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State Estimation for Parabolic PDEs with Reactive-Convective Non-Linearities

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Abstract— **An extended Luenberger observer is proposed for the solution of the state estimation problem for semi-linear parabolic PDEs with reactive-convective non-linearities. Here, the backstepping method is applied to the linearised observer error dynamics to determine the observer gains. This, however, requires a successive evaluation of the so-called Hopf-Cole transformation allowing to transform the PDE of the linearised observer error into a normal form, for which backstepping can be directly used. Moreover, the computational efficiency of determination of the gains is improved by combining the direct numerical solution approach with the sample-and-hold implementation. Finally, the observer error convergence is analysed both theoretically and by means of numerical simulations.**

I. Introduction and problem formulation

The need for state estimation algorithms for distributed-parameter systems is justified by the fact many applications ranging from advanced control schemes to process monitoring and diagnostics require full state information. For finite-dimensional systems it is well known that the state estimation problem induces significant difficulties in the non-linear case. Consequently, it is not surprising that this is especially true when dealing with the observer design for systems governed by partial differential equations (PDEs).

For the solution of the state estimation problem for systems governed by non-linear PDEs, different observer and filter design techniques are proposed. In [15], a distributed-parameter observer of Luenberger structure is considered for a semi-linear model of a chemical fixed-bed reactor. This concept is adopted in similar observer design procedures in various applications, see, e.g., [5], [7]. On the other hand, an observer design approach for infinite-dimensional dissipative bilinear systems based on semigroup theory is addressed, e.g., in [3]. Alternatively, design methods based on optimal estimation and filter techniques are suggested in, e.g., [1], [9]. Finally, backstepping-based state estimation for a non-linear Navier-Stokes PDE can be found in [14] and for a semi-linear parabolic PDE in [10].

Subsequently, an extension of [10] consisting of the combination of the extended Luenberger observer design approach and the backstepping method is presented for semi-linear parabolic PDEs with a reactive-convective non-linearity and linear BCs. Here, the extended linearisation of the observer error dynamics around the estimated state is combined with the successive evaluation of the so-called Hopf-Cole transformation (see, e.g., [4]) such that for the resulting linearised observer error dynamics the backstepping method (see, e.g., [13]) can be directly applied. Thereby, in view of the successive determination of the observer gains an efficient numerical solution for the corresponding kernel-PDE is considered as proposed in [6].

The paper is organized as follows: In Section II the state estimation problem is formulated for a semi-linear parabolic PDE, for which the extended Luenberger observer is developed and observer gains are computed. Convergence of both linearised and semi-linear observer error dynamics is addressed in Section III. An efficient computational implementation of the proposed observer scheme by means of the sample-and-hold approach is presented in Section IV. Section V provides simulation results of an exemplary set-up. Some final remarks close the paper.

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II. Observer design

In the following, the state estimation problem is considered for a semi-linear scalar parabolic PDE given by

$$\partial_t x(z,t) = \partial_z^2 x(z,t) + f(z,t,x,\partial_z x) \tag{1}$$

defined on $(z,t) \in (0,L) \times \mathbb{R}^+_{t_0}$ with $\mathbb{R}^+_{t_0} = \{t \in \mathbb{R}^+ \mid t > t_0\}$ with BCs

$$\partial_z x(0,t) = 0, \quad t \in \mathbb{R}^+_{t_0} \tag{2a}$$

$$\partial_z x(L,t) + q x(L,t) = 0, \quad t \in \mathbb{R}^+_{t_0}, \tag{2b}$$

with an arbitrary constant parameter q and IC according to

$$x(z,t_0) = x_0(z) \quad z \in [0,L]. \tag{3}$$

The system output is defined as the system state at the boundary $z = 0$, i.e.,

$$y(t) = x(0,t), \quad t \in \mathbb{R}^+_{t_0}. \tag{4}$$

In view of the well-posedness of (1)–(3) the following assumption is made.

Assumption 1: It is assumed that there exists a unique solution to the considered initial value problem (1)–(3). For results on the existence and uniqueness of solutions for such PDEs, the interested reader is referred to, e.g., [8] and [12].

Remark 1: Note that the subsequent observer design approach is not restricted to autonomous systems as it is here exemplarily illustrated for homogeneous BCs (2) of Neumann or mixed type, respectively. However, in case of different configuration including Dirichlet BCs and the presence of inhomogeneities being known functions of time representing exogenous signals the following procedure is in principle identical.

A. Extended Luenberger observer

Following the idea of [10], the combination of the backstepping method, the extended linearisation and the Hopf-Cole transformation is considered for the observer design.

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For this, a distributed-parameter Luenberger-type observer in the observer state $\hat{x}(z,t)$ is set up according to

$$\partial_t \hat{x}(z,t) = \partial_z^2 \hat{x}(z,t) + f(z,t,\hat{x},\partial_z \hat{x}) + l(z,t)\tilde{y}(t) \tag{5}$$

with $\tilde{y}(t) = y(t) - \hat{y}(t)$ defined on $(z,t) \in (0,L) \times \mathbb{R}^+_{t_0}$. The BCs are assigned as

$$\partial_z \hat{x}(0,t) = l_0(t)\tilde{y}(t) \tag{6a}$$
$$\partial_z \hat{x}(L,t) + q\hat{x}(L,t) = 0 \tag{6b}$$

for $t \in \mathbb{R}^+_{t_0}$ with the IC

$$\hat{x}(z,t_0) = \hat{x}_0(z) \quad z \in [0,L]. \tag{7}$$

The output estimate is given by

$$\hat{y}(t) = \hat{x}(0,t), \tag{8}$$

while $l(z,t)$ and $l_0(t)$ entering both the PDE (5) and the BC (6a) denote the observer gains to be determined. Considering the plant (1)–(3) with the state observer (5)–(7) it follows that the dynamics of the observer error $e(z,t) = x(z,t) - \hat{x}(z,t)$ is governed by

$$\begin{aligned}\partial_t e(z,t) = \partial_z^2 e(z,t) + f(z,t,x,\partial_z x) - f(z,t,\hat{x},\partial_z \hat{x}) \\ - l(z,t)e(0,t)\end{aligned} \tag{9}$$

