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Model predictive control for tracking random references

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Abstract— **This paper proposes a simple model predictive control scheme for linear systems, tracking a random reference and analysis its performance. In such situations it is usual to assume that the reference eventually converges to a constant in which case convergence to zero of the tracking error can be established. In this note we characterize the set to which the tracking error converges and the associated region of convergence when the reference does not converge to a constant.**

I. Introduction

If a dynamic linear system subject to constraints is required to track a time varying reference it is possible, under certain conditions, to establish, under model predictive control, asymptotic or exponential stability of a robust invariant set. However establishing such stability is often difficult and it is common, therefore, to assume that the reference signal converges to a constant; this assumption simplifies the analysis considerably and usually permits asymptotic or exponential stability of a robust invariant set to be established. In the present paper we examine performance, in the context of Model Predictive control (MPC) for linear systems subject to constraints, when this simplifying assumption is *not* made. The feasibility problem for time varying references was originally tackled using an error governor [1] and subsequently a reference governor [2]–[4] and a controller designed a priori. The key idea is to attenuate the reference command to avoid constraint violation [5]. Interesting control approaches for the tracking of piece-wise constant references, have been studied in [6]–[12] and in [13], [14] for asymptotically constant references. In [9] the problem of tracking sinusoids and ramps has been solved using a generalization of the disturbance estimation approach. In this note an MPC algorithm, that permits constrained linear systems to track time varying references that are not assumed to be generated by a known finite dimensional exogenous system, is presented. In particular we seek to guarantee the confinement of the trajectories in a compact set by the addition of a constraint. The tracking performance in this framework can be improved adopting the method proposed in [9] to achieve offset-free tracking for reference signals such as sinusoids and ramps but we do no follow this approach here in order to focus on on the confinement of the trajectories in a compact set when the reference is varying arbitrarily in a compact set. Preliminary studies have been carried out for nonlinear systems [15] where a MPC scheme, with an artificial reference similar to that in [16], exploited a bound on the value function introduced directly in the optimization problem was exploited. Here, contrary to the approach in [15], the optimization problem solved at each time instant is a vanilla Model Predictive Control problem where the virtual reference, that is close to the actual reference, guarantees the confinement of all trajectories to a compact set.

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II. A tracking problem

We consider the problem of MPC control when the target state varies randomly but remains in a compact set and the system is subject to state and control constraints. The system to be controlled is described by

$$\begin{aligned} x^+ &= Ax + Bu \\ r^+ &= r + \xi \\ y &= Cx \end{aligned} \qquad (1)$$

where $x \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state of the system being controlled, $u \in \mathbb{U} \subset \mathbb{R}^m$ is the control input, $r \in \mathbb{R}^p$ with $p \leq m$ is the reference output and ξ is a random signal that takes values in the compact set Ξ that contains the origin and is such that r remains in a compact set $\mathcal{R}$; it is required that the output y track a reference r so that, ideally, $y = r$. Assume that, for each $r \in \mathcal{R}$, there exists an equilibrium state-control pair $(x_s(r), u_s(r))$ satisfying

$$\begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \quad u_s \in \mathbb{U} \quad x_s \in \mathbb{X} \quad (2)$$

where $\mathbb{X}$ is a closed subset of $\mathbb{R}^n$ and $\mathbb{U}$ a compact subset of $\mathbb{R}^m$. For sake of clarity, let assume that the solution to (2) can be parametrized [7] as

$$(x_s(r), u_s(r)) = [M_x, M_u]r. \qquad (3)$$

where $M_x \in \mathbb{R}^{n\times p}$ and $M_u \in \mathbb{R}^{m\times p}$ are constant matrices. The parametrization (3) can be determined if the solution to (2) is unique. More generally, it is possible to select an equilibrium pair $(x_s(\hat{r}), u_s(\hat{r})) \in \mathbb{R}^{n+m}$ satisfying input and state constraints and minimizing the distance between $\hat{r}$ and the desired reference r. If r is constant, the control is required to steer the initial state of the system to an equilibrium state. However, the reference output r may vary and is assumed to take values in a discrete set $\mathcal{R}$; it may be regarded as a discrete state Markov process and we do not necessarily assume that r converges to a constant. Notice that state component r is not stabilizable. It is assumed the state and control constraint sets $\mathbb{X}$ and $\mathbb{U}$ are, respectively, a polyhedron and a polytope and that they contain the desired target pairs $(x_s(r), u_s(r))$.
Given the current state x and the reference $r \in \mathcal{R}$, the

