

A Decomposition Approach to Multi-Region Optimal Power Flow in Electricity Networks

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Abstract—We present a decomposition approach to a class of social welfare optimization problems for optimal power flow in multi-region electricity networks. The electricity network is decomposed into multiple regions which decide independently over the amount of power produced within the region and exchanged with neighboring regions. We decompose the overall power flow (or social welfare) optimization into region-based optimization problems (namely, power flow game), which is based on the introduction of dual variables representing nodal and link prices. Due to the interdependencies between regions' utilities, the social welfare maximizer might not necessarily correspond to a Nash equilibrium of the power flow game. We derive conditions under which the social welfare maximizer is a Nash equilibrium of the game, and investigate uniqueness of Nash equilibria. Finally, we examine whether convergence to the social welfare maximizer may occur under natural best-response dynamics.

I. INTRODUCTION

Recently there has been significant effort on exploring alternative ways for restructuring the power market. As pointed out in [1], *the difficulty of regulating open access to the transmission grid is related to the allocation of transmission capacity for the efficient use of the grid*. Following this line of research, this paper is also concerned with a pricing mechanism for transmission rights in multi-region electricity markets, where each region acts myopically trying to maximize its local objective function. Due to the physical laws of electricity transmission, interdependencies between regions are present in their objective functions, resulting to strong strategic interactions.

More specifically, we wish to derive a pricing mechanism for transmission rights that will lead to an *efficient* operation of the electricity network, where efficient operation corresponds to the maximization of an aggregate objective function (or *social welfare*). This work is mostly related to market mechanisms for power transmission, e.g., [2], [3], where the connection between a form of *competitive* (or *market*) equilibrium and *social optimum* is investigated under a linearized direct-current (DC) transmission approximation. For example, in [2], it is shown that when nodal prices correspond to the Kuhn-Tucker coefficients of the energy balance constraints, a maximizer of the social welfare corresponds to a *competitive equilibrium*. Since the analysis in [2],

[3] is *static* (investigating the relation between equilibrium notions), a follow-up question is whether a *distributed* and *dynamic* optimization scheme can lead to the social welfare maximizer under a pricing mechanism for transmission rights.

To this end, this paper introduces a decomposition approach for the multi-region power flow problem through which we investigate a) *the relation between locally optimum allocations (or Nash equilibria) and social welfare maximizers*, and b) *the transient behavior of the market due to the strategic behavior of the regions*. The problem of decomposing a social welfare optimization for multi-region optimal power flow is not recent. In fact, several approaches has been introduced including [4], [5], [6], [7]. In comparison with this line of research, we consider a simplified model for power transmission that allows for a better intuition of market power analysis, since more detailed models (as, e.g., in [6]) does not lead to closed-form solutions. Also, we avoid the introduction of fictitious intermediate buses (as, e.g., in [4], [5], [7]) to deal with the interdependencies between neighboring nodes, and instead we analyze the decomposed system as it is.

Furthermore, the decomposition proposed herein for the multi-region power flow problem is fundamentally different from prior approaches for single-region optimal power flow, such as the ones in [8], [9], since here each region is directly controlling the amount of power exchanged with neighboring regions. However, as we shall see the conclusions are comparable. Lastly, the decomposition of the power market into regions is assumed given and we are not concerned with the problem of optimally defining such regions (or zones) [10].

The remainder of the paper is organized as follows. Section II presents the necessary terminology for graphical games. Sections III–IV present the class of social welfare optimization problems considered herein and provide characteristics of the social welfare maximizers. Section V presents a decomposition approach for the multi-region optimal power flow problem and Section VI analyzes the resulting strategic interaction between regions. Finally, Section VII presents concluding remarks.

II. TERMINOLOGY

A. Graph

We consider a finite set of nodes \mathcal{I} connected through a set of links \mathcal{L} defined as $\mathcal{L} \subseteq \{(ij) : i, j \in \mathcal{I}\}$, where (ij) represents an (undirected) link between nodes i and j . We will denote by $|\mathcal{I}|$ the cardinality of the set \mathcal{I} . Furthermore, let \mathcal{N} denote the set of neighboring nodes, i.e., $\mathcal{N}(i) \triangleq$

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$\{j \in \mathcal{I} : (ij) \in \mathcal{L}\} \cup \{i\}$. The pair $\mathfrak{G} \triangleq \{\mathcal{I}, \mathcal{L}\}$ will denote a *graph*.

B. Graphical Game

Throughout the paper, we will deal with a class of *strategic-form games* (cf. [11]) on graphs, namely *graphical games* [12], with a continuum of actions. In particular, each node $i \in \mathcal{I}$ has a set of actions, denoted \mathcal{A}_i , which is a non-empty closed and bounded interval in \mathbb{R} , i.e., of the form $[\underline{\alpha}_i, \bar{\alpha}_i]$ for some $\underline{\alpha}_i, \bar{\alpha}_i \in \mathbb{R}$. Let also α_i be a representative element of this set. Define \mathcal{A} to be the Cartesian product $\mathcal{A} \triangleq \mathcal{A}_1 \times \dots \times \mathcal{A}_{|\mathcal{I}|}$. For some set of indices $J \subseteq \mathcal{I}$, we will also use the notation $\alpha_J \triangleq \{\alpha_i\}_{i \in J}$ and $\mathcal{A}_J \triangleq \times_{i \in J} \mathcal{A}_i$.

We will further assume that each node $i \in \mathcal{I}$ is a strategic player whose preferences can be represented by a *utility function* $u_i : \mathcal{A}_{\mathcal{N}(i)} \rightarrow \mathbb{R}$ which depends only on the actions of the nodes within $\mathcal{N}(i)$. In this paper, we will consider utility functions that are continuously differentiable in their domain.

Let $-i$ denote the complementary set $\mathcal{N}(i) \setminus i$. We will often split the argument of a function in this way, e.g., $F(x) = F(x_i, x_{-i})$.