defined on $(z,t) \in (0,L) \times \mathbb{R}^+_{t_0}$. The BCs follow as

$$\partial_z e(0,t) + l_0(t)e(0,t) = 0, \quad t \in \mathbb{R}^+_{t_0} \tag{10a}$$
$$\partial_z e(L,t) + qe(L,t) = 0, \quad t \in \mathbb{R}^+_{t_0} \tag{10b}$$

and the IC is given by

$$e(z,t_0) = e_0(z), \quad z \in [0,L] \tag{11}$$

with $e_0(z) = x_0(z) - \hat{x}_0(z)$. For the determination of the observer gains $l(z,t)$ and $l_0(t)$ such that (9)–(11) converges in the L^2-norm the governing equations of the observer error dynamics are linearised with respect to the current observer state $\hat{x}(z,t)$ and its spatial derivative $\partial_z \hat{x}(z,t)$. Considering that $x(z,t) = \hat{x}(z,t) + e(z,t)$ this yields[1]

$$\begin{aligned}f(z,t,x,\partial_z x) = f(z,t,\hat{x},\partial_z \hat{x}) + \partial_x f(z,t,\hat{x},\partial_z \hat{x})e(z,t) \\ + \partial_{x_z} f(z,t,\hat{x},\partial_z \hat{x})\partial_z e(z,t).\end{aligned} \tag{12}$$

Substituting (12) into (9) results in the linearised observer error dynamics in $\tilde{e}(z,t)$ of the form

$$\begin{aligned}\partial_t \tilde{e}(z,t) = \partial_z^2 \tilde{e}(z,t) + \tilde{b}(z,t)\partial_z \tilde{e}(z,t) \\ + \tilde{c}(z,t)\tilde{e}(z,t) - l(z,t)\tilde{e}(0,t)\end{aligned} \tag{13}$$

with $\tilde{b}(z,t) = \partial_{x_z} f(z,t,\hat{x},\partial_z \hat{x})$, $\tilde{c}(z,t) = \partial_x f(z,t,\hat{x},\partial_z \hat{x})$. Here, BCs and the IC remain unchanged and follow according (10), (11) as

$$\partial_z \tilde{e}(0,t) + l_0(t)\tilde{e}(0,t) = 0, \qquad t \in \mathbb{R}^+_{t_0} \tag{14a}$$
$$\partial_z \tilde{e}(L,t) + q\tilde{e}(L,t) = 0, \qquad t \in \mathbb{R}^+_{t_0} \tag{14b}$$
$$\tilde{e}(z,t_0) = \tilde{e}_0(z), \qquad z \in [0,L]. \tag{14c}$$

[1]Here and in the following, ∂_x and ∂_{x_z} denote partial derivatives with respect to $x(z,t)$ and $\partial_z x(z,t)$, respectively.

B. Hopf-Cole transformation

For the determination of the observer gains $l(z,t)$ and $l_0(t)$ the governing equations (13), (14) are transferred to a simpler form by applying a suitable coordinate and state transformation. Introducing the coordinate transformation

$$z \mapsto \zeta = \frac{1}{L}z, \qquad t \mapsto \tau = \frac{1}{L^2}(t - t_0) \tag{15}$$

and taking into account the Hopf-Cole transformation

$$\tilde{e}(z,t) \mapsto \breve{e}(\zeta,\tau) = \tilde{e}(L\zeta, L^2\tau + t_0)\exp(\chi(\zeta,\tau)), \tag{16}$$

where $\chi(\zeta,\tau) = \int_0^\zeta \bar{b}(s,\tau)\mathrm{d}s$ with $\bar{b}(\zeta,\tau) = L\tilde{b}(L\zeta, L^2\tau + t_0)/2$ it is possible to eliminate the convection term $\tilde{b}(z,t)\partial_z \tilde{e}(z,t)$ in (13), while the spatial coordinate z is scaled to unity.

As a result, instead of (13), (14) it is hence equivalent to analyse the following diffusion-reaction PDE with only a single spatially and time-varying reaction parameter

$$\partial_\tau \breve{e}(\zeta,\tau) = \partial_\zeta^2 \breve{e}(\zeta,\tau) + \breve{c}(\zeta,\tau)\breve{e}(\zeta,\tau) - \breve{l}(\zeta,\tau)\breve{e}(0,\tau) \tag{17}$$

defined on $(\zeta,\tau) \in (0,1) \times \mathbb{R}^+_0$, $\mathbb{R}^+_0 = \{\tau \in \mathbb{R} \mid \tau > 0\}$ with

$$\partial_\zeta \breve{e}(0,\tau) + \big(\breve{p}(\tau) + \breve{l}_0(\tau)\big)\breve{e}(0,\tau) = 0, \qquad \tau \in \mathbb{R}^+_0 \tag{18a}$$
$$\partial_\zeta \breve{e}(1,\tau) + \breve{q}(\tau)\breve{e}(1,\tau) = 0, \qquad \tau \in \mathbb{R}^+_0 \tag{18b}$$
$$\breve{e}(\zeta,0) = \breve{e}_0(\zeta), \zeta \in [0,1]. \tag{18c}$$

Here, the transformed reaction parameter is given by $\breve{c}(\zeta,\tau) = \int_0^\zeta \partial_\tau \bar{b}(s,\tau)\mathrm{d}s - \bar{b}^2(\zeta,\tau) - \partial_\zeta \bar{b}(\zeta,\tau) + \bar{c}(\zeta,\tau)$ with $\bar{c}(\zeta,\tau) = L^2\tilde{c}(L\zeta, L^2\tau + t_0)$, $\breve{p}(\tau) = -\bar{b}(0,\tau)$, $\breve{q}(\tau) = Lq - \bar{b}(1,\tau)$ and the mapping of the observer gains is defined by

$$l(z,t) \mapsto \breve{l}(\zeta,\tau) = L^2 l(L\zeta, L^2\tau + t_0)\exp(\chi(\zeta,\tau)) \tag{19a}$$
$$l_0(t) \mapsto \breve{l}_0(\tau) = l_0(L^2\tau + t_0). \tag{19b}$$

Remark 2: Note that due to the Hopf-Cole transformation additional time dependency in form of $\breve{p}(\tau)$ in (18a) and $\breve{q}(\tau)$ in (18b) is induced compared to (14a) and (14b). This has to be taken into account in the subsequent design of the observer gains.