model predictive control action is obtained by solving a finite horizon optimal control problem $\mathbb{P}_N$ that includes a virtual reference $\hat{r}$ characterized below. The composite state is $w \triangleq (x, \hat{r})$. Because future values of the reference are unknown, it is assumed constant along the prediction horizon. This leads to the following optimization problem $\mathbb{P}_N(w)$:

$$V_N^0(w) = \min_{\mathbf{u}}\{V_N(w, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(w)\}$$

where

$$V_N(w, \mathbf{u}) \triangleq V_f(\tilde{x}(N; \hat{r}), \hat{r}) + \sum_{i=0}^{N-1} \ell(\tilde{x}(i; \hat{r}), \tilde{u}(i; \hat{r})),$$

and

$$\begin{aligned}\mathcal{U}_N(w) \triangleq \quad &\{\mathbf{u} \mid x(i) \in \mathbb{X},\ u(i) \in \mathbb{U}, \\ &\forall i \in \mathbb{I}_{0:N-1}; x(N) \in \mathbb{X}_f(\hat{r})\}\end{aligned}$$

where, for each $i \in \mathbb{I}_{\geq 0}$, $\tilde{x}(i; r) \triangleq x(i) - x_s(r)$ and $\tilde{u}(i; r) \triangleq u(i) - u_s(r)$. In the two definitions immediately above, $x(i) \triangleq x(i; x, \mathbf{u}, r)$, the solution to $x^+ = Ax + Bu$ at time i if the initial state $x(0) = x$ and the control sequence is $\mathbf{u} \triangleq \{u(0), u(1), \ldots, u(N-1)\}$ so that the argument of V_N is $(w, \mathbf{u})$ and the argument of $\mathcal{U}_N$ is w as indicated; also $\mathbb{I}_{\geq 0} \triangleq \{0, 1, 2, \ldots\}$ and, for all integers j, k, the integer set $\mathbb{I}_{j,k} \triangleq \{j, j+1, \ldots, k\}$. The solution to $\mathbb{P}_N(w)$ is $\mathbf{u}^0(w)$ and the implicit model predictive control law is κ_N defined by

$$\kappa_N(w) = u(0; w) \tag{4}$$

where $u(0; w)$ is the first element of the sequence $\mathbf{u}^0(w)$. We also assume the existence, for each $r \in \mathcal{R}$, of a terminal constraint set $\mathbb{X}_f(r)$ and a terminal cost function $V_f(\cdot, r)$ satisfying the usual stability condition

Assumption 1: For all $r \in \mathcal{R}$ and all $x \in \mathbb{X}_f(r)$: $\min_{u \in \mathbb{U}}\{V_f(x^+ - x_s(r), r) \mid x^+ \in \mathbb{X}_f(r)\} \leq V_f(x - x_s(r), r) - \ell(x - x_s(r), u - u_s(r))$ with $\mathbb{X}_f(r) \triangleq \{x \in \mathbb{R}^n \mid V_f(\cdot, r) \leq c_r\}$ a sublevel set of $V_f(\cdot, r)$ and the condition $V_f(0, r) = 0$.

A reference output r is said to be admissible if $\mathbb{X}_f(r) \subseteq \mathbb{X}$ satisfying assumption 1 is non-empty. Let $\tilde{x}(r) \triangleq x - x_s(r)$ and $\tilde{u}(r) \triangleq u - u_s(r)$, we assume that ℓ, $\mathbb{X}_f(r)$ and V_f satisfy

Assumption 2: There exist $\mathcal{K}_\infty$ functions α_1 and α_2 such that $\ell(\tilde{x}(r), \tilde{u}(r)) \geq \alpha_1(|\tilde{x}(r)|)$ for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$, $V_f(\tilde{x}(r), r) \leq \alpha_2(|\tilde{x}(r)|)$ for all $x \in \mathbb{X}_f(r)$, $r \in \mathcal{R}$.

If r is constant ($\xi \equiv 0$) then, as is well known, the assumptions 1 and 2 imply that $x_s(r)$ is asymptotically stable with a region of attraction $\mathcal{X}_N(r) \triangleq \{x \mid \mathcal{U}_N(x, r) \neq \emptyset\}$. Since the adopted model (1) is linear a convex cost is computationally advantageous and in particular the quadratic one with $\ell(\tilde{x}(i; \hat{r}), \tilde{u}(i; \hat{r})) \triangleq |x - x_s(\hat{r})|_Q^2 + |u - u_s(\hat{r})|_R^2$ and $V_f(\tilde{x}(N; \hat{r}), \hat{r}) \triangleq |x(N) - x_s(\hat{r})|_P^2$ where $Q > 0$, $R > 0$ and $P > 0$ is such that assumption 1 is satisfied.