The quadruple $\Gamma(\mathfrak{G}) \triangleq \{\mathcal{I}, \mathcal{L}, \mathcal{A}, \{u_i\}_i\}$ will define a *graphical game*.

1) *Better reply and Nash equilibria*: The following defines the notion of *better reply* to graphical games with a continuum of actions.

Definition 2.1 (Better reply): An agent i exhibits a *better reply* at an action profile $\alpha_{\mathcal{N}(i)} = (\alpha_i, \alpha_{-i})$ if either one of the following conditions are satisfied:¹

- (B1) $\nabla_{\alpha_i} u_i(\alpha_{\mathcal{N}(i)}) \neq 0 \wedge \alpha_i \in (\underline{\alpha}_i, \bar{\alpha}_i)$;
- (B2) $\nabla_{\alpha_i} u_i(\alpha_{\mathcal{N}(i)}) < 0 \wedge \alpha_i = \bar{\alpha}_i$;
- (B3) $\nabla_{\alpha_i} u_i(\alpha_{\mathcal{N}(i)}) > 0 \wedge \alpha_i = \underline{\alpha}_i$.

Definition 2.2 (Nash equilibrium): An action profile α^* is a *Nash equilibrium* of the graphical game $\Gamma(\mathfrak{G})$, if there is no agent which exhibits a *better reply*.

2) *Efficient action profiles*: We also define the *value* of the graphical game $\Gamma(\mathfrak{G})$ at action profile α as:

$$\mathcal{W}(\alpha) \triangleq \sum_{i \in \mathcal{I}} u_i(\alpha_{\mathcal{N}(i)}) = \sum_{i \in \mathcal{I}} u_i(\alpha_i, \alpha_{-i}). \quad (1)$$

Then, an *efficient action profile* is defined as follows:

Definition 2.3 (Efficient action profile): An action profile α^* is an *efficient action profile* if $\alpha^* \in \arg \max_{\alpha \in \mathcal{A}} \mathcal{W}(\alpha)$.

III. SOCIAL WELFARE OPTIMIZATION FOR DC-OPF

A. Social welfare optimization

Several optimization problems related to the *optimal power flow* (OPF) in power systems can be formulated as a social welfare optimization. In particular, a power network \mathfrak{G} can be represented as a set of nodes \mathcal{I} connected through a set of links \mathcal{L} , where the links here represent the physical interconnections between nodes. Nodes represent large network components or regions, similarly to the framework of

¹The gradient of the utility function is defined appropriately to accommodate also the case where the action of agent i is located at the boundary of its state space \mathcal{A}_i .

[4], [5]. Each region contains one or more generation units and decides about the total amount of power p_i produced to meet the power demand d_i and the amount of power p_{ij} that is exchanged with any neighboring node $j \in \mathcal{N}(i)$, which is subject to Kirchhoff's current law.

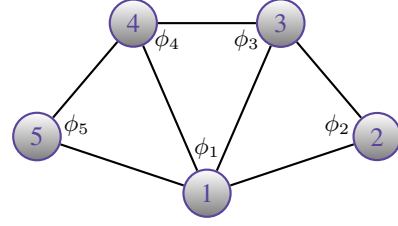


Fig. 1. A schematic of a multi-region power system.

A *social welfare optimization* (SW) can be formulated as:

$$(SW) : \begin{cases} \max & \sum_{i \in \mathcal{I}} [f_i(d_i) - g_i(p_i)] \\ \text{s.t.} & p_i - \sum_{j \in \mathcal{N}(i) \setminus i} p_{ij}(\phi_i, \phi_j) = d_i, \quad \forall i \in \mathcal{I} \\ & p_{ij}(\phi_i, \phi_j) \leq t_{ij}, \quad \forall (ij) \in \mathcal{L} \\ \text{var.} & p_i \in \mathcal{P}_i, d_i \in \mathcal{D}_i, \phi_i \in \Phi_i, \quad \forall i \in \mathcal{I} \end{cases}$$

where $g_i : \mathcal{P}_i \rightarrow \mathbb{R}$ denotes the *cost function* for producing power p_i in region i and $f_i : \mathcal{D}_i \rightarrow \mathbb{R}$ denotes the *utility of consumption* of power demand d_i . The overall objective function represents a standard social-welfare criterion for multi-region optimal power flow (see, e.g., [13]).

Furthermore, $\mathcal{P}_i \triangleq [0, \infty)$ and $\mathcal{D}_i \triangleq [0, \infty)$. Although infinite production or demand is not feasible, we indirectly impose any upper bound constraints on p_i and d_i by appropriately defining the utility f_i and the cost g_i .

The term p_{ij} represents power flow from node i to node j , and as it will become obvious in a forthcoming section, can be represented as a function of physical quantities ϕ_i and ϕ_j characterizing nodes i and j , respectively. We also assume that $\Phi_i \triangleq [\underline{\phi}_i, \bar{\phi}_i]$, representing generic bounds on the physical quantity ϕ_i for each $i \in \mathcal{I}$.

Variations of the centralized optimization (SW) have been considered to define shadow prices for electricity in a power network through the Lagrange multipliers of the dual optimization problem (see, e.g., [13], [14], [8], [9]).

B. Objective

The question we wish to answer in this paper is whether the above centralized optimization (SW) can be decomposed into natural node-based optimization problems, the solutions of which correspond to social welfare maximizers. In other words, the goal of this paper is to address the following decentralized optimization problem:

Problem 3.1: Are there prices λ_{ij}^* , $(ij) \in \mathcal{L}$, for power exchange, such that the allocation $\{(p_i^*, d_i^*, \phi_i^*)\}_i$, where (p_i^*, d_i^*, ϕ_i^*) is a solution to region i 's optimization:

$$(DO) : \begin{cases} \max & f_i(d_i) - g_i(p_i) + \sum_{j \in \mathcal{N}(i) \setminus i} \lambda_{ij}^* p_{ij}(\phi_i, \phi_j) \\ \text{s.t.} & p_i - \sum_{j \in \mathcal{N}(i) \setminus i} p_{ij}(\phi_i, \phi_j) = d_i \\ \text{var.} & p_i \in \mathcal{P}_i, d_i \in \mathcal{D}_i, \phi_i \in \Phi_i \end{cases}$$

is a solution to the centralized optimization (SW)?

