrightarrow \forall i, j \left(X_{i,j} = 0 \Leftrightarrow Y_{i,j} = 0\right).$$

The relation $\sim$ is an equivalence relation on $\mathbb{R}^{n\times m}$ and the equivalence classes are $n \times m$ structural matrices

$$[X]_\sim = \left\{Y \in \mathbb{R}^{n\times m} \middle| X \sim Y\right\}.$$

The class $[X]_\sim \in \mathbb{R}^{n\times m}/\sim$ is said to be the structure of $X \in \mathbb{R}^{n\times m}$. Given a structural matrix $\mathcal{X} \in \mathbb{R}^{n\times m}/\sim$ and an interval $\mathbb{T}$, we define the set

$$\mathcal{F}_{\mathcal{X},\mathbb{T}} := \left\{X : \mathbb{T} \to \mathbb{R}^{n\times m} \middle| \begin{array}{l} X \text{ analytic and } \forall t \in \mathbb{T}, \\ X(t) \in \mathcal{X} \in \mathbb{R}^{n\times m}/\sim \end{array}\right\}.$$

A class $\mathcal{X} \in \mathbb{R}^{n\times m}/\sim$ is represented with a $n\times m$ matrix, in which non-zero entries are depicted with asterisks, and zeros with "0". For example,

$$\text{if } \mathcal{X} = \begin{pmatrix} * & 0 & * \\ * & 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 3}/\sim \text{ and } \mathbb{T} =]0, 1[$$

$$\text{then e.g. } X(t) = \begin{pmatrix} 1 & 0 & \sin(\pi \cdot t) \\ e^{-t} & 0 & 0 \end{pmatrix} \in \mathcal{F}_{\mathcal{X},\mathbb{T}}.$$

Of course, if we take the interval $\mathbb{T}' = [0, 1]$, then $X \notin \mathcal{F}_{\mathcal{X},\mathbb{T}'}$ because the element $X_{1,3}(0) = X_{1,3}(1) = 0$.

For convenience, $\mathcal{I}_n := [I_n]_\sim$ and given any structural matrix $\mathcal{X}$, then $\mathcal{G}(\mathcal{X}) := \mathcal{G}(X)$ for some $X \in \mathcal{X}$.

B. Structural properties

Application of the usual matrix operations to structural matrices does not make much sense. However, for two structural matrices $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{n\times m}/\sim$, the operation of addition may be defined as

$$(\mathcal{X} + \mathcal{Y})_{i,j} := \begin{cases} 0 & \text{if } \mathcal{X}_{i,j} = \mathcal{Y}_{i,j} = 0 \\ * & \text{else} \end{cases}$$

and the transformation of a structural matrix by the permutation matrices P_1 and P_2 may be defined as

$$P_1^T[X]_\sim P_2 := [P_1^T X P_2]_\sim.$$

Given any $\mathcal{A} \in \mathbb{R}^{n\times n}/\sim$ and any $\mathcal{B} \in \mathbb{R}^{n\times r}/\sim$, the structural pair $(\mathcal{A}, \mathcal{B})$ is said to be in *Form I* if there exists a permutation matrix P such that

$$P^T \cdot \mathcal{A} \cdot P = \begin{pmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix} \quad \text{and} \quad P^T \cdot \mathcal{B} = \begin{pmatrix} 0 \\ \mathcal{B}_2 \end{pmatrix}, \quad (11)$$

in which $\mathcal{A}_{11} \in \mathbb{R}^{q\times q}/\sim$, $\mathcal{A}_{21} \in \mathbb{R}^{(n-q)\times q}/\sim$, $\mathcal{A}_{22} \in \mathbb{R}^{(n-q)\times(n-q)}/\sim$, $\mathcal{B}_2 \in \mathbb{R}^{(n-q)\times r}/\sim$ and $q > 0$.

Given any structural matrix $\mathcal{X} \in \mathbb{R}^{n\times m}/\sim$. $\mathcal{X}$ is said to be in *Form II* if all $X \in \mathcal{X}$ are not of full row rank n and $\mathcal{X}$ is said to be in *Form III* if there exists two permutation matrices P_1 and P_2 such that

$$P_1^T \cdot \mathcal{X} \cdot P_2 = \begin{pmatrix} x & \cdots & x & * & & & 0 \\ \vdots & & \vdots & \ddots & * & & \\ \vdots & & \vdots & & \ddots & \ddots & \\ x & \cdots & x & \cdots & \cdots & x & * \end{pmatrix}, \quad (12)$$

where each x-entry is either $*$ or 0. See [12, 17].