Let $\mathcal{X} \triangleq \bigcup_{r \in \mathcal{R}} \mathbb{X}_f(r)$. At time 0 if $x \notin \mathcal{X}$ select $\hat{r} \in \text{interior}(\mathcal{R})$ closest to the current r and keep $\hat{r}$ constant while the measured state $x \notin \mathcal{X}$. When $x \in \mathcal{X}$ the reference $\hat{r}$ is the virtual reference closest to r such that $x \in \mathbb{X}_f(\hat{r})$. Then, given the current state x and the reference $r \in \mathcal{R}$, the control action, obtained by solving the optimal control problem $\mathbb{P}_N$ with virtual reference $\hat{r}$, guarantees that the successor state x^+ lies in a set denoted by $\mathcal{X}'$ and characterized below. For each $(r, c) \in \mathcal{R} \times \mathbb{R}_{\geq 0}$, let $\mathcal{L}(r, c) \triangleq \{x \mid V_N^0(x, r) \leq c\}$ and

$$\underline{c}_r \triangleq \min_{c \in \mathbb{R}_{\geq 0}} \{c \mid Ax + B\kappa_N(x, r) \in \mathcal{L}(r, c),\ \forall\ x \in \mathbb{X}_f(r)\}. \tag{5}$$

where $\kappa_N(x, r)$ is given by (4). For each $r \in \mathcal{R}$, let $\mathbb{X}_f'(r) \triangleq \{x \in \mathbb{R}^n \mid V_N^0(x, r) \leq \underline{c}_r\}$. Thus $x \in \mathbb{X}_f(r)$ implies $x^+ \in \mathbb{X}_f'(r)$. Since we are interested in satisfying prescribed bounds on the tracking error it is possible to choose the size of $\mathbb{X}_f(r)$ in a such away that $\mathcal{X}' \triangleq \bigcup_{r \in \mathcal{R}} \mathbb{X}_f'(r)$ guarantees the desired tracking performance.

III. The tracking algorithm

In this section we formulate and discuss the control approach for tracking an unknown randomly varying reference. Roughly speaking, the main idea of the proposed strategy consists in changing the reference used by the MPC optimization problem only if the state is close enough to the steady state characteristic in order to guarantee that the successor state does not leave the set $\mathcal{X}'$. In this way it is possible to ensure that the tracking error converges to a specific region. The proposed approach is computationally attractive since it merely requires the solution of a conventional convex model predictive control problem to compute the feedback law and to determine $\hat{r}$. This desired virtual reference $\hat{r}$ minimizes its distance from the measured reference and satisfies $x \in \mathbb{X}_f(\hat{r})$; it can be computed by solving an optimization problem. The additional computational load introduced by the solution of the latter problem depends on the choice of $\mathbb{X}_f(\hat{r})$. Possible, computationally appealing, choices of $\mathbb{X}_f(\hat{r})$ are ellipsoids or polytopes. Here, we choose the ellipsoidal description since it is computationally more suitable for high dimensional systems even if it gives a more conservative domain of attraction. The ellipsoidal set parameterized with respect to the reference can be chosen as $\mathbb{X}_f(\hat{r}) = \{x \mid (x - M_x\hat{r})'P(x - M_x\hat{r}) \leq c_{\hat{r}}\}$ with $P > 0$ and $c_{\hat{r}}$ such that assumptions 1 and 2 are satisfied. The choice of P is driven by performance requirements and the existence of a linear feedback law $u = K(x - M_x\hat{r}) + M_u\hat{r}$ that stabilizes the operating points when the constraints are not active. Then, it is of interest to find the ellipsoid of maximum volume that lies inside the polyhedral set $C \triangleq \{x \mid a_i x \leq b_i(\hat{r})\ i = 1, \ldots d\} \triangleq \{x - M_x\hat{r} \in \mathbb{X},\ K(x - M_x\hat{r}) + M_u\hat{r} \in \mathbb{U}\}$ which we assume is bounded and has nonempty interior following the procedure in [17, page 414]. In particular given the actual x and r the current virtual reference $\hat{r}$ and $\mathbb{X}_f(\hat{r})$ can be obtained solving the following optimization problem