With this, for the determination of the observer gains $\breve{l}(\zeta,\tau)$ and $\breve{l}_0(\tau)$ to stabilize (17), (18) the backstepping method is applied.

C. Stabilization of the linearized observer error dynamics

An essential feature of the backstepping-based observer design is the determination of the observer gains in such a way that the observer error dynamics follows the behaviour of a predefined target system. In the following, the desired target system for the behaviour of the linearised observer error is specified.

Selection of the target system: The observer error dynamics (17)–(18) is enforced to behave like the target system

$$\partial_\tau w(\zeta,\tau) = \partial_\zeta^2 w(\zeta,\tau) - \mu(\tau)w(\zeta,\tau) \tag{20}$$

defined on $(z,\tau) \in (0,1) \times \mathbb{R}^+_\tau$ with corresponding BCs

$$\partial_\zeta w(0,\tau) = 0, \quad \tau \in \mathbb{R}^+_0 \tag{21a}$$
$$\partial_\zeta w(1,\tau) = 0, \quad \tau \in \mathbb{R}^+_0 \tag{21b}$$

and the IC

$$w(\zeta,0)=w_0(\zeta),\quad \zeta\in[0,1]. \tag{22}$$

Here, $\mu(\tau)$ denotes a design parameter, whose appropriate choice guarantees the exponential stability of the target system. It can be shown, see, e.g., [11], that the parabolic PDE (20)–(22) is exponentially stable in the L^2-norm, if the inequality $\mu(\tau)+\lambda_{\text{min}}\geq\epsilon$ is satisfied for some $\epsilon>0$, where λ_{min} denotes the smallest eigenvalue of the Sturm-Liouville problem $\partial_\zeta^2 w(\zeta,t)+\lambda w(\zeta,t)=0$ with BCs (21). As a result, it follows that

$$\|w(\zeta,\tau)\|_{L^2}\leq\exp(-\kappa(\tau))\|w_0(\zeta)\|_{L^2} \tag{23}$$

with $\kappa(\tau)=\int_0^\tau(\mu(s)+\lambda_{\text{min}})\,\mathrm{d}s$. With this, the evaluation of the backstepping transformation relating the desired target dynamics with the linearised observer error PDE is carried out as sketched below.

Determination of the kernel-PDE: By making use of the Volterra integral transformation

$$\breve{e}(\zeta,\tau)=w(\zeta,\tau)-\int_0^\zeta\breve{g}(\zeta,s,\tau)w(s,\tau)\mathrm{d}s \tag{24}$$

with the integral kernel $\breve{g}(\zeta,s,\tau)$, the mapping of the target system (20)–(22) to the observer error dynamics (17)–(18) is realized. To determine the PDE governing the evolution of the kernel $\breve{g}(\zeta,s,\tau)$ expressions for the observer error (24) and its partial derivatives are substituted into (17), (18). After some intermediate calculations this allows to deduce the PDE governing the evolution of the kernel $\breve{g}(\zeta,s,\tau)$, i.e.,

$$\begin{aligned}\partial_\tau\breve{g}(\zeta,s,\tau)&=\partial_\zeta^2\breve{g}(\zeta,s,\tau)-\partial_s^2\breve{g}(\zeta,s,\tau)\\&\quad+\breve{\gamma}(\zeta,\tau)\breve{g}(\zeta,s,\tau)\end{aligned} \tag{25a}$$

$$\mathrm{d}_z\breve{g}(\zeta,\zeta,\tau)=\frac{\breve{\gamma}(\zeta,\tau)}{2} \tag{25b}$$

$$\partial_\zeta\breve{g}(1,s,\tau)=-\breve{q}(\tau)\breve{g}(1,s,\tau) \tag{25c}$$

$$\breve{g}(1,1,\tau)=\breve{q}(\tau) \tag{25d}$$

defined on the triangular spatial domain $(\zeta,s)\in\{(\zeta,s)\in\mathbb{R}^2\mid s\in[0,1],\ \zeta\in[s,1]\}$ with $\breve{\gamma}(\zeta,\tau)=\breve{c}(\zeta,\tau)+\mu(\tau)$, see, e.g., [6]. Moreover, the observer gains $\breve{l}(\zeta,\tau)$ and $\breve{l}_0(\tau)$ follow in the form

$$\breve{l}(\zeta,\tau)=\partial_s\breve{g}(\zeta,0,\tau)\quad \breve{l}_0(\tau)=\breve{g}(0,0,\tau)-\breve{p}(\tau). \tag{26}$$

By considering the inverse of (19) the observer gains $l(z,t)$ and $l_0(t)$ of the linearised observer error dynamics in (z,t)-coordinates follow as

$$l(z,t)=\frac{1}{L^2}\Big(\breve{l}(\zeta,\tau)\exp(-\chi(\zeta,\tau))\Big)\Big|_{\substack{\zeta=z/L\\ \tau=\frac{t-t_0}{L^2}}} \tag{27a}$$

$$l_0(t)=\breve{l}_0(\tau)|_{\tau=\frac{t-t_0}{L^2}}. \tag{27b}$$

This guarantees the exponential stability of (13)–(14).

Obviously, an explicit solution of (25) is required to determine the observer gains $\breve{l}(\zeta,\tau)$ and $\breve{l}_0(\tau)$. The classical solution of such PDEs traces back to [2], where the kernel is obtained in terms of integral operators followed by successive series approximation. Under assumption of certain Gevrey regularity for time-varying parameters this method allows to construct a strong solution of the kernel-PDE by approximating $\breve{g}(\zeta,s,\tau)$ by means of an infinite series. However, the recursive determination of the series coefficients significantly increases the computing time. For this reason, subsequently an efficient numerical method introduced in [6] is used by applying a formal discretization of the kernel integral formulation and a numerical time integration of the resulting system of ODEs.