Note that this problem is fundamentally different from the decomposition schemes proposed in [8], [9], since each region i decides directly about the amount of power exchanged with the neighboring nodes via adjusting ϕ_i .

In the forthcoming Section V, we propose one such decomposition approach for an approximation of the social welfare optimization (SW). This question is also related to the existence of sustainable *bilateral contracts* between regions as discussed in [2].

C. DC approximation

Before addressing this problem, we adopt a direct-current (DC) approximation of the centralized optimization (SW) (see, e.g., [13], [15]). In particular, given a phasor representation of power transmission, the total real power injected in line (ij) can be approximated by² $p_{ij} \approx -|y_{ij}|(\phi_i - \phi_j)$, where y_{ij} is the admittance of line (ij) and ϕ_i is the phase of the voltage phasor at node i .

Under this approximation, the social welfare optimization can be formulated as follows:

$$\begin{cases} \max & \sum_{i \in \mathcal{I}} [f_i(d_i) - g_i(p_i)] \\ \text{s.t.} & p_i + \sum_{j \in \mathcal{N}(i) \setminus i} |y_{ij}|(\phi_i - \phi_j) = d_i, \quad \forall i \in \mathcal{I} \\ & -|y_{ij}|(\phi_i - \phi_j) \leq t_{ij}, \quad \forall (ij) \in \mathcal{L} \\ \text{var.} & p_i \in \mathcal{P}_i, \phi_i \in \Phi_i, d_i \in \mathcal{D}_i, \quad \forall i \in \mathcal{I} \end{cases}$$

and will be denoted by (SW-DC).

We will assume that the functions f_i and $-g_i$ are concave in \mathbb{R}_+ . In most cases, both f_i and g_i are assumed quadratic (see, e.g., [16]). Then, it is trivial to check that the above social welfare optimization corresponds to a convex optimization problem, and therefore, (SW-DC) admits a solution.

Definition 3.1 (Social welfare maximizer): A social welfare maximizer (p^*, d^*, ϕ^*) is a solution to (SW-DC).

IV. SOCIAL WELFARE MAXIMIZERS

A. Dual optimization and strong duality

A natural way of decomposing (SW-DC) is by decomposing the corresponding dual problem. In particular, the Lagrangian of the social welfare optimization is:

$$\begin{aligned} L(p, d, \phi, \lambda, \mu) \triangleq & \sum_{i \in \mathcal{I}} [f_i(d_i) - g_i(p_i)] + \\ & \sum_{i \in \mathcal{I}} \lambda_i \left(p_i - \sum_{j \in \mathcal{N}(i) \setminus i} p_{ij} - d_i \right) - \\ & \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{N}(i) \setminus i} \mu_{ij} (p_{ij} - t_{ij}) \end{aligned}$$

where $\lambda \triangleq [\lambda_i]_i$ and $\mu \triangleq [\mu_{ij}]_{(ij)}$ denote the dual variables associated with the energy balance and transmission constraints, respectively. The Lagrangian of the social welfare optimization can be written equivalently as:

$$L(p, d, \phi, \lambda, \mu) =$$

²This approximation is subject to the assumption that the voltage magnitude remains constant throughout the network and it is normalized to unity and that the voltage phase angle difference $\phi_i - \phi_j$ in any line (ij) is sufficiently small (see, e.g., [13]).

$$\begin{aligned} & \sum_{i \in \mathcal{I}} [-g_i(p_i) + \lambda_i p_i] + \sum_{i \in \mathcal{I}} [f_i(d_i) - \lambda_i d_i] - \\ & \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{N}(i) \setminus i} (\lambda_i + \mu_{ij}) p_{ij} + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{N}(i) \setminus i} \mu_{ij} t_{ij}. \end{aligned} \quad (2)$$

A useful property for defining prices for electricity exchange is *strong duality*, defined as follows:

Definition 4.1 (Strong duality): The social welfare optimization (SW-DC) satisfies strong duality if there exist vectors λ^* and μ^* such that the supremum of the concave function $L(p^*, d^*, \phi^*, \lambda^*, \mu^*)$ in $\mathcal{P} \times \mathcal{D} \times \Phi$ is finite and equal to the optimal value of (SW-DC). We will also refer to vectors λ^*, μ^* as the Kuhn-Tucker coefficients.

Claim 4.1: If both f_i and $-g_i$ are concave functions in \mathbb{R}_+ for all $i \in \mathcal{I}$, then the social welfare optimization (SW-DC) satisfies strong duality.

Proof: The optimization (SW-DC) is an (ordinary) convex program when both f_i and $-g_i$ are concave functions in \mathbb{R}_+ for all $i \in \mathcal{I}$. Note that all the constraints in (SW-DC) are affine with respect to the optimization variables. Then, by [17, Theorem 28.2], the optimization (SW-DC) satisfies strong duality. ■

The following proposition [17, Theorem 28.1] provides a way to compute the solutions to the primal optimization (SW-DC), by first computing the maximizers of the Lagrangian over $\mathcal{P} \times \mathcal{D} \times \Phi$ and then eliminating those points which do not satisfy certain conditions.