$\mathbb{P}_r(x,r)$

$$\begin{array}{ll}\min\limits_{r_e\in\mathcal{R},\gamma>0} & \|r_e-r\|^2\\ & (x-M_xr_e)'P(x-M_xr_e)\le\gamma\\ & \gamma^{1/2}\|P^{-1/2}a_i'\|+a_iM_xr_e\le b_i(r_e)\ i=1,\dots d\end{array}\tag{6}$$

The optimization problem $\mathbb{P}_r(x,r)$ is not convex. Indeed, the constraints in $\mathbb{P}_r(x,r)$ for a given i can be rewritten in a quadratic form as follows

$$(x-M_xr_e)'P(x-M_xr_e)-(b_i(r_e)-a_iM_xr_e)^2d\le 0 \tag{7}$$

where $d\triangleq 1/\|P^{-1/2}a_i'\|^2$ is a constant and $b_i(r_e)$ depends affinely on r_e since describes the bounds on the input variables. Let $b_i(r_e)=b_i^0-b_i^1r_e$. Since the quadratic form (7) is convex if and only if the quadratic term $M_x'(P-a_i'a_i)M_x-b_i^{1'}b_i^1+2b_i^{1'}a_iM_x$ is semidefinite positive [17], the optimization problem $\mathbb{P}_r(x,r)$ is not convex in general. A simplified optimization problem is $\mathbb{P}_{\hat\gamma}(x,r)$

$$\begin{array}{ll}\min\limits_{r_e\in\mathcal{R}} & \|r_e-r\|^2\\ & (x-M_xr_e)'P(x-M_xr_e)\le\hat\gamma\end{array}\tag{8}$$

where $\hat\gamma$ is the smallest value of γ for all $r_e\in\mathcal{R}$; $\hat\gamma$ is easily computable exploiting convexity.

If $P>0$ in (8) is the solution of the algebraic Riccati equation of the optimization problem $\mathbb{P}_N(x,r)$ with $Q>0$ and $R>0$ without constraints, the set $\mathcal{X}'$ with $\underline{c}_r=c_r$ coincides exactly with $\mathcal{X}$. This is a consequence of the fact that the solution of the infinite time linear quadratic optimal control problem subject to constraints is equivalent to a finite dimensional cost with $V_f(\tilde x(N;\hat r),\hat r)\triangleq|x(N)-x_s(\hat r)|_P^2$ (see [18], [19]).

The adopted control algorithm is:

Tracking algorithm

Initialization: At time 0, given the current values of x and r, select

- $\hat r\in\mathcal{R}$ closest to r, if $x\notin\mathcal{X}$
- $\hat r\in\mathcal{R}$ closest to r such that $x\in\mathbb{X}_f(\hat r)$, if $x\in\mathcal{X}$.

Step 1 (Compute control:) Compute $u=\kappa_N(x,\hat r)$ by solving $\mathbb{P}_N(x,\hat r)$ and apply u to the system being controlled.

Step 2 (Update $\hat r$ and x)

Measure the successor state x^+ and set $x=x^+$. Set $i=i+1$ and update $\hat r$ as follows

$$\hat r=\begin{cases}\hat r(i-1) & \text{if } x\notin\mathcal{X}\\ \bar r & \text{if } x\in\mathcal{X}\end{cases}\tag{9}$$

where $\bar r$ is the solution to problem $\mathbb{P}_{\hat\gamma}(x,r)$ or$\mathbb{P}_r(x,r)$. Then go to Step 1.

Notice that the actual x belongs to $\mathcal{X}$ if the optimization problem $\mathbb{P}_{\hat\gamma}(x,r)$ is feasible.

Remark 1: The initialization of the "Tracking algorithm" can be carried out with different techniques. At the first time instant it can be advantageous to solve the optimization problem proposed in [7] where an admissible reference $\hat r$ is computed together with the control law allowing a larger domain of attraction.

Remark 2: Whenever the obtained set $\mathcal{X}'$ is too large for the problem under consideration, the condition $x\in\mathbb{X}_f(\hat r)$ could be replaced by $x\in\mathcal{B}(x_s(\hat r),\varepsilon)$ where $\mathcal{B}(x_s,\varepsilon)\triangleq\{x\mid |x-x_s|\le\varepsilon\}$ with $\mathcal{B}(x_s(\hat r),\varepsilon)\subset\mathbb{X}_f(\hat r)$ for suitably chosen ε.

IV. Analysis

We now examine the properties of the control strategy introduced in the previous section when r takes arbitrary values in $\mathcal{R}$. Feasibility of $\mathbb{P}_N(x,\hat r)$ is guaranteed for all $t>0$ once it is obtained at $t=0$.