C. Structural systems

Let the structural matrices $\mathcal{A} \in \mathbb{R}^{n\times n}/\sim$ and $\mathcal{B} \in \mathbb{R}^{n\times r}/\sim$ be given, the *class of linear time-invariant systems* (1) *of structure* $(\mathcal{A}, \mathcal{B})$ is the set of systems (1) which satisfy $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Given additionally any continuous interval $\mathbb{T}$, the *class of linear time-varying systems* (3) *of structure* $(\mathcal{A}, \mathcal{B})$ *on* $\mathbb{T}$ is the set of systems (3) which satisfy $A \in \mathcal{F}_{\mathcal{A},\mathbb{T}}$ and $B \in \mathcal{F}_{\mathcal{B},\mathbb{T}}$.

We now present some definition and results on structural and strong structural controllability of linear time-invariant systems (1).

Definition 6 : The class of linear time-invariant systems (1) of structure $(\mathcal{A}, \mathcal{B})$ is *structurally controllable (SC)* if there exists some $A \in \mathcal{A}$ and some $B \in \mathcal{B}$, such that (1) is controllable.

Theorem 7 ([3]–[5]): For structural matrices $\mathcal{A} \in \mathbb{R}^{n\times n}/\sim$ and $\mathcal{B} \in \mathbb{R}^{n\times r}/\sim$ the following are equivalent.

(i) The class of linear time-invariant systems (1) of structure $(\mathcal{A}, \mathcal{B})$ is SC.
(ii) The graph $\mathcal{G}(\mathcal{A}, \mathcal{B})$ contains no non-accessible vertex, and no dilation.
(iii) The structural pair $(\mathcal{A}, \mathcal{B})$ is not in Form I, and the structural matrix $\mathcal{M} = (\mathcal{A}, \mathcal{B})$ is not in Form II.

Definition 8 : The class of linear time-invariant systems (1) of structure $(\mathcal{A}, \mathcal{B})$ is *strongly structurally controllable (SSC)* if for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$ (1) is controllable.

Theorem 9 ([6, 7]): For structural matrices $\mathcal{A} \in \mathbb{R}^{n\times n}/\sim$ and $\mathcal{B} \in \mathbb{R}^{n\times r}/\sim$ the following are equivalent.

(i) The class of linear time-invariant systems (1) of structure $(\mathcal{A}, \mathcal{B})$ is SSC.
(ii) The graph $\mathcal{G}(\mathcal{A}, \mathcal{B})$ satisfies the following two properties:
  a) For every non-empty subset $N \subseteq \{1, \ldots, n\}$ of vertices of $\mathcal{G}(\mathcal{A}, \mathcal{B})$ there is some vertex j such that N contains exactly one successor of j in $\mathcal{G}(\mathcal{A}, \mathcal{B})$.
  b) For every non-empty subset $N \subseteq \{1, \ldots, n\}$ of vertices of $\mathcal{G}(\mathcal{A}, \mathcal{B})$ that satisfies $N \subseteq T(N)$, there is some vertex $j \in T(N) \setminus N$ such that N contains exactly one successor of j in $\mathcal{G}(\mathcal{A}, \mathcal{B})$.
(iii) The structural matrix $(\mathcal{A}, \mathcal{B})$ is in Form III, and the structural matrix $(\mathcal{I}_n + \mathcal{A}, \mathcal{B})$ can be permuted into the form (12) in such a way that the following holds: If $\mathcal{A}_{i,i} = *$, then the entry $(\mathcal{I}_n + \mathcal{A})_{i,i}$ is not permuted to a $*$-entry in (12).

Furthermore, we will need a results of [13], which can be deduced from Theorem 9 with the system $(\mathcal{A}', \mathcal{B}) = (\mathcal{I}_n + \mathcal{A}, \mathcal{B})$.

Proposition 10 ([13]): For structural matrices $\mathcal{A} \in \mathbb{R}^{n\times n}/\sim$ and $\mathcal{B} \in \mathbb{R}^{n\times r}/\sim$ the following are equivalent.