Proposition 4.1 (Optimality Conditions): Let λ and μ be Kuhn-Tucker coefficients for (SW-DC). Let also D be the set of points where the Lagrangian L attains its supremum over $\mathcal{P} \times \mathcal{D} \times \Phi$. If D_0 is the set of points $(p^*, d^*, \phi^*) \in D$ such that for all $i \in \mathcal{I}$ and $(ij) \in \mathcal{L}$, the following conditions are satisfied

$$\begin{aligned} \text{(K1)} & \quad p_i^* - \sum_{j \in \mathcal{N}(i) \setminus i} p_{ij}^* - d_i = 0, \\ \text{(K2)} & \quad \mu_{ij}(p_{ij}^* - t_{ij}) = 0, \quad \mu_{ij} \geq 0, \quad p_{ij}^* - t_{ij} \leq 0, \end{aligned}$$

then, D_0 is the set of optimal solutions to (SW-DC).

B. Characterization of Social Welfare Maximizers

In this section, we derive properties of the social welfare maximizers, i.e., the solutions of (SW-DC).

We will define the matrix $G = [G(i, j)] \in \mathbb{R}^{|\mathcal{I}|}$, such that

$$G(i, j) \triangleq \begin{cases} |y_{ij}| / \bar{y}_i, & (ij) \in \mathcal{L} \\ 0, & \text{else} \end{cases},$$

where $\bar{y}_i \triangleq \sum_{k \in \mathcal{N}(i) \setminus i} |y_{ik}|$. We will also denote G_i to be the i th row of matrix G . Let also $\bar{Y} \triangleq \text{diag}\{\bar{y}_i\}_i$.

Furthermore, we will assume a quadratic cost function for the production of power in each region, i.e.: $g_i(p_i) \triangleq \beta_i p_i + \frac{1}{2} \gamma_i p_i^2$, for all $i \in \mathcal{I}$, for some positive constants β_i, γ_i . Also, we will assume a quadratic form for the utility of consumption, i.e., $f_i(d_i) \triangleq \zeta_i d_i - \frac{1}{2} \eta_i d_i^2$, for all $i \in \mathcal{I}$, for some positive constants ζ_i and η_i .

Lemma 4.1 (Kuhn-Tucker coefficients): Any Kuhn-Tucker coefficients of the social welfare optimization (SW-DC), λ^* and μ^* , satisfy the following condition:

$$(e_i^T - G_i) \lambda^* + G_i \theta_i^* = 0. \quad (3)$$

where $\theta_i^* \triangleq [\mu_{ij}^*]_j - [\mu_{ji}^*]_j$.

Proof: According to Definition 4.1 and since (SW-DC) satisfies strong duality (by Claim 4.1), if λ^* is a Kuhn-Tucker vector for (SW-DC), then the supremum of the Lagrangian (2) is finite and attained for some feasible allocation (p^*, d^*, ϕ^*) . Note that the gradient of L with respect to ϕ_i satisfies:

$$\begin{aligned} \nabla_{\phi_i} L(p, d, \phi, \lambda, \mu) &= - \sum_{j \in \mathcal{N}(i) \setminus i} (\lambda_i^* + \mu_{ij}^*) |y_{ij}| + \sum_{j \in \mathcal{N}(i) \setminus i} (\lambda_j^* + \mu_{ji}^*) |y_{ij}| \\ &= \bar{y}_i [(-e_i^T + G_i) \lambda^* - G_i \theta_i^*]. \end{aligned}$$

Since (SW-DC) was derived under the assumption that the phase angle differences are sufficiently small, and L attains its supremum at (p^*, d^*, ϕ^*) , the conclusion follows. ■

Lemma 4.1 provides a necessary condition for the vectors λ^* and μ^* to be Kuhn-Tucker coefficients for (SW-DC). Condition (3) reveals the interdependencies of each λ_i^* with i) the neighboring coefficients λ_j^* , ii) the congestion coefficients, μ_{ij}^* , of neighboring transmission lines, and iii) the admittances $|y_{ij}|$ of neighboring transmission lines.³ Note also that in the trivial case where no transmission constraint is reached, i.e., $p_{ij}^* < t_{ij}$ for all $(ij) \in \mathcal{L}$, then from the optimality condition (K2), we have that $\mu_{ij}^* = \mu_{ji}^* = 0$ for all $(ij) \in \mathcal{L}$, or, equivalently, $\theta_i^* = 0$, for all $i \in \mathcal{I}$. In that case, it is straightforward to check that condition (3) is satisfied for coefficients $\lambda_1^* = \dots = \lambda_n^*$.

Lemma 4.2 (Uniqueness): Let λ^* and μ^* be Kuhn-Tucker coefficients for (SW-DC). Then, the supremum of the Lagrangian L is attained for

$$p_i^* = \begin{cases} \frac{\lambda_i^* - \beta_i}{\gamma_i}, & \lambda_i^* \geq \beta_i \\ 0, & \text{else} \end{cases}, \quad d_i^* = \begin{cases} \frac{\zeta_i - \lambda_i^*}{\eta_i}, & \zeta_i \geq \lambda_i^* \\ 0, & \text{else} \end{cases} \quad (4)$$

for all $i \in \mathcal{I}$. Also, the allocation $\{p_i^*, d_i^*\}_i$ is the unique solution of (SW-DC). Moreover, the optimal phase angles, ϕ^* , may not be unique and satisfy $(I - G)\phi^* = \bar{Y}^{-1}w$, where $w \triangleq [d_i^* - p_i^*]_i$.

Proof: Condition (4) defines the unique points at which the gradient of the Lagrangian with respect to p_i and d_i vanishes. Thus, according to [17, Corollary 28.1.1], $\{(p_i^*, d_i^*)\}_i$ is the unique solution for (SW-DC). Furthermore, condition (K1) of Proposition 4.1 implies that the optimal phase angle ϕ^* satisfies:

$$\sum_{j \in \mathcal{N}(i) \setminus i} |y_{ij}| (\phi_i^* - \phi_j^*) = d_i^* - p_i^* \triangleq w_i,$$

which can be written equivalently as $(I - G)\phi^* = \bar{Y}^{-1}w$. Note that G is a row stochastic matrix, since its elements are nonnegative and the sum of all elements of each row is 1. Thus, by Perron-Frobenius Theorem (cf., [18, Theorem 6.1.1]), the maximum eigenvalue of G is $\rho_{\max}(G) = 1$. Since G is a symmetric matrix, $(I - G)$ is singular. Thus, we conclude that there might be more than a single allocation of angles ϕ^* that attains the supremum of (SW-DC). ■

³An interesting discussion about the consequences of such interdependencies can be found at [2, Section 3].