Proposition 1: Suppose assumptions 1 and 2 are satisfied. If $\mathbb{P}_N(x,\hat r)$ is feasible then $\mathbb{P}_N(x^+,\hat r^+)$ is feasible where $x^+=Ax+B\kappa_N(x,\hat r)$ and $\hat r^+,\hat r\in\mathcal{R}$ satisfy (9)

The proposed approach enjoys the following property

Theorem 1: Suppose assumptions 1 and 2 are satisfied. The control strategy proposed in section III guarantees

(i) The set $\mathcal{X}'$ is positively invariant for the controlled system $x^+=Ax+B\kappa_N(x,\hat r)$ with $\hat r\in\mathcal{R}$ solution to (9) and $t\ge t_c+1$ where $t_c\triangleq\min\{t\ge 0\mid x(t)\in\mathcal{X}\}$.

(ii) Every trajectory of the controlled system $x^+=Ax+B\kappa_N(x,\hat r)$ commencing in $\mathcal{X}_N(\hat r(0))$ remains in it and converges to $\mathbb{X}_f(\hat r)$ for all possible realizations of the reference sequence r.

If the reference trajectory takes values in the compact set $\mathcal{R}$, then the state x converges to the set $\mathcal{X}$ and does not leave the set $\mathcal{X}'$. The set $\mathcal{X}$ needs to be large enough to guarantee fast tracking but not too much to ensure a reasonable confinement of the tracking error. This trade of is not easily achievable and it depends on the system under control. The addition of a constraint while guarantees the confinement of the trajectories in a compact set on the other side generate a slower response of the closed loop system and consequently a loss in performance. The simulation experiments have shown that the introduced control strategy performs reasonably well when the set of references is not excessively large. Notice that when $x\in\mathcal{X}$ the loss of performance, when compared with [7], [20], which can occur in some situations, when fast tracking is required, is not significant as illustrated in the examples. The overall performance of the proposed algorithm depends on the choice of $\mathbb{X}_f(\hat r)$ and its tuning may not be easy.

V. Examples

In both examples the proposed model predictive controller is compared with the approach proposed in [7], here denoted by PW tracking MPC, for tracking piecewise constant references since it preserves feasibility of the controller for varying reference signals. The comparison analyses the behaviour of the proposed strategy for slowly varying and piecewise constant references for which the PW tracking MPC was designed. The terminal ingredients $\mathbb{X}_f(r)$ and $V_f(\cdot,r)$ satisfying assumption 1 are the same for the proposed algorithm and the PW tracking MPC. The PW tracking MPC algorithm has an additional term in the cost penalizing the deviation between the artificial steady state and the target steady state that has been chosen as $\|x_s(\hat{r})-x_s(r)\|_{V_T}$ where $V_T = 5000$ to allow fast tracking.

The four tank example

A state-space continuous time model of the quadruple-tank process system is described in [6], [21]. This problem is an interesting example for demonstrating the effectiveness of the proposed tracking approach since the system transfer function has multivariable transmission zeros in the right hand side of the complex plane. A linear model describing the plant is given by

$$\begin{aligned} \dot{x} = & \begin{bmatrix} a_1 & 0 & c_1 & 0 \\ 0 & a_2 & 0 & c_2 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix} x + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & b_3 \\ b_4 & 0 \end{bmatrix} u \\ y = & \quad Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x \end{aligned}$$

where $x \triangleq h - h_e$ and $u \triangleq q - q_e$ are deviation variables around the operating point $h_e = [0.627,\ 0.636,\ 0.652,\ 0.633]$ and $q_e = [1.6429,\ 2]/3600$ denoting, respectively, the levels and inflow in the tanks (see [6] for more details). The model parameters are $a_1 = -0.0063$, $a_2 = -0.0071$, $a_3 = -0.0043$, $a_4 = -0.0042$, $c_1 = 0.0043$, $c_2 = 0.0042$, $b_1 = 5.0000$, $b_2 = 6.6667$, $b_3 = 10.0000$ and $b_4 = 11.6667$ and they are provided in the appropriate units where the length is specified in meters and the time in seconds. The state and control constraint sets in the physical coordinates are