(i) The class of linear time-invariant systems (1) of structure $(\mathcal{I}_n + \mathcal{A}, \mathcal{B})$ is SSC.
(ii) For every non-empty subset $N \subseteq \{1, \ldots, n\}$ of vertices of $\mathcal{G}(\mathcal{A}, \mathcal{B})$, there is some vertex $j \in \{1, \ldots, n + r\} \setminus N$ such that N contains exactly one successor of j in $\mathcal{G}(\mathcal{A}, \mathcal{B})$.
(iii) The structural matrix $(\mathcal{I}_n + \mathcal{A}, \mathcal{B})$ can be permuted into the form (12) in such a way that none of the $(\mathcal{I}_n + \mathcal{A})_{i,i}$ is permuted to a $*$-entry in (12).

IV. Main Result: Structural and Strong Structural Controllability of Time-Varying Systems

In this section, we define the structural and the strong structural controllability for the class of linear time-varying systems (3) and we present our main result in Theorem 18.

For any structural matrices $\mathcal{A} \in \mathbb{R}^{n\times n}/\sim$ and $\mathcal{B} \in \mathbb{R}^{n\times r}/\sim$ and any point $t \in \mathbb{R}$, we define the structural and the strong structural controllability of linear time-varying systems (3) as follows.

Definition 11 : The class of linear time-varying systems (3) of structure $(\mathcal{A}, \mathcal{B})$ on $\mathbb{T} = \mathbb{I}_t$ is *structurally controllable (SC)* if there exists some $A \in \mathcal{F}_{\mathcal{A},\mathbb{T}}$ and some $B \in \mathcal{F}_{\mathcal{B},\mathbb{T}}$, such that (3) is totally controllable on $\mathbb{T}$.

Definition 12 : The class of linear time-varying systems (3) of structure $(\mathcal{A}, \mathcal{B})$ on $\mathbb{T} = \mathbb{I}_t$ is *strongly structurally controllable (SSC)* if for all $A \in \mathcal{F}_{\mathcal{A},\mathbb{T}}$ and all $B \in \mathcal{F}_{\mathcal{B},\mathbb{T}}$ (3) is totally controllable on $\mathbb{T}$.

The SC and the SSC of time-varying systems is defined on a sufficient small interval $\mathbb{T}$ around a point t. This is due to the fact, that polynomials of any degree should be allowed as analytic functions in the matrices A and B, in which usually several zero points exists. Hence, we have restricted the interval $\mathbb{T}$ to be not arbitrarily long to ensure that $A \in \mathcal{F}_{\mathcal{A},\mathbb{T}}$ and $B \in \mathcal{F}_{\mathcal{B},\mathbb{T}}$ is satisfied even if the entries are polynomials. See the discussion after Lemma 16.

A. Structural controllability of time-varying systems

With the following Lemma 13, we give a condition for total controllability of the linear time-varying system (A, B) by analyzing the total controllability of a subsystem of (A, B).

Lemma 13 : Let any interval $\mathbb{T}$ and any time-varying coefficient matrices $A_0 \colon \mathbb{T} \to \mathbb{R}^{n\times n}$, $B_1 \colon \mathbb{T} \to \mathbb{R}^{n\times r_1}$ and $B_2 \colon \mathbb{T} \to \mathbb{R}^{n\times r}$ be given, in which the entries of A_0, B_1 and B_2 are analytic on $\mathbb{T}$. If the linear time-varying system (3) with A_0 and $B_0 = (B_1, B_2)$ is not totally controllable on $\mathbb{T}$, then the linear time-varying systems (3) with

$$A = \begin{pmatrix} A_0 & B_1 \\ X_1 & X_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_2 \\ X_3 \end{pmatrix}$$

is not totally controllable on $\mathbb{T}$, independent of the analytic matrices $X_1 \colon \mathbb{T} \to \mathbb{R}^{r_1\times n}$, $X_2 \colon \mathbb{T} \to \mathbb{R}^{r_1\times r_1}$ and $X_3 \colon \mathbb{T} \to \mathbb{R}^{r_1\times r}$.