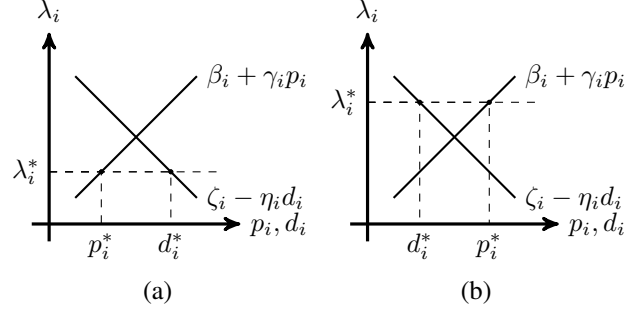


Fig. 2. Marginal supply-demand curves and optimal price selection when power (a) imported, and (b) exported.

Lemma 4.2 states that the optimal allocation of phase angles, ϕ^* , may not be unique. However, the allocation of power production, $\{p_i^*\}$, and power demand, $\{d_i^*\}$, is unique. Consequently, the allocation of power imported or exported from each node will also be unique.

An immediate consequence of Lemma 4.2 is the following.

Proposition 4.2 (Optimal prices): Let p^* and d^* denote the unique solution of (SW-DC) according to Lemma 4.2. If, either $p_i^* > 0$ or $d_i^* > 0$, then the Kuhn-Tucker coefficients $\{\lambda_i^*\}$ are unique and satisfy either one of the conditions:

- (P1) $\lambda_i^* = \gamma_i p_i^* + \beta_i = -\eta_i d_i^* + \zeta_i$, if $p_i^* > 0$ and $d_i^* > 0$;
- (P2) $\lambda_i^* = \gamma_i p_i^* + \beta_i > \zeta_i$, if $p_i^* > 0$ and $d_i^* = 0$;
- (P3) $\lambda_i^* = -\eta_i d_i^* + \zeta_i < \beta_i$, if $p_i^* = 0$ and $d_i^* > 0$.

In the trivial case where $p_i^* = d_i^* = 0$, then from Lemma 4.1, we conclude that there might be more than one Kuhn-Tucker vector λ^* .

V. DECOMPOSITION

In this section, we decompose the dual optimization problem of (SW-DC) into region-based optimizations that capture naturally the power exchanges within the network and provide an answer to Problem 3.1. Note that the Lagrange multipliers λ_i , $i \in \mathcal{I}$, can naturally be interpreted as the nodal prices for power exchange. This definition of prices also suggests a natural definition of utility functions for both the generators and the consumers in each region.

In particular, the actuation variables for each region include the power production p_i , the power demand d_i and the phase angle ϕ_i . Since the consumers act independently and, in fact, drive the production, we further decompose the objective function of each region into a) the objective of the consumers at node i , and b) the objective of the generators at node i . There is also a third agent, the network operator, which is performing the power transfer between regions.

A. The consumers' objective

The objective of the consumers at node i , is:

$$(\text{DO-}C_i): \quad \left\{ \max_{d_i \in \mathcal{D}_i} u_{c,i}(d_i) = f_i(d_i) - \lambda_i d_i, \quad (5) \right.$$

which consists of:

- (1) the utility of consumption $f_i(d_i)$ at demand d_i , and
- (2) the cost of purchased power d_i .

Note that the demand d_i may include power that is imported from neighboring nodes. The price of power at node i is λ_i .

B. The generators' objective

Given the demand, d_i , and the neighboring phase angles, $\{\phi_j\}_{j \in \mathcal{N}(i) \setminus i}$, the objective of the generators at node i is:

$$\begin{cases} \max & \lambda_i p_i - g_i(p_i) \\ \text{s.t.} & p_i = d_i - \sum_{j \in \mathcal{N}(i) \setminus i} |y_{ij}| (\phi_i - \phi_j) \\ & -|y_{ij}| (\phi_i - \phi_j) \leq t_{ij}, \forall j \in \mathcal{N}(i) \setminus i, \\ \text{var} & p_i \in \mathcal{P}_i, \phi_i \in \Phi_i. \end{cases}$$

which consists of:

- (1) the cost of production $g_i(p_i)$ of power p_i , and
- (2) the benefit from selling power p_i at price λ_i .

Note that the total produced power p_i at node i may include power that is injected to neighboring links at price λ_i .

Let us define the set:

$$\begin{aligned} \mathcal{F}_i(d_i, \phi_{\mathcal{N}(i) \setminus i}) &\triangleq \\ &\left\{ \phi_i \in [\underline{\phi}, \bar{\phi}] : \sum_{j \in \mathcal{N}(i) \setminus i} |y_{ij}| (\phi_i - \phi_j) \leq d_i, \wedge \right. \\ &\left. |y_{ij}| (\phi_i - \phi_j) \geq -t_{ij}, \forall j \in \mathcal{N}(i) \setminus i \right\}, \end{aligned}$$

which defines a closed interval within $[\underline{\phi}, \bar{\phi}]$. In several cases, we will use the notation $\mathcal{F} = \mathcal{F}(d, \phi)$ to denote the *joint action space*. Then, the optimization of the generation units at node i can be rewritten as follows:

$$(\text{DO-}G_i): \left\{ \max_{\phi_i \in \mathcal{F}_i(d_i, \phi_{\mathcal{N}(i) \setminus i})} u_{g,i}(d_i, \phi_{\mathcal{N}(i)}) \right\} \quad (6)$$

where $u_{g,i}(d_i, \phi_{\mathcal{N}(i)}) \triangleq \lambda_i p_i - g_i(p_i)$ is evaluated at $p_i = d_i - \sum_{j \in \mathcal{N}(i) \setminus i} |y_{ij}| (\phi_i - \phi_j)$.