$$\begin{aligned} \mathbb{X} = & \ \{h \in \mathbb{R}^4 \mid h_1 \in [0.3,\ 1.36],\ h_2 \in [0.3,\ 1.36]\ , \\ & \ h_3 \in [0.3,\ 1.3],\ h_4 \in [0.3,\ 1.3]\}\ , \\ \mathbb{U} = & \ \{q \in \mathbb{R}^2 \mid q_1 \in [0,\ 5.2/3600],\ q_2 \in [0,\ 4.85/3600]\}. \end{aligned}$$

while the set of references is required to lie in

$$\mathcal{R} = \{r \in \mathbb{R}^2 \mid r_i \in [0.61,\ 0.65]\ i = 1, 2\}\ ,$$

The discrete-time model is implicitly defined via the optimization process using a period of 5 seconds and the controller uses $N = 40$ corresponding to a prediction horizon $T_N = 200s$. The selected stage cost is $\ell(\tilde{x}(r), \tilde{u}(r)) = |x - x_s(r)|_Q^2 + |u - u_s(r))|_R^2$ where $Q = C'C$ and $R = 10^{(-4)} I_2$.

The ellipsoidal set $\mathbb{X}_f(r)$ used in (8) is defined by $P > 0$ solution of the algebraic Riccati equation with the given Q and R. Figures 1 and 2 show the system response when the reference is a square wave. Notice that the reference signals for h_1 and h_2 jump at different times and all state variables move away from their positions when a change in one of the reference signals occurs. The large changes in the reference can cause the trajectories of the controlled process to deviate considerably from reference trajectory during the transient. The proposed approach prevents the occurrence of large deviations from the set of equilibrium pairs using a suitable virtual reference as shown in figure 3. The adopted policy generates smaller spikes in the applied inputs as illustrated in figure 4 since essentially the reaction to a change of the reference is contained. This can give loss of performance in some situations where it is not necessary to limit the

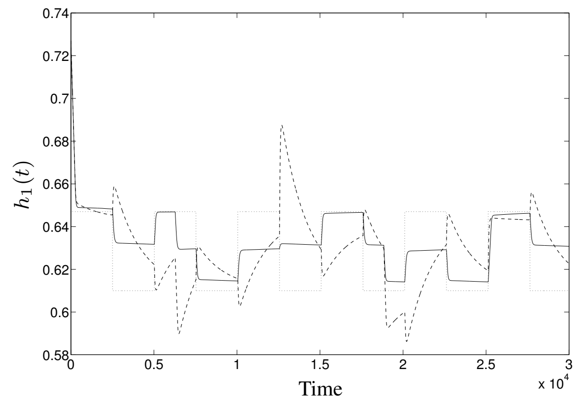

Fig. 1: State variables $h_1(t)$ vs time: references trajectories (dotted), proposed algorithm (solid), PW tracking MPC (dashed).

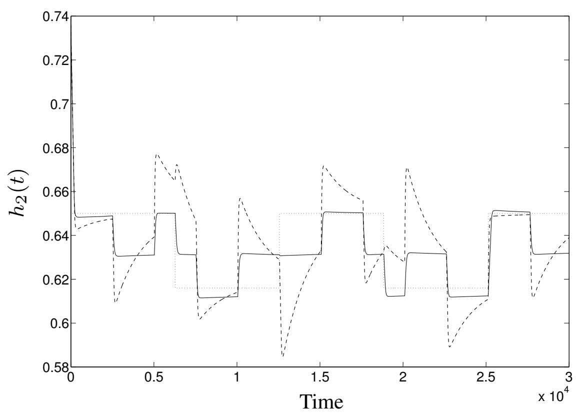

Fig. 2: State variables $h_2(t)$ vs time: references trajectories (dotted), proposed algorithm (solid), PW tracking MPC (dashed).

change of the reference as demonstrated in the next example. However this is not always the case as is shown in figures 1, 2 and 5 where the closed loop system has been simulated using randomly varying references.

Continuous stirred tank reactor

The linear model of the continuous stirred tank reactor is based on the model proposed in [22], [23]. The system is described by

$$\begin{aligned} \dot{x}_1 &= (1/\theta)(1 - x_1) - kx_1 e^{-(M/x_2)} \\ \dot{x}_2 &= (1/\theta)(x_f - x_2) + kx_1 e^{(-M/x_2)} - \alpha u(x_2 - x_c) \\ y &= x_2 \end{aligned} \qquad (10)$$

where x_1 is the product concentration, x_2 is the temperature and u is the coolant flow rate and it has been linearized at the locally unstable operating point $x_e = (0.2632, 0.6519)$. The model parameters are $\theta = 20$, $k = 300$, $M = 5$, $x_f = 0.3947$, $x_c = 0.3816$ and $\alpha = 0.117$. The state and control