Proof. 1) If the linear time-varying system (3) with A_0 and B_0 is not totally controllable on $\mathbb{T}$, then by Theorem 5.(iii) the matrix Q_{n,A_0,B_0} has not rank n on the interval $\mathbb{T}$ and there is a time-varying row vector $w \neq 0$ such that $w \cdot Q_{n,A_0,B_0} = 0 = w \cdot (Q_{n,A_0,B_1}, Q_{n,A_0,B_2})$. This implies that $w \cdot P_{i,A_0,B_1} = 0$ and $w \cdot P_{i,A_0,B_2} = 0$ for all $i \in [0; n-1]$ and since the time-varying entries of the matrices are analytic, this is also true for all $i \geq n$ (see [1]).

We show now, that $\tilde{w} \cdot Q_{n+r_1,A,B} = 0$ or rather $\tilde{w} \cdot P_{i,A,B} = 0$ for all $i \geq 0$ with $\tilde{w} = (w, 0, \ldots, 0)$ is then satisfied.

2a) To this end, we define M_i as the upper $(n \times r)$ and N_i as the lower $r_1 \times r$ submatrix of $P_{i,A,B}$ so that $P_{i,A,B}^T = \left(M_i^T, N_i^T\right)$. We first prove that

$$M_i = P_{i,A_0,B_2} - \sum_{j=0}^{i-1} P_{j,A_0,B_1} \cdot K_{i,j} \tag{13}$$

with $K_{0,0} = 0$ and some clear defined $K_{i,j}$ using an induction over i.

2b) For $i = 0$, $M_0 = B_2 = P_{0,A_0,B_2}$ and (13) is true. We now assume that if (13) is satisfied for some $k \geq 0$ then (13) is also satisfied for $i = k + 1$.

With equation (10), we get $M_{k+1} = -A_0 M_k + \dot{M}_k - B_1 N_k$. The derivative of equation (13) yields $\dot{M}_k = \dot{P}_{k,A_0,B_2} - \sum_{j=0}^{k-1} \dot{P}_{j,A_0,B_1} K_{k,j} - \sum_{j=0}^{k-1} P_{j,A_0,B_1} \dot{K}_{k,j}$ so that M_{k+1} is represented by $M_{k+1} = P_{k+1,A_0,B_2} - \sum_{j=0}^{k-1} P_{j+1,A_0,B_1} K_{k,j} - \sum_{j=0}^{k-1} P_{j,A_0,B_1} \dot{K}_{k,j} - B_1 N_k$. We define

$$K_{k+1,j} = \begin{cases} \dot{K}_{k,0} + N_k & \text{for } j = 0 \\ K_{k,j-1} + \dot{K}_{k,j} & \text{for } j \in [1; k-1] \\ K_{k,k-1} & \text{for } j = k \end{cases}$$

and M_{k+1} can be expressed by $M_{k+1} = P_{k+1,A_0,B_2} - \sum_{j=0}^{k} P_{j+1,A_0,B_1} K_{k+1,j}$. This result is equal to equation (13) for $i = k + 1$, which completes the proof of 2a).

3) If we take $\tilde{w}$ defined as above, $\tilde{w} \cdot P_{i,A,B}$ equals $w \cdot M_i$ and with (13) it is equal to $wP_{i,A_0,B_2} - \sum_{j=0}^{i-1} wP_{j,A_0,B_1} K_{i,j}$. By the assumption, $wP_{i,A_0,B_1} = 0$ and $wP_{i,A_0,B_2} = 0$ for all $i \geq 0$ so that $\tilde{w} \cdot P_{i,A,B} = 0$ for all $i \geq 0$ and hence, $\tilde{w} \cdot Q_{n+r,A,B} = 0$. Using Theorem 5.(iii), the linear time-varying system (3) with A and B is not totally controllable on $\mathbb{T}$. □

We now present a necessary condition on SC of linear time-varying systems (3).

Theorem 14 : Let structural matrices $\mathcal{A} \in \mathbb{R}^{n\times n}/\sim$ and $\mathcal{B} \in \mathbb{R}^{n\times r}/\sim$ and any interval $\mathbb{T}$ be given. The class of linear time-varying systems (3) of structure $(\mathcal{A}, \mathcal{B})$ on $\mathbb{T}$ is SC only if the pair $(\mathcal{A}, \mathcal{B})$ is not in Form I.