Numerical solution of the kernel-PDE: Following the solution method from [6], in a first step formal integration is applied to the kernel PDE (25a). Therefore, scattering coordinates are introduced $\xi=2-\zeta-s,\quad \eta=\zeta-s$ such that $\breve{g}(\zeta,s,\tau)=\bar{g}(\xi(\zeta,s),\eta(\zeta,s),\tau)$. With this, it follows from (25) that $\bar{g}(\xi,\eta,\tau)$ has to satisfy the PDE

$$\partial_\eta\partial_\xi\bar{g}(\xi,\eta,\tau)=\frac{1}{4}\mathcal{B}_{\bar{g}}(\xi,\eta,\tau) \tag{28a}$$

$$\partial_\xi\bar{g}(\xi,0,\tau)=-\frac{1}{4}\breve{\gamma}\Big(1-\frac{\xi}{2},\tau\Big) \tag{28b}$$

$$\partial_\eta\bar{g}(\eta,\eta,\tau)-\partial_\xi\bar{g}(\eta,\eta,\tau)=-\breve{q}(\tau)\bar{g}(\eta,\eta,\tau) \tag{28c}$$

$$\bar{g}(0,0,\tau)=\breve{q}(\tau) \tag{28d}$$

with $\mathcal{B}_{\bar{g}}(\xi,\eta,\tau)=-\partial_\tau\bar{g}(\xi,\eta,\tau)+\breve{\gamma}\big(1-\frac{\xi-\eta}{2},\tau\big)\bar{g}(\xi,\eta,\tau)$ defined on $(\xi,\eta)\in\{(\xi,\eta)\in\mathbb{R}^2\mid\eta\in[0,1],\ \xi\in[\eta,2-\eta]\}$. Integrating (28a) with respect to η from 0 to η and afterwards over ξ from η to ξ allows together with (28b)–(28d) to determine an implicit integral formulation of the kernel-PDE, i.e.,

$$\begin{aligned}\bar{g}(\xi,\eta,\tau)&=\mathcal{A}(\xi,\eta,\tau)+\frac{1}{4}\int_\eta^\xi\int_0^\eta\mathcal{B}_{\bar{g}}(\beta,\alpha,\tau)\mathrm{d}\alpha\mathrm{d}\beta\\&\quad+\frac{1}{2}\int_0^\eta\int_0^\beta\mathcal{B}_{\bar{g}}(\beta,\alpha,\tau)\mathrm{d}\alpha\mathrm{d}\beta-\breve{q}(\tau)\int_0^\eta\bar{g}(\beta,\beta,\tau)\mathrm{d}\beta,\end{aligned} \tag{29}$$

where $\mathcal{A}(\xi,\eta,\tau)=-\frac{1}{2}\int_0^\eta\breve{\gamma}\big(1-\frac{\beta}{2},\tau\big)\mathrm{d}\beta-\frac{1}{4}\int_\eta^\xi\breve{\gamma}\big(1-\frac{\beta}{2},\tau\big)\mathrm{d}\beta+\breve{q}(\tau)$. Discretization of the kernel integral equation (29), approximation of the integral terms by means of the composite trapezoidal rule and appropriate indexing of the discretized domain yields the formulation for a vector kernel $\bar{\mathbf{g}}(\tau)$ in terms of coupled first-order ODEs

$$-\mathbf{M}\dot{\bar{\mathbf{g}}}(\tau)=(\mathbf{D}(\tau)-\mathbf{I}_\mathcal{N})\,\bar{\mathbf{g}}(\tau)+\mathbf{b}(\tau) \tag{30}$$

with $\mathbf{I}_\mathcal{N}$ the identity matrix and matrices $\mathbf{M}$, $\mathbf{D}(\tau)$ and a vector $\mathbf{b}(\tau)$ as defined in the Appendix, where to keep the paper self-contained a detailed determination of (30) is presented.

In this way, the original problem of solving (29) is transformed into an initial-value problem (IVP), which can be solved numerically for a given initial condition $\bar{\mathbf{g}}(0)=\bar{\mathbf{g}}_0$. Subsequently, the initial condition $\bar{\mathbf{g}}_0$ is chosen as a stationary solution of (30) at $\tau=0$, i.e., $\bar{\mathbf{g}}_0=-\left(\mathbf{D}(0)-\mathbf{I}_\mathcal{N}\right)^{-1}\mathbf{b}(0)$.

Remark 3: Note that in case of a time-independent parameter $\breve{\gamma}(z,\tau)\equiv\breve{\gamma}(z)$ and a constant parameter $\tilde{q}(\tau)\equiv\tilde{q}=$ const., the IVP (30) reduces to a set of algebraic equations given by $\bar{\mathbf{g}}=-\left(\mathbf{D}-\mathbf{I}_\mathcal{N}\right)^{-1}\mathbf{b}$ providing an efficient solution procedure.

III. Convergence of the observer error dynamics

Subsequently, the verification of the exponential stability of the linearised observer error dynamics of the closed-loop observer is based on the analysis of the inverse backstepping transformation from the $\breve{e}(\zeta,\tau)$-system to the $w(\zeta,\tau)$-PDE. In particular, it can be shown that the stability of the target system (20)–(22) implies the exponential decay of the linearised observer error dynamics (13), (14). The theoretical stability assertions of the semi-linear observer error dynamics of the form (9)–(11) are still an open question such that the observer error convergence is studied in a numerical simulation scenario.

A. Inverse backstepping transformation

As mentioned above, the stability of the observer error dynamics (17)–(18) can be directly deduced taking into account the inverse backstepping transformation. For this, in the following an inverse to (24) is introduced in the form

$$w(\zeta,\tau) = \breve{e}(\zeta,\tau) + \int_0^\zeta \breve{m}(\zeta,s,\tau)\breve{e}(s,\tau)\mathrm{d}s, \qquad (31)$$

with the inverse kernel $\breve{m}(\zeta,s,\tau)$ mapping (17)–(18) to (20)–(22). Therefore, proceeding similarly as in Section II-C the kernel-PDE governing the evolution of $\breve{m}(\zeta,s,\tau)$ can be determined in a form similar to $\breve{g}(\zeta,s,\tau)$. Hence, the presented solution method can be similarly applied to $\breve{m}(\zeta,s,\tau)$. Following this argumentation, the existence of a bounded strong solution $\breve{m}(\zeta,s,\tau)$ to the corresponding kernel-PDE is guaranteed.