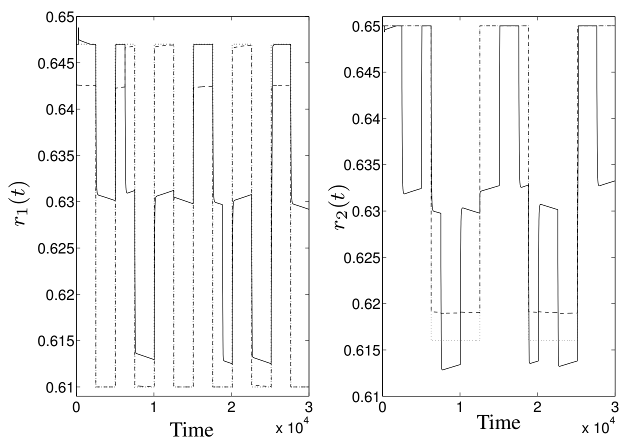

Fig. 3: References trajectories (dotted), virtual reference for the proposed algorithm (solid), artificial reference for the PW tracking MPC (dashed) in the original coordinate space

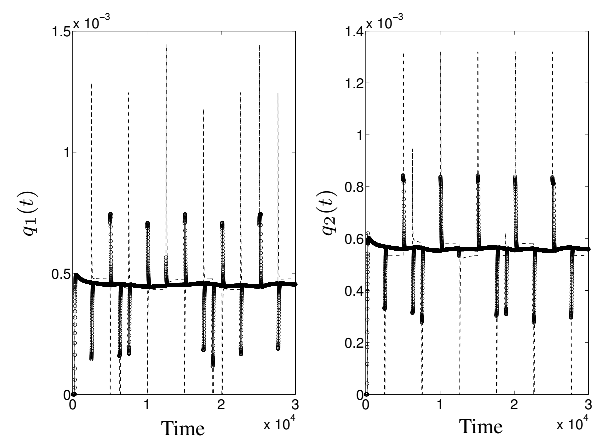

Fig. 4: Input variables q vs time: proposed algorithm (solid with circles) and PW tracking MPC (dashed)

constraint sets are

$$\mathbb{X} = \{x \in \mathbb{R}^2 \mid x_1 \in [0,1],\ x_2 \in [0,1]\},$$
$$\mathbb{U} = [0,2],$$

The controller uses $N = 420$ with a prediction horizon $T_N = 420s$ corresponding to a sampling period of 1 second. The selected stage cost is $\ell(\tilde{x}(r), \tilde{u}(r)) = |x - x_s(r)|_Q^2 + |u - u_s(r))|_R^2$ where $Q = [1\ 0;\ 0\ 10^4]$ and $R = 1$.

The ellipsoidal set $\mathbb{X}_f(r)$ described in 8 is defined by $P > 0$ solution of the algebraic Riccati equation with the selected Q and R. The simulation experiments carried out using the linearized system show that, in this case, a rapidly varying reference trajectory does not cause the trajectories to move far away from the set of admissible references

$$\mathcal{R} = \{r \in \mathbb{R} \mid r \in [0.55,\ 0.65]\}.$$

In this case it would not be necessary to apply our approach. Figures 6 and 7 demonstrate that the behaviour of the two algorithms is similar while $x \in \mathcal{X}$ and the trajectory is

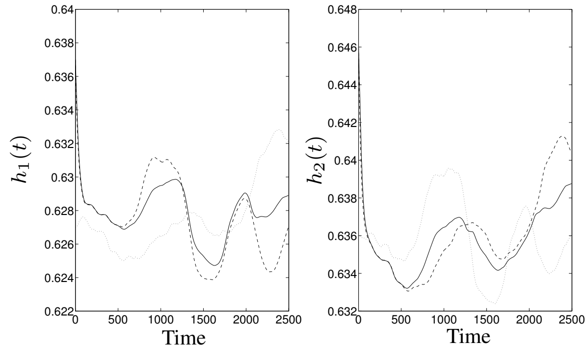

Fig. 5: State variables $h_1(t)$ and $h_2(t)$ vs time: reference trajectory (dotted), proposed algorithm (solid), PW tracking MPC (dashed).