Proof. Assume that the condition is not true. Then there exists a permutation matrix P such that $P^T \mathcal{A} P = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix}$ and $P^T \mathcal{B} = \begin{pmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{pmatrix}$ with $\mathcal{A}_{12} = 0$ and $\mathcal{B}_1 = 0$. It is obvious that all systems with $A_0 \in \mathcal{F}_{\mathcal{A}_{11},\mathbb{T}}$ and $B_0 \in \mathcal{F}_{(\mathcal{A}_{12},\mathcal{B}_1),\mathbb{T}} = \mathcal{F}_{(0,0),\mathbb{T}}$ are not controllable and of course not totally controllable on any interval $\mathbb{T}$. From Lemma 13 follows, that all linear time-varying systems (3) with $A \in \mathcal{F}_{\mathcal{A},\mathbb{T}}$ and $B \in \mathcal{F}_{\mathcal{B},\mathbb{T}}$ are also not totally controllable on $\mathbb{T}$. □

The algebraic condition of Theorem 14 is equal to the following graph theoretic condition (See [3]).

Corollary 15 : Let structural matrices $\mathcal{A} \in \mathbb{R}^{n\times n}/\sim$ and $\mathcal{B} \in \mathbb{R}^{n\times r}/\sim$ and any interval $\mathbb{T}$ be given. The class of linear time-varying systems (3) of structure $(\mathcal{A}, \mathcal{B})$ on $\mathbb{T}$ is SC only if the graph $\mathcal{G}(\mathcal{A}, \mathcal{B})$ contains no non-accessible vertex.

B. Strong structural controllability of time-varying systems

Clearly, since the conditions of Theorem 14 and Corollary 15 are necessary for SC of linear time-varying systems (3), these conditions are also necessary for SSC of linear time-varying systems (3) per definition. In the following Lemma 16, we provide a further necessary condition for SSC of linear time-varying systems (3) based on the structure of the input matrix B. Together with Lemma 13, we finally deduce our main result in Theorem 18, which is combined with the result of [13] in Theorem 19 a sufficient and necessary condition for SSC of linear time-varying systems (3).

Lemma 16 : Let any structural matrices $\mathcal{A} \in \mathbb{R}^{n\times n}/\sim$ and $\mathcal{B} \in \mathbb{R}^{n\times r}/\sim$ and any point $t \in \mathbb{R}$ be given. The class of linear time-varying systems (3) of structure $(\mathcal{A}, \mathcal{B})$ on $\mathbb{I}_t$ is SSC only if for the set of state vertices $Z = \{1, \dots, n\}$ of $\mathcal{G}(\mathcal{A}, \mathcal{B})$ there is a input vertex $j \in \{n+1, \dots, n+r\}$ such that Z contains exactly one successor of j in $\mathcal{G}(\mathcal{A}, \mathcal{B})$.

Proof. Assume that the condition is not true. Then there is no column in $\mathcal{B}$ with just one element. Take any constant $A \in \mathcal{A}$ and generate $\Psi_A^{-1}(t) = e^{-A\cdot t}$. Since $\Psi_A^{-1}(t)$ is nonsingular for all t, a row vector $v = (v_1, v_2 \dots, v_n) \in \mathbb{R}^{1\times n}$ can be selected such that the time-varying row vector $w(t) = v\Psi_A^{-1}(t)$ has no column w_i with $w_i(t) = 0$ on a sufficient small interval $\mathbb{T}$ around the point t. Each column $i \in [1; r]$ of $\mathcal{B}$ has $q_i \geq 2$ or $q_i = 0$ elements, so that we can select the time-varying entries of $B \in \mathcal{F}_{\mathcal{B},\mathbb{T}}$ such that $w(t)B(t) = 0$ for all $t \in \mathbb{T}$. The rows of $\Theta_{A,B} = \Psi_A^{-1} B$ are then linear dependent on every subinterval of $\mathbb{T}$ and with Theorem 5.(ii) the linear time-varying system (3) with A and B is not totally controllable on $\mathbb{T}$. Hence, the class of linear time-varying systems (3) of structure $(\mathcal{A}, \mathcal{B})$ on $\mathbb{T}$ is not SSC. □

The proof of Lemma 16 can be seen as an instruction to create a not totally controllable time-varying system (3) if the condition is not satisfied. This is done by selecting the time-varying elements of B such that the rows of $\Theta_{A,B}$ are linear dependent. To illustrate the proof and the necessity of a sufficient small interval within the proof, we will now deduce a not totally controllable time-varying system (3) from the structure of the example in equation (2).