B. Stability of the linearized observer error dynamics

Considering that $\breve{g}(\zeta,s,\tau)$ and $\breve{m}(\zeta,s,\tau)$ are bounded strong solutions of the corresponding kernel-PDEs, the subsequent estimates follow by application of the Cauchy-Schwarz inequality. Evaluating the L^2-norm of the integral term in (24) results in

$$\Big\| \int_0^\zeta \breve{g}(\zeta,s,\tau)w(s,\tau)\mathrm{d}s \Big\|_{L^2}^2 \le C_{\breve{g}} \|w(\zeta,\tau)\|_{L^2}^2 \qquad (32)$$

with $C_{\breve{g}} = \sup_{\zeta,s,\tau} \breve{g}^2(\zeta,s,\tau)$, while a similar calculation for the integral term in (31) evaluated at $\tau = 0$ yields

$$\Big\| \int_0^\zeta \breve{m}(\zeta,s,0)\breve{e}(s,0)\mathrm{d}s \Big\|_{L^2}^2 \le C_{\breve{m}} \|\breve{e}_0(\zeta)\|_{L^2}^2, \qquad (33)$$

where $C_{\breve{m}} = \sup_{\zeta,s} \breve{m}^2(\zeta,s,0)$. In addition, taking into account the boundedness of $\breve{m}(\zeta,s,\tau)$ it follows from (31) that

$$\|w_0(\zeta)\|_{L^2} \le \left(1+\sqrt{C_{\breve{m}}}\right) \|\breve{e}_0(\zeta)\|_{L^2}. \qquad (34)$$

Hence, evaluating the L^2-norm of (24) in view of the estimates (32)–(34) and the stability of the $w(\zeta,\tau)$-system according to (23), it is easy to deduce that

$$\|\breve{e}(\zeta,\tau)\|_{L^2} \le C_{\breve{e}} \exp(-\kappa(\tau)) \|\breve{e}_0(\zeta)\|_{L^2} \qquad (35)$$

with $C_{\breve{e}} = \left(1+\sqrt{C_{\breve{g}}}\right)\left(1+\sqrt{C_{\breve{m}}}\right)$. This guarantees the exponential decay in the L^2-norm of the linearised observer error dynamics (17)–(18) with the observer gains $\breve{l}(\zeta,\tau)$ and $\breve{l}_0(\tau)$ from (26). Consequently, considering (16) and changing the limits of integration it follows that

$$\begin{aligned}\|\breve{e}(\zeta,\tau)\|_{L^2}^2 &= \int_0^1 \big(\tilde{e}(L\zeta, L^2\tau + t_0)\exp(\chi(\zeta,\tau))\big)^2 \mathrm{d}\zeta \\ &\ge C_{\inf} \|\tilde{e}(z,t)\|_{L^2}^2 \end{aligned} \qquad (36)$$

with $C_{\inf} = \exp\big(2\inf_{\zeta,\tau}\chi(\zeta,\tau)\big)/L$ and

$$\begin{aligned}\|\breve{e}_0(\zeta)\|_{L^2}^2 &= \int_0^1 \big(\tilde{e}(L\zeta, t_0)\exp(\chi(\zeta,0))\big)^2 \mathrm{d}\zeta \\ &\le C_{\sup} \|\tilde{e}_0(z)\|_{L^2}^2 \end{aligned} \qquad (37)$$

with $C_{\sup} = \exp\big(2\sup_\zeta \chi(\zeta,0)\big)/L$. Hence, evaluating (35) in view of (36) and (37) yields

$$\|\tilde{e}(z,t)\|_{L^2} \le C_{\tilde{e}} \exp(-\kappa(t)) \|\tilde{e}_0(z)\|_{L^2} \qquad (38)$$

with $\kappa(t) = \kappa(\tau)|_{\tau=(t-t_0)/L^2}$ and $C_{\tilde{e}} = C_{\breve{e}}\sqrt{C_{\sup}}/\sqrt{C_{\inf}}$. With this it is shown that the observer error $\tilde{e}(z,t)$ of the linearised observer error dynamics (13), (14) with observer gains $l(z,t)$ and $l_0(t)$ according to (27) decays exponentially over time t in the L^2-norm.

C. Convergence of the semi-linear observer error dynamics

A rigorous proof of the exponential stability of the semi-linear observer error dynamics (9)–(11) is still an open problem. The challenges are twofold. First, evaluation of (9)–(11) with the backstepping transformation (24) implies that the dynamics of the resulting effective target system is governed by a non-linear PDE. Hence, this precludes the use of the stability assertion as obtained for the linear target PDE (20)–(22) and requires that the stability of the non-linear target system has to be considered separately. On the other hand note that for the design of the observer gains $l(z,t)$ and $l_0(t)$ the additional coordinate and state transformation (15), (16) is applied and the target system is assigned in the transformed (ζ,τ)-coordinates. This means for the non-linear target system expressions corresponding to the linear $\breve{w}(\zeta,\tau)$-system (20)–(22) and the backstepping transformation (24), that at first (20)–(22) and (24) have to be formulated in the (z,t)-coordinates by means of the inverse of (15)–(16) and then substituted into (9)–(11).

Remark 4: For parabolic PDEs (1)–(3) with the non-linearity $f(z,t,x,\partial_z x) \equiv f(z,t,x)$ the theoretical analysis of the stability of the semi-linear observer error dynamics is presented in [10]. In particular, the exponential convergence of the semi-linear observer error PDE is shown for locally and uniformly Lipschitz continuous non-linearities $f(z,t,x)$.

IV. Implementation for real-time applications

The continuous update of the extended linearisation of the semi-linear observer error dynamics (13), (14) involves a significant computational cost. Moreover, in practical applications measurements are often obtained at discrete times only. Motivated by this, in the following a sample-and-hold approach is employed for the efficient and real-time capable implementation of the considered observer scheme.