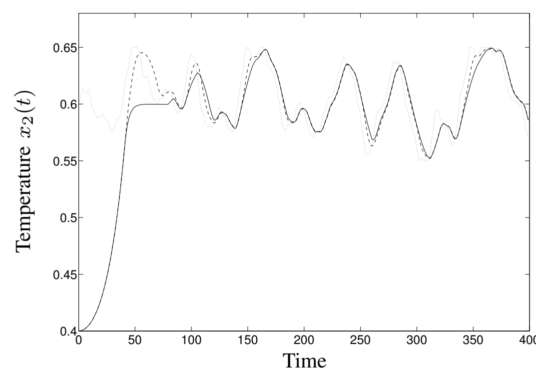

Fig. 6: State variable x_2 vs time: reference trajectory (dotted), proposed algorithm (solid) and PW tracking MPC (dashed)

distant from the set $\mathcal{X}$. A delay in the tracking response occurs when the trajectory is approaching $\mathcal{X}$ but is not inside the set as it is visible in figures 6, 7 and 8. However, the simulations performed using the nonlinear model show that the proposed technique is effective and considerably reduces the occurrence of spikes slowing down the response of the system as shown in figures 9 and 10.

VI. Conclusion

Whenever the output reference for a system is constant or converges to some constant, standard model predictive control may be used to ensure stability of the controlled system and asymptotic tracking. This paper study the case when the reference output ranges randomly over a set of values and shows that, when this happens, the state can move far away from the sets of operating points if a virtual reference is not chosen carefully. The proposed approach selects a virtual reference that guarantees the trajectory remains in a specific set.

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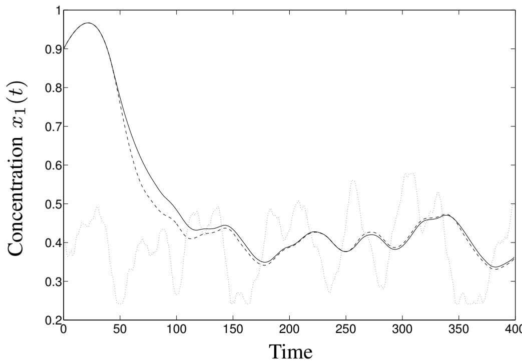

Fig. 7: State variable x_1 vs time: steady state values given by the reference trajectory (dotted), proposed algorithm (solid) and PW tracking MPC (dashed)

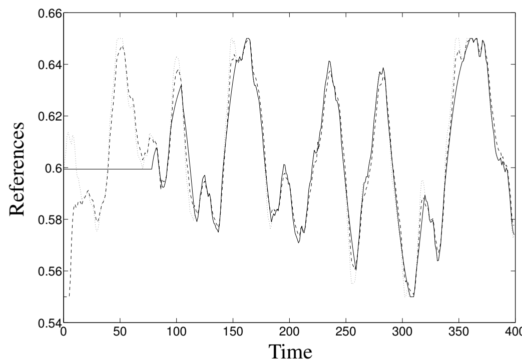

Fig. 8: Reference trajectory (dotted), Virtual reference for the proposed algorithm (solid) and artificial reference for PW tracking MPC (dashed)

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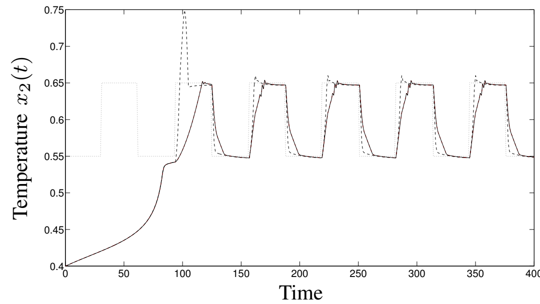

Fig. 9: State variable x_2 vs time: reference trajectory (dotted), proposed algorithm (solid) and PW tracking MPC (dashed)

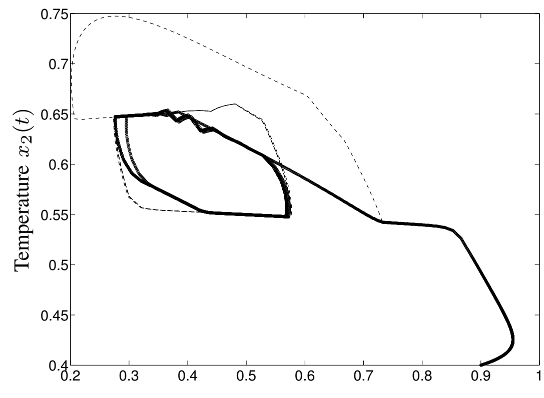

Fig. 10: State variable x_2 vs x_1: proposed algorithm (solid) and PW tracking MPC (dashed)

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