There is no column in $\mathcal{B}$ with just one element, so that the condition of Lemma 16 is not satisfied and we select the point $t_0 \in \mathbb{R}$. The inverse of the fundamental matrix is given by

$$\Psi_A^{-1}(t) = e^{-A\cdot t} = \begin{pmatrix} 1 & -\rho_1 \cdot t \\ 0 & 1 \end{pmatrix},$$

so that we can pre-multiply a vector $v = (v_1, v_2)$ to get $w(t) = (v_1, -v_1 \cdot \rho_1 \cdot t + v_2)$. To ensure that both, w_1 and w_2, are non-zero on the interval $\mathbb{T} = [t_0 - c, t_0 + c]$ for some $c \in \mathbb{R}, c > 0$, we define $v_1 \neq 0$ and $v_2 = -v_1 \cdot \rho_1 \cdot (t_0 - c + 1)$. Finally, we have to find a time-varying $B^T = (b_1, b_2)^T$ such that

$$v_1 b_1(t) - v_1 \rho_1 (t + t_0 - c + 1) b_2(t) = 0, \quad \forall t \in \mathbb{T}.$$

If we now define $b_2 = w_1 = v_1 = 1$ and $b_1(t) = -w_2(t) = \rho_1(t + t_0 - c + 1)$ with $t_0 = c = \rho = 1$, we get (4), which is not (completely) controllable on $\mathbb{T} = [0, 2]$ as has been shown in the introduction.

It is quite obvious that we have enough freedom in v_1, v_2, b_1 and b_2 to create a set N of analytic, not totally controllable systems (3) for time-invariant matrices A. But since $b_1(t) \overset{!}{=} \rho_1(t-\tilde{c})b_2(t)$ ($\tilde{c} = \frac{v_2}{v_1\rho_1}$) needs to be satisfied for all systems in N, the structure of these systems is not defined on $\mathbb{T}_\infty =]-\infty, \infty[$, because $b_1(t)$ will always have at least one zero point at $t = \tilde{c}$.

To clarify, it is possible to find not totally controllable systems (3) with the structure of (2) for the interval $\mathbb{T}_\infty$, but this can not be done with a constant ρ_1 (see e.g. $\rho_1(t) = e^t, b_1 = \rho_1 b_2$).

Remark 17 : For SSC of linear time-varying systems (3) with a single input ($r = 1$), the matrix B has to be a column vector with just one nonzero element. This is not needed for SSC in the case of linear time-invariant systems (1) as shown with the example in (2).

We now present the main result of this paper, which is a necessary condition for SSC of linear time-varying systems (3).

Theorem 18 : Let any structural matrices $\mathcal{A} \in \mathbb{R}^{n\times n}/\sim$ and $\mathcal{B} \in \mathbb{R}^{n\times r}/\sim$ and any point $t \in \mathbb{R}$ be given. The class of linear time-varying systems (3) of structure $(\mathcal{A}, \mathcal{B})$ on $\mathbb{I}_t$ is SSC only if the class of linear time-invariant systems (3) of structure $(\mathcal{A} + \mathcal{I}_n, \mathcal{B})$ is SSC.

Proof. Due to space limitations we omit the proof. □

In the following, we give the main result of [13], which is a sufficient condition for SSC of linear time-varying systems.

Theorem 19 ([13]): Let any structural matrices $\mathcal{A} \in \mathbb{R}^{n\times n}/\sim$ and $\mathcal{B} \in \mathbb{R}^{n\times r}/\sim$ and any interval $\mathbb{T}$ be given. The class of linear time-varying systems (3) of structure $(\mathcal{A}, \mathcal{B})$ on $\mathbb{T}$ is SSC if the class of linear time-invariant systems (3) of structure $(\mathcal{A} + \mathcal{I}_n, \mathcal{B})$ is SSC.

From Theorem 18, Theorem 19 and Proposition 10, we conclude with the following Corollary 20.

Corollary 20 : For any structural matrices $\mathcal{A} \in \mathbb{R}^{n\times n}/\sim$ and $\mathcal{B} \in \mathbb{R}^{n\times r}/\sim$ and any point $t \in \mathbb{R}$ the following are equivalent.