Therefore, it is subsequently assumed that the output (4) and the linearisation algorithm are updated only at discrete times $t_k = kT_a + t_0$, $k \in \mathbb{N}_0$ with the sampling time T_a.With this, the determination of the observer gains $l_k(z) = l(z,t_k)$ and $l_{0,k} = l_0(t_k)$ is performed for each sampling period $[t_k, t_{k+1})$ depending on the previous result $\hat{x}_k(z) = \hat{x}(z,t_k)$ and the sampled output $y_k = y(t_k)$.

In particular, in view of the extended linearisation of (9)–(11) at $t = t_k$ let $\tilde{b}_k(z) := \tilde{b}(z,t_k)$ and $\tilde{c}_k(z) = \tilde{c}(z,t_k)$. Consequently, the reformulation of the time scaling in (15) leads to $t_k \mapsto \tau_k = kT_a/L^2$ while the Hopf-Cole transformation (16) is evaluated at every sampling step t_k, $k \in \mathbb{N}_0$

according to $\tilde{e}(z,t_k) \mapsto \breve{e}_k(\zeta) = \tilde{e}(L\zeta,t_k)\exp(\chi_k(\zeta))$, where $\chi_k(\zeta) = \int_0^\zeta \bar{b}_k(s)\mathrm{d}s$ with $\bar{b}_k(\zeta) = L\tilde{b}_k(L\zeta)/2$. Hence, the observer gains are calculated by means of backstepping based on the time-invariant kernel-PDE (25) with $\breve{\gamma}_k(\zeta) = \breve{\gamma}(\zeta,\tau_k)$ and $\breve{q}_k = \breve{q}(\tau_k)$ in every sampling period $[\tau_k,\tau_{k+1}]$. This allows to neglect the differentiation with respect to τ in (29), which results in an efficient solution procedure for the kernel-PDE as already mentioned in Remark 3.

V. Simulation results

Subsequently, numerical results are presented to evaluate the backstepping-based state observer for the parabolic PDE with a non-linear reaction-convection term.

The numerical solution is performed using the `pdepe`-algorithm of MATLAB. Here, the spatial coordinate $z \in [0,L]$ with $L = 1$ of the system (1)–(3) and the state observer (5)–(7) is split into $N = 10$ equidistant intervals of length $\Delta z = 1/N = 0.1$. The reaction-convection non-linearity is thereby exemplarily assigned according to

$$f(z,t,x,\partial_z x) = \sin\bigl(2\pi x\bigr) + x(x-1)(\partial_z x)^3,$$

while the boundary parameter in (2b) is defined by $q = 1$. Hence, the PDE (1)–(3) is characterized by a homogeneous Neumann BC at $z = 0$ and a mixed BC at $z = 1$. The IC with $t_0 = 0$ is chosen as $x_0(z) = 0.1(\cos(\pi z) + 1)$. Moreover, the target system (20)–(22) is parametrized by $\mu(\tau) \equiv \mu \in \{0,1\}$.

Simulation results are presented in Fig. 1. Thereby, results of the state estimation based on an open-loop observer (simulator) are compared with the determined backstepping-based observer. In particular, Figs. 1(a)–(b) show the profiles of the plant and the observer error due to a simple simulator ($l(z,t) = l_0(t) = 0$) with the IC $\hat{x}_0(z) = -0.7x_0(z)$. Note that due to the deviating initial condition the state evolution of the simulator converges to another equilibrium of the plant (differing from this in Fig. 1(a)) such that a non-zero stationary estimation error evolves (Fig. 1(b)). With the application of the introduced observer design approach the resulting profile of the evolving observer error is shown in Fig. 1(c) for $\mu = 1$. Here, the observer error evolution converges to the zero equilibrium $e(z,t) \equiv 0$ providing a very accurate state estimation. Fig. 1(d) addresses the corresponding evolution of the observer gains $l(z,t)$ at $z = 0.5$ and $l_0(t)$ with a sampling time $T_a = 0.02$ for $\mu = 1$. Figs. 1(e)–(f) clearly indicate the superiority of the proposed backstepping observer compared to a simple simulator. In particular, Fig. 1(e) shows the evolution of $\|e(z,t)\|_{L^2}$ for $\mu = 0$ (dashed) and $\mu = 1$ (gray). The dash-dotted line addresses the evolution of the L^2-norm of the simulator error. Obviously, by increasing the design parameter μ the decay behaviour of the observer error can be improved. A similar decay behaviour of the observer error in terms of the sup-norm is presented in Fig. 1(f).

VI. Conclusions

In this contribution, a backstepping-based solution of the state estimation problem is presented for parabolic PDEs with reactive-convective non-linearities. For this, assuming a sufficiently small initial observer error an extended linearisation of the observer error dynamics is performed to obtain a linear PDE.This serves as the basis for the design of the observer gains, where the Hopf-Cole transformation is applied to transform the linearised observer error PDE to a suitable normal form. The design and the numerical computation of the observer gains is carried out in the transformed coordinates. Convergence of the observer error dynamics is discussed for the linearised and the semi-linear PDE. In view of a real-time implementation of the state observer, a sample-and-hold approach is employed, which significantly reduces the computing costs for the observer gains. The performance of the proposed observer is illustrated in a simulation scenario, which confirms a very accurate state estimation.