Each node i needs to compute the optimal phase angle ϕ_i , given d_i and $\phi_{\mathcal{N}(i) \setminus i}$. Note that this computation does not necessarily requires that nodes communicate their phase angles to their neighboring nodes, since the neighboring phase angles can be indirectly computed through the power flow in each transmission line.

C. The network operator

The final part of this decomposition is the utility of the (fictitious) network operator captured by the residual part of the Lagrangian of (SW-DC), i.e.,

$$\begin{aligned} u_n(\phi) &\triangleq - \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{N}(i) \setminus i} \lambda_i p_{ij}(\phi_i, \phi_j) - \\ &\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{N}(i) \setminus i} \mu_{ij} p_{ij}(\phi_i, \phi_j) \end{aligned}$$

where the first term corresponds to the *merchandizing surplus* and the second term to the *congestion rent*.

The (fictitious) network operator does not have any control over the exchanged power, since the phase angles are being decided by each region independently. This is an implication of the decomposition and a distinctive feature of this model.

VI. POWER FLOW GAME (PFG)

A. Preliminaries

The decomposition of (SW-DC) introduces a (graphical) game among the consumers and the generators of the network given a price vector $\lambda = (\lambda_1, \dots, \lambda_{|\mathcal{I}|})$. We will call this game the *power flow game* (PFG). We denote such graphical game by $\Gamma(\mathfrak{G})$ where the utility functions of the consumers and the generators are defined by (5)–(6).

A *Nash equilibrium* of the (PFG) is defined as follows:

Definition 6.1 (Nash equilibrium): An allocation (d^*, ϕ^*) is a *Nash equilibrium of the (PFG)* if, for each $i \in \mathcal{I}$, both the consumers and the generators do not exhibit a better reply at d^* and ϕ^* with respect to their utility functions $u_{c,i}$ and $u_{g,i}$, respectively.

We may define the *value* of the (PFG) at (d, ϕ) as:

$$W_{\text{PFG}}(d, \phi) \triangleq \sum_{i \in \mathcal{I}} \{u_{c,i}(d_i) + u_{g,i}(d_i, \phi_{\mathcal{N}(i)})\},$$

a maximizer of which will correspond to an *efficient allocation*. The value of (PFG) corresponds to the welfare function of the participants in the game, and may not coincide with the value of the social welfare. Obviously, an efficient allocation (d^*, ϕ^*) may not be a Nash equilibrium of the (PFG).

B. Social welfare maximizers vs Nash equilibria

The following theorem provides conditions under which a social welfare maximizer of (SW-DC) is a Nash equilibrium of the (PFG) and vice versa.

Theorem 6.1: Let λ^* and μ^* be Kuhn-Tucker coefficients for the (SW-DC). Then, the following hold:

- (1) Let (p^*, d^*, ϕ^*) be a social welfare maximizer such that $p_i^* > 0$ or $d_i^* > 0$ for all $i \in \mathcal{I}$. Then, (d^*, ϕ^*) , evaluated at λ^* is a Nash equilibrium of $\Gamma(\mathfrak{G})$;
- (2) Let (d^*, ϕ^*) be a Nash equilibrium of $\Gamma(\mathfrak{G})$ evaluated at λ^* . Define $p_i^* = p_i^*(d_i^*, \phi_{\mathcal{N}(i)}^*)$ satisfying (K1). If $p_i^* > 0$ for all $i \in \mathcal{I}$, then the allocation (p^*, d^*, ϕ^*) is a social welfare maximizer.

In other words, Theorem 6.1 states that in non-trivial cases, the social welfare maximizer corresponds to a Nash equilibrium of the (PFG). The converse is also true if the corresponding power production is non-zero at all nodes.

Proof: (1) Consider any allocation (p^*, d^*, ϕ^*) which is a social welfare maximizer and satisfies $p_i^* > 0$ or $d_i^* > 0$ for all $i \in \mathcal{I}$. By optimality condition (K1), the optimal allocation of power production p_i^* satisfies the energy balance constraint for each node $i \in \mathcal{I}$. Also, the consumers' and generators' utility satisfy:

$$\frac{\partial u_{c,i}}{\partial d_i}(d_i^*) = -\eta_i d_i^* + \zeta_i - \lambda_i^*, \quad (7)$$

and

$$\frac{\partial u_{g,i}}{\partial \phi_i}(\phi_{\mathcal{N}(i)}^*) = (\gamma_i p_i^* + \beta_i - \lambda_i^*) \sum_{j \in \mathcal{N}(i) \setminus i} |y_{ij}|, \quad (8)$$

respectively. According to Proposition 4.2, we may identify the following cases regarding the optimal prices λ_i^* :

- (1a) $p_i^* > 0$, $d_i^* > 0$ and $\lambda_i^* = \gamma_i p_i^* + \beta_i = -\eta_i d_i^* + \zeta_i$;
(1b) $p_i^* > 0$, $d_i^* = 0$ and $\lambda_i^* = \gamma_i p_i^* + \beta_i$;
(1c) $p_i^* = 0$, $d_i^* > 0$ and $\lambda_i^* = -\eta_i d_i^* + \zeta_i$.

Note that in case (1a), both partial derivatives (7)–(8) vanish at (d^*, ϕ^*) , implying that it corresponds to a Nash equilibrium of the (PFG).

In case (1b), the partial derivative (8) vanishes. Also, $\frac{\partial u_{c,i}}{\partial d_i}(d_i^*) < 0$, i.e., consumers may increase their utility by decreasing the demand (which is not possible since $d_i^* = 0$). Thus, neither generators nor consumers can increase their utility, which implies that (d^*, ϕ^*) is a Nash equilibrium.