(i) The class of linear time-varying systems (3) of structure $(\mathcal{A}, \mathcal{B})$ on $\mathbb{I}_t$ is SSC.
(ii) The class of linear time-invariant systems (1) of structure $(\mathcal{I}_n + \mathcal{A}, \mathcal{B})$ is SSC.
(iii) For every non-empty subset $N \subseteq \{1, \ldots, n\}$ of vertices of $\mathcal{G}(\mathcal{A}, \mathcal{B})$, there is some vertex $j \in \{1, \ldots, n + r\} \setminus N$ such that N contains exactly one successor of j in $\mathcal{G}(\mathcal{A}, \mathcal{B})$.
(iv) The structural matrix $(\mathcal{I}_n + \mathcal{A}, \mathcal{B})$ can be permuted into the form (12) in such a way that none of the $(\mathcal{I}_n + \mathcal{A})_{i,i}$ is permuted to a $*$-entry in (12).

V. Conclusion

In this paper we have extended the notion of structural controllability of linear time-invariant systems (1) to the time-varying case (3). In two Examples, we have shown that neither the conditions for structural controllability of time-invariant systems are necessary, nor that the conditions for strong structural controllability of time-invariant systems (1) are sufficient for the controllability of time-varying systems (3). We present a necessary condition for structural controllability of linear time-varying systems (3) in Theorem 14 and in our main result, Theorem 18, a necessary condition for strong structural controllability of linear time-invariant systems (3) is given.

References

[1] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. New York: Springer, 1998.

[2] K. Murota, *Matrices and Matroids for Systems Analysis*. Berlin: Springer, 2000.

[3] C.-T. Lin, "Structural controllability," *IEEE Trans. Automat. Contr.*, vol. AC-19, no. 3, pp. 201–208, 1974.

[4] R. W. Shields and J. B. Pearson, "Structural controllability of multiinput linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-21, no. 2, pp. 203–212, 1976.

[5] K. Glover and L. M. Silverman, "Characterization of structural controllability," *IEEE Trans. Automat. Contr.*, vol. AC-21, no. 4, pp. 534–537, 1976.

[6] H. Mayeda and T. Yamada, "Strong structural controllability," *SIAM J. Control Optimiz.*, vol. 17, no. 1, pp. 123–138, 1979.

[7] C. Hartung, G. Reissig, and F. Svaricek, "Characterization of strong structural controllability of uncertain linear time-varying discrete-time systems," in *Proc. of the 51st IEEE Conference on Decision and Control*, Maui, Hawaii, USA, 2012, pp. 2189–2194.

[8] K. J. Reinschke, *Multivariable Control – A Graph–theoretic Approach*. Berlin: Springer, 1988.

[9] G. Reißig and U. Feldmann, "Computing the generic index of the circuit equations of linear active networks," in *Proc. of the IEEE International Symposium on Circuits and Systems, ISCAS'96, Connecting the World*, vol. 3, 1996, pp. 190–193.

[10] ——, "A simple and general method for detecting structural inconsistencies in large electrical networks," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications,*, vol. 50, no. 11, pp. 1482–1485, 2003.

[11] G. Reißig, "On the maximum rank of toeplitz block matrices of blocks of a given pattern," *Proc. of the 16th Int. Symp. on Math. Th. of Networks and Systems (MTNS), Leuven, Belgium*, 2004.

[12] J. C. Jarczyk, F. Svaricek, and B. Alt, "Strong structural controllability of linear systems revisited," in *Proc. of the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, Orlando, Florida, USA, 2011, pp. 1213–1218.

[13] C. Hartung, G. Reissig, and F. Svaricek, "Sufficient conditions for strong structural controllability of uncertain linear time-varying systems," in *Proc. of the American Control Conference 2013, Washington, DC*, Washington, DC, USA, 2013.

[14] L. Silverman and H. Meadows, "Controllability and observability in time-variable linear systems," *SIAM J. on Control*, vol. 5, pp. 64–73, 1967.

[15] H. D'Angelo, *Linear Time-varying Systems: Analysis and Synthesis*. Boston: Allyn and Bacon, 1970.

[16] E. Freund, *Zeitvariable Mehrgrößensysteme*. Berlin: Springer, 1971.

[17] K. J. Reinschke, F. Svaricek, and H.-D. Wend, "On strong structural controllability of linear systems," in *Proc. of the 31th IEEE Conference on Decision and Control*, Tucson, Arizona, USA, 1992, pp. 203–208.