Appendix

To approximate the integral terms in (29) by means of the composite trapezoidal rule, in a first step the spatial domain $\Omega_{\bar{g}} = \{(\xi,\eta) \in \mathbb{R}^2 \mid \eta \in [0,1],\ \xi \in [\eta, 2-\eta]\}$ is discretized. For this, $\Omega_{\bar{g}}$ is split into $N_\delta - 1$, $N_\delta \geq 3$ intervals in the η-direction and into $2N_\delta - 2$ intervals in the ξ-direction with an equidistant interval length $\delta = 1/(N_\delta - 1)$ implying that $0 = \xi_1 < \xi_2 < \cdots < \xi_{2N_\delta - 1} = 2$ and $0 = \eta_1 < \eta_2 < \cdots < \eta_{N_\delta} = 1$ with $\xi_i = \delta(i-1)$, $i = 1,\ldots,2N_\delta - 1$ and $\eta_j = \delta(j-1)$, $j = 1,\ldots,N_\delta$. Thereby, the integrals in (29) are approximated by using discrete values of their integrands evaluated only at grid points $(i,j) \in \mathcal{I}$ with $\mathcal{I} := \{(i,j) \in \mathbb{N}^2 \mid j = 1,\ldots,N_\delta, i = j,\ldots,2N_\delta - j\}$. Application of the composite trapezoidal rule to each of the integrals in (29) with spatially discretized kernel $\bar{g}_{i,j}(\tau) = \bar{g}(\xi_i,\eta_j,\tau)$, $(i,j) \in \mathcal{I}$ and discretized function $\breve{\gamma}_{i,j}(\tau) = \breve{\gamma}\bigl(1 - \frac{\xi_i - \eta_j}{2},\tau\bigr)$, $(i,j) \in \mathcal{I}$ yields a pointwise approximation of the kernel integral equation (29), i.e.,

$$\bar{g}_{i,j}(\tau) = \bar{\mathcal{A}}_{i,j}(\tau) + \bar{\mathcal{B}}_{i,j}(\tau), \quad (i,j) \in \mathcal{I}. \tag{39}$$

Here, $\bar{\mathcal{A}}_{i,j}(\tau) = \mathcal{A}(\xi_i,\eta_j,\tau)$ and $\bar{\mathcal{B}}_{i,j}(\tau) = \frac{1}{4}\int_{\eta_j}^{\xi_i}\int_0^{\eta_j} \mathcal{B}_{\bar{g}}(\beta,\alpha,\tau)\mathrm{d}\alpha\mathrm{d}\beta + \frac{1}{2}\int_0^{\eta_j}\int_0^{\beta} \mathcal{B}_{\bar{g}}(\beta,\alpha,\tau)\mathrm{d}\alpha\mathrm{d}\beta - \breve{q}(\tau)\int_0^{\eta_j} \bar{g}(\beta,\beta,\tau)\mathrm{d}\beta$ denote approximations of the integral terms in the integral formulation (29) according to

$$\begin{aligned}
\bar{\mathcal{B}}_{i,j}(\tau) = {} & \frac{\delta^2}{4}\Bigl[\frac{1}{4}\bigl(\varkappa_{j,j}(\tau) + \varkappa_{i,j}(\tau)\bigr) + \frac{1}{2}\sum_{\mathbb{k}=2}^{j-1}{}^{*}\bigl(\varkappa_{j,\mathbb{k}}(\tau) \\
& + \varkappa_{i,\mathbb{k}}(\tau)\bigr) + \frac{1}{2}\sum_{\mathbb{h}=j+1}^{i-1}{}^{*}\varkappa_{\mathbb{h},j}(\tau) + \sum_{\mathbb{h}=j+1}^{i-1}{}^{*}\sum_{\mathbb{k}=2}^{j-1}{}^{*}\varkappa_{\mathbb{h},\mathbb{k}}(\tau)\Bigr] \\
& + \frac{\delta^2}{2}\Bigl[\frac{1}{4}\varkappa_{j,j}(\tau) + \frac{1}{2}\sum_{\mathbb{k}=2}^{j-1}{}^{*}\varkappa_{j,\mathbb{k}}(\tau) + \frac{1}{2}\sum_{\mathbb{h}=2}^{j-1}{}^{*}\varkappa_{\mathbb{h},\mathbb{h}}(\tau) \\
& + \sum_{\mathbb{h}=2}^{j-1}{}^{*}\sum_{\mathbb{k}=2}^{\mathbb{h}-1}{}^{*}\varkappa_{\mathbb{h},\mathbb{k}}(\tau)\Bigr] - \breve{q}(\tau)\delta\Bigl[\frac{1}{2}\bar{g}_{j,j}(\tau) + \sum_{\mathbb{k}=2}^{j-1}{}^{*}\bar{g}_{\mathbb{k},\mathbb{k}}(\tau)\Bigr] \\
& + \frac{\delta^2}{4}\Bigl[\frac{1}{4}\bigl(\varkappa_{j,1}(\tau) + \varkappa_{i,1}(\tau)\bigr) + \frac{1}{2}\sum_{\mathbb{h}=j+1}^{i-1}{}^{*}\varkappa_{\mathbb{h},1}(\tau)\Bigr] \\
& + \frac{\delta^2}{2}\Bigl[\frac{1}{4}\varkappa_{j,1}(\tau) + \frac{1}{2}\sum_{\mathbb{h}=2}^{j-1}{}^{*}\varkappa_{\mathbb{h},1}(\tau)\Bigr] - \frac{\delta}{2}\breve{q}(\tau)\bar{g}_{1,1}(\tau)
\end{aligned}$$