In case (1c), the partial derivative (7) vanishes at d_i^* . Also, $\frac{\partial u_{g,i}}{\partial \phi_i}(\phi_{\mathcal{N}(i)}^*) > 0$, which implies that generators at i can only increase their utility by increasing ϕ_i . However, increasing ϕ_i implies reducing the power production p_i^* even further, which is not possible since $p_i^* = 0$. Thus, neither generators nor consumers can increase their utility, which implies that (d^*, ϕ^*) is a Nash equilibrium.

We conclude that if (p^*, d^*, ϕ^*) is a social welfare maximizer such that either $p_i^* > 0$ or $d_i^* > 0$ for all $i \in \mathcal{I}$, then it defines a Nash equilibrium for (PFG), namely (d^*, ϕ^*) .

(2) Let us consider any allocation (d^*, ϕ^*) which corresponds to a Nash equilibrium for the (PFG). Let also p^* be defined according to (K1) and belong to \mathcal{P} . If we substitute $p_i^* = p_i^*(d_i^*, \phi_{\mathcal{N}(i)}^*)$, given by (K1), into the Lagrangian L and we compute its gradient with respect to ϕ , we get:

$$\begin{aligned} \nabla_{\phi_i} L(d^*, \phi^*) \\ = \sum_{j \in \mathcal{N}(i) \setminus i} (\lambda_i^* + \mu_{ij}^*) |y_{ij}| - \sum_{j \in \mathcal{N}(i) \setminus i} (\lambda_j^* + \mu_{ji}^*) |y_{ij}|. \end{aligned}$$

which results from the fact that λ^* is a Kuhn-Tucker vector for (SW-DC) and satisfies property (P2) (since $p_i^* > 0$ for all $i \in \mathcal{I}$). Finally, according to Lemma 4.1, the last expression is identically zero. Thus, L attains its supremum with respect to ϕ_i for every i . Furthermore,

$$\nabla_{d_i} L(d^*, \phi^*) = -\eta_i d_i^* - \gamma_i p_i^* - \beta_i + \zeta_i,$$

which vanishes in case (1a). In case (1b), we have $\nabla_{d_i} L(d^*, \phi^*) = \zeta_i - \lambda_i^* < 0$ at $d_i^* = 0$, which implies that L achieves its supremum. Thus, in either case (1a) or (1b) at which $p_i^* > 0$ for all $i \in \mathcal{I}$, we conclude that the Nash equilibrium defines a social welfare maximizer. ■

Theorem 6.1 provides conditions under which a social welfare maximizer (p^*, d^*, ϕ^*) is a Nash equilibrium, and vice versa. However, this theorem does not characterize explicitly the set of Nash equilibria. In fact, we do not know whether there exist Nash equilibria (when utilities are evaluated at λ^*) which will *not* be social welfare maximizers. In certain cases, the set of Nash equilibria can be characterized explicitly.

Lemma 6.1 (Positive production): Let λ^* , μ^* be Kuhn-Tucker coefficients for the (SW-DC), such that $\beta_i < \lambda_i^* < \zeta_i$ for all $i \in \mathcal{I}$. If (d^*, ϕ^*) is a Nash equilibrium of the (PFG) evaluated at λ^* , then the corresponding production $p_i^* = p_i^*(d_i^*, \phi_{\mathcal{N}(i)}^*)$ given by (K1) satisfies $p_i^* > 0$ for all i . The condition $\beta_i < \lambda_i^* < \zeta_i$ reflects a solution of the (SW-DC) that corresponds to either one of the cases of Fig. 2.

For such price vector, it is shown that a Nash equilibrium may only assign a non-zero power production at each node.

Proof: First, note that at a Nash equilibrium and when $\lambda_i^* < \zeta_i$, $d_i^* > 0$. We will show that an allocation (d^*, ϕ^*) at which $p_i^* = p_i^*(d_i^*, \phi_{\mathcal{N}(i)}^*) = 0$ may not correspond to a Nash equilibrium allocation at λ^* . In particular, for each generator i , we have:

$$\frac{\partial u_{g,i}}{\partial \phi_i}(\phi_{\mathcal{N}(i)}^*) = (\gamma_i p_i^* + \beta_i - \lambda_i^*) \sum_{j \in \mathcal{N}(i) \setminus i} |y_{ij}|.$$

If $p_i^* = p_i^*(d_i^*, \phi_{\mathcal{N}(i)}^*) = 0$, then the above gradient is strictly negative implying that $u_{g,i}$ can increase only if ϕ_i reduces even further. Thus, if $\phi_i^* > \phi$, then (d^*, ϕ^*) cannot be a Nash equilibrium. If, instead, $\phi_i^* = \phi$, then $p_i^* \geq d_i^* > 0$, which is a contradiction. We conclude that an allocation at which $p_i^*(d_i^*, \phi_{\mathcal{N}(i)}^*) = 0$ may not correspond to a Nash equilibrium of the (PFG). ■

Theorem 6.2: Under the hypotheses of Lemma 6.1, the following hold:

- (1) the set of Nash equilibria is isomorphic with the set of social welfare maximizers;
- (2) the utility each participant receives at any Nash equilibrium allocation is unique;
- (3) any Nash equilibrium allocation is also an efficient allocation.

In other words, Theorem 6.2 identifies a condition on the price vector under which the set of Nash equilibria of the (PFG) is completely characterized and, in fact, is isomorphic with the set of social welfare maximizers. *Proof:*

(1) Let (d^*, ϕ^*) be a Nash equilibrium of the power flow game evaluated at the Kuhn-Tucker coefficients λ^* and satisfying the hypotheses of Lemma 6.1. According to Lemma 6.1, we should have $p_i^* = p_i^*(d_i^*, \phi_{\mathcal{N}(i) \setminus i}^*) > 0$. Thus, according to Theorem 6.1, the Nash equilibrium (d^*, ϕ^*) defines (uniquely) a social welfare maximizer.