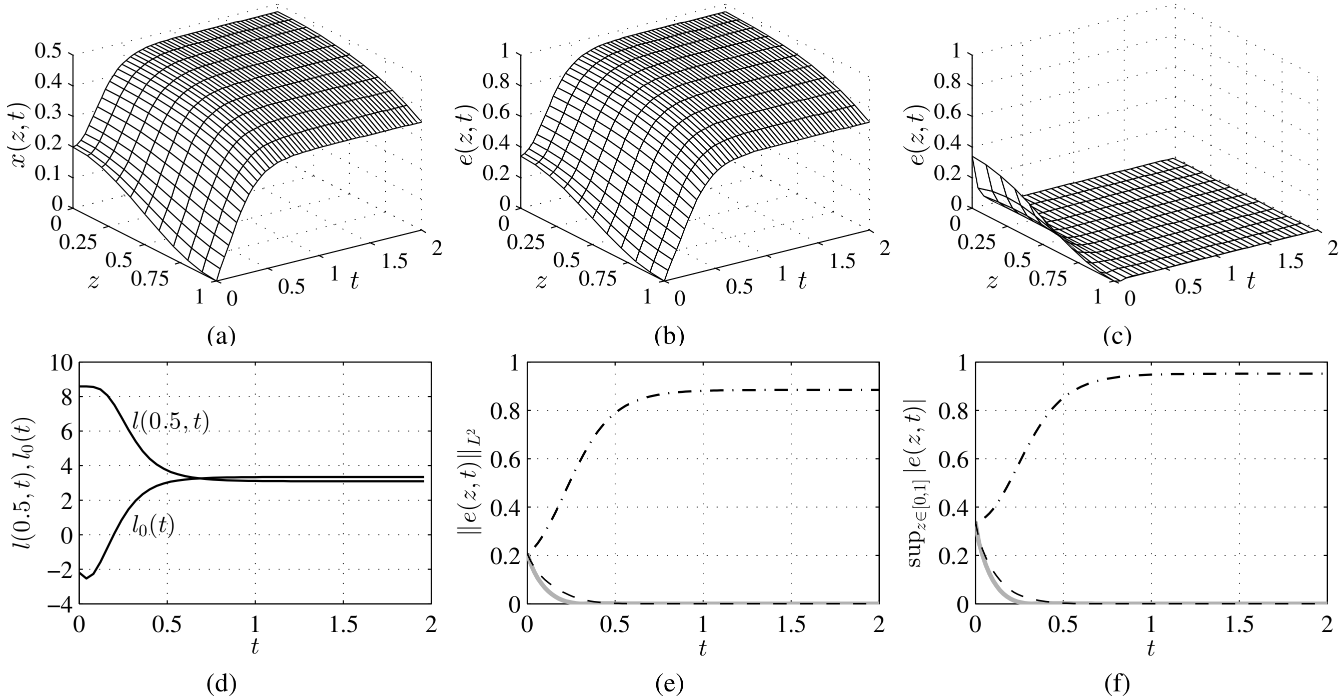

Fig. 1. Numerical results for the state estimation problem. (a) Solution $x(z,t)$; (b) Simulator error $e(z,t)$; (c) Observer error $e(z,t)$ for $\mu = 1$; (d) Observer gains $l(0.5,t)$ and $l_0(t)$ for $\mu = 1$; (e) L^2-norm of the simulator error (dash-dotted) and of the observer error with $\mu = 0$ (dashed), $\mu = 1$ (gray); (f) sup-norm of the simulator error (dash-dotted) and of the observer error with $\mu = 0$ (dashed), $\mu = 1$ (gray);.

$$\begin{aligned}\bar{\mathcal{A}}_{i,j}(\tau) = &-\frac{\delta}{2}\Big[\frac{1}{2}\big(\breve{\gamma}_{1,1}(\tau)+\breve{\gamma}_{j,1}(\tau)\big) + {\sum_{\Bbbk=2}^{j-1}}^{*}\breve{\gamma}_{\Bbbk,1}(\tau)\Big] \\ &- \frac{\delta}{4}\Big[\frac{1}{2}\big(\breve{\gamma}_{j,1}(\tau)+\breve{\gamma}_{i,1}(\tau)\big) + {\sum_{\Bbbk=j+1}^{i-1}}^{*}\breve{\gamma}_{\Bbbk,1}(\tau)\Big] + \breve{q}(\tau)\end{aligned}$$

where $\varkappa_{i,j}(\tau) = -\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}_{i,j}(\tau) + \breve{\gamma}_{i,j}(\tau)\bar{g}_{i,j}(\tau)$, $(i,j) \in \mathcal{I}$ and the summation operator $\sum_n^{*,m} = \sum_n^m$ if $n-m<1$, $\sum_n^{*,m} = 0$ if $n-m=1$ and $\sum_n^{*,m} = -\sum_{m+1}^{n-1}$ if $n-m>1$.

With this, in a next step (39) is reformulated in terms of a set of first-order ODEs evaluated at the discrete grid points $(i,j) \in \hat{\mathcal{I}}$ with $\hat{\mathcal{I}} := \{(i,j) \in \mathcal{I}\} \setminus \{(i,j) \mid j = 1\}$. Therefore, a kernel vector $\bar{\mathbf{g}}(\tau) = [\bar{g}_n(\tau)]_{n=1,\ldots,\mathcal{N}}$ of the length $\mathcal{N} = (N_\delta - 1)^2$ is set up by an appropriate indexing of $\hat{\mathcal{I}}$ such that $n = \mathfrak{I} + j$ for $i \le N_\delta$ and $n = \mathfrak{I} + j - (i - N_\delta)(i - N_\delta - 1)$ for $i > N_\delta$ with $\mathfrak{I} = (i-2)(i-1)/2 - 1$ implying that each $\bar{g}_n(\tau)_{n=1,\ldots,\mathcal{N}}$ corresponds to the respective $\bar{g}_{i,j}(\tau)$. A similar indexing scheme for the discretized integral operators results in $\bar{\mathcal{A}}_n(\tau) = \bar{\mathcal{A}}_{i,j}(\tau)$, $\bar{\mathcal{B}}_n(\tau) = \bar{\mathcal{B}}_{i,j}(\tau)$ and $\bar{\mathcal{A}}^*_n(\tau) = \bar{\mathcal{A}}^*_{i,j}(\tau)$, $(i,j) \in \hat{\mathcal{I}}$, where $\bar{\mathcal{A}}^*_{i,j}(\tau)$ combines the last two lines of $\bar{\mathcal{B}}_{i,j}(\tau)$. Hence, it is possible to formulate (39) by means of a set of $\mathcal{N}$ coupled first-order ODEs (30) with identity matrix $\mathbf{I}_{\mathcal{N}}$, matrices $\mathbf{M} = [M_{n,m}]$, $\mathbf{D}(\tau) = [D_{n,m}(\tau)]$, $m,n = 1,\ldots,\mathcal{N}$ formally given by $M_{n,m} = \partial\bar{\mathcal{B}}_n(\tau)/\partial\dot{\bar{g}}_m$, $D_{n,m}(\tau) = \partial\bar{\mathcal{B}}_n(\tau)/\partial\bar{g}_m$ and vector $\mathbf{b}(\tau) = [b_n(\tau)]_{n=1,\ldots,\mathcal{N}}$ defined as $b_n(\tau) = \bar{\mathcal{A}}_n(\tau) + \bar{\mathcal{A}}^*_n(\tau)$.

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