On the other hand, since the social welfare optimization (SW-DC) admits a Kuhn-Tucker vector which satisfies the condition $\beta_i < \lambda_i^* < \zeta_i$ for all $i \in \mathcal{I}$, then according to Lemma 4.2, we should have that any social welfare maximizer satisfies $p_i^* > 0$ and $d_i^* > 0$ for all $i \in \mathcal{I}$. Thus, according to Theorem 6.1, any social welfare maximizer defines (uniquely) a Nash equilibrium.

(2) According to Lemma 4.2, any social welfare maximizer provides a unique allocation of (p^*, d^*) , therefore, by (1) we have that the utilities of each participant at the corresponding Nash equilibrium allocation will be unique.

(3) From (1), we have that a Nash equilibrium evaluated at the Kuhn-Tucker coefficients defines a social welfare maximizer. Since the Kuhn-Tucker coefficients of (SW-DC) satisfy condition (3), it is straightforward to show that $u_n(\phi^*) \equiv 0$ (simply multiply (3) with ϕ_i^* for all i , and then add all these identities to get $u_n(\phi^*)$). Thus, since at a Nash equilibrium the social welfare is maximized, we conclude that the value of the (PFG), \mathcal{W}_{PFG} , is also maximized, i.e., a Nash equilibrium defines an efficient allocation. ■

Remark 6.1: The conclusions of Theorems 6.1–6.2 pro-

vide an answer to Problem 3.1. In fact, when the network operator selects the prices of exchanged power to be the Kuhn-Tucker coefficients of the nodal energy preservation constraints $\{\lambda_i^*\}$, then Nash equilibrium allocations correspond to social welfare maximizers in non-trivial cases. Furthermore, if the congestion rent prices correspond to the Kuhn-Tucker coefficients of the line transmission constraints $\{\mu_{ij}^*\}$, then the overall network utility is zero, as shown in the proof of Theorem 6.2(3), i.e., the merchandizing surplus will coincide with the congestion rent.

C. Learning Optimal Phase Allocation

In this section, we discuss convergence to Nash equilibria of the (PFG) presented above.

We introduce a class of natural best-response dynamics (or *adaptive play*) for the evolution of phase angles $\{\phi_i\}$:⁴

$$\phi_i(t+1) = \Pi_{\mathcal{F}_i} [\phi_i(t) + \epsilon(t) \nabla_{\phi_i} u_{g,i}(\phi_i(t), \phi_{-i}(t)) + \epsilon(t) \xi_i(t)]$$

for a step-size sequence of the form $\epsilon(t) = 1/t+1$ and for an i.i.d. Gaussian noise sequence $\{\xi_i(t)\}$ which satisfies $\mathbb{E}[\xi_i(t)|\xi_i(\tau), \tau < t, \phi(0)] = 0$ and $\mathbb{E}[\xi_i^2(t)] = \sigma_i^2 > 0$.

Proposition 6.1 (Adaptive Play Convergence): *Consider the power flow game $\Gamma(\mathfrak{G})$ and let the action profile ϕ evolve according to the adaptive play dynamics. If $\{\phi(t)\}$ converges⁵ to ϕ^* , then ϕ^* is a Nash equilibrium of the power flow game $\Gamma(\mathfrak{G})$.*

Proof: By [19, Theorem 5.2.1], we have that $\{\phi(t)\}$ converges almost surely to some invariant set of the differential equation

$$\dot{\phi}_i = \nabla_{\phi_i} u_{g,i}(\phi_i, \phi_{-i}) + z_i(\phi_{\mathcal{N}(i)}(t)), \quad i \in \mathcal{I},$$

where $z_i(\phi_{\mathcal{N}(i)}(t))$ is the projection of $\nabla_{\phi_i} u_{g,i}(\cdot)$ onto the set $\mathcal{F}_i(d_i^*, \phi_{\mathcal{N}(i)}(t))$. If $\{\phi(t)\}$ converges to allocation ϕ^* in \mathcal{F} , then ϕ^* necessarily satisfies $\nabla_{\phi_i} u_{g,i}(\phi_i^*, \phi_{-i}^*) + z_i(\phi_{\mathcal{N}(i)}^*) = 0$, since ϕ^* is a stationary point. In this case, and according to Definition 2.1, there is no participant i that exhibits a better reply at ϕ^* . Thus, ϕ^* is necessarily a Nash equilibrium of the power flow game. ■

Proposition 6.1 does not address the question of whether the dynamics converge, however, stronger convergence arguments can be derived due to fact that the presented power-flow game belongs to the general class of games in networks with strategic complementarities (see, e.g., [20]).

VII. CONCLUSIONS AND FUTURE WORK

We proposed a distributed optimization approach to a class of social welfare optimization problems for multi-region optimal power flow. The analysis was based on a simplified DC approximation of the power transmission, and we derived conditions under which locally optimal solutions (or Nash equilibria) coincide with social welfare maximizers.

⁴The gradient ∇_{ϕ_i} is defined appropriately to accommodate for the case where ϕ_i is located at the boundary of Φ_i .

⁵By convergence $\phi(t) \rightarrow A$ for some set A , we mean $\lim_{t \rightarrow \infty} \text{dist}(\phi(t), A) = 0$, where $\text{dist}(\phi, A) = \min_{x \in A} |\phi - x|$.

There are several questions emerging from such a decomposition approach. In particular, the current formulation does not account for the effect of transmission losses in the electricity market. Furthermore, it is implicitly assumed throughout the analysis that the cost curves of the generation plants and the utility function of the consumption units are truthfully declared by all participants, which may not be realistic. A richer formulation that will include a market for transmission losses and will incentivize truthful revelation of utility functions will eliminate deviations from the social optimum behavior and will provide a better understanding of the strategic interactions in electricity markets.

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