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Passivity of Plane Poiseuille Flow

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Abstract— **This paper considers the passivity of plane Poiseuille flow, which is the incompressible flow observed between two parallel plates that are assumed to be of infinite extent. A model in which the flow is considered as the feedback connection between a linear time-invariant system and a static, memoryless nonlinear system is used. It is well known that the nonlinearity of plane Poiseuille flow is not only passive but lossless, which means that it does not generate or consume energy and the only effect of the nonlinearity is to move energy from one flow mode to another. However, little has been addressed about the passivity of the entire flow system. The primary aim of this paper is to find a subcritical Reynolds number below which the linearized flow is always strictly passive, which means that the origin of the full nonlinear system is globally asymptotically stable when the Reynolds number does not exceed the subcritical number. Our results show that the subcritical Reynolds number obtained by the passivity approach is equal to the energy Reynolds number, which is derived by the classical energy approach. This result indicates that the passivity approach is closely related to the energy approach and is a valuable tool to study the stability of fluid flows.**

I. INTRODUCTION

Plane Poiseuille flow is the incompressible flow with a parabolic velocity profile between two parallel, stationary and infinite plates. Despite the efforts of numerous physicists and applied mathematicians for over a century, the mechanisms of instability of the flow are still not fully understood [1].

Usually the starting point of studying the stability of the viscous channel flow is linear stability analysis, which consists of three steps: (i) The Navier-Stokes equations describing the evolution of the flow are linearized around the steady state; (ii) The resulting Orr-Sommerfeld equation is discretized by replacing all the partial differential operators with appropriate matrices; (iii) The stability is determined by examining the eigenvalues of the discretized Orr-Sommerfeld operator. The flow is linearly stable if all the eigenvalues lie in the open left half of complex plane. Orszag found plane Poiseuille flow is linearly stable when the Reynolds number is less than $\mathrm{Re}_{\nabla} = 5772.22$ [2]. However, experiments show that the Reynolds number at which the transition to turbulent flow occurs is as low as $\mathrm{Re} \approx 1000$ [3], [4].

The invalidity of the linearized model has long been believed to be one of the reasons for the discrepancy between the theoretical prediction and experimental results [1]. Indeed, if the linearization fails to approximate the Navier-Stokes equations accurately, it is unlikely that the method can provide a satisfactory result. When the disturbance grows and is no longer of infinitesimal magnitude, the nonlinearity of the flow plays a more important role and its effects can not be ignored [5]. It was also suggested that transient energy growth which arises from the nonnormality of the Orr-Sommerfeld operator is a possible explanation for the failure of eigenvalue analysis [1], [6].

Another standard tool of analyzing the stability of the flow is the energy approach [7], which is based on variational calculus and aims to find the Reynolds number below which there is no energy growth for perturbations of arbitrary magnitudes. It has been shown that the energy Reynolds number for plane Poiseuille flow is $\mathrm{Re}_E = 49.6$ [8], [5]. It is not surprising that the result is quite conservative when compared to $\mathrm{Re} \approx 1000$ given that the monotonic decay requirement is likely to be unnecessarily restrictive.

A concept closely related to stability and energy of a system is passivity. It is well known that the nonlinearity of the flow is lossless in the sense that the nonlinearity does not create or consume energy and is responsible for the interaction between different modes [5]. Although it is pointed out in [9], [10] that introducing feedback control to remove the nonnormality and make the linear part passive can stabilize the flow, little attention has been paid to the passivity of the uncontrolled flow.

This paper employs a model that regards plane Poiseuille flow as the interconnection of a linear time-invariant dynamical system and a memoryless lossless nonlinear system. The linear part of the flow is converted into a finite number of decoupled systems which are continuous in the wall-normal direction, by first applying Fourier transformations in the streamwise and spanwise directions, and then truncating the systems at sufficiently high wavenumber pairs. A Chebyshev spectral discretization method is then used in the wall-normal direction. This leads to a set of subsystems and the passivity of each is examined. According to the passivity theorem [11], the feedback connection of two passive systems is passive, and if the dynamical linear part is strictly passive and the memoryless nonlinear part is passive, then the origin of the closed-loop system is asymptotically stable. In addition, the origin is globally asymptotically stable if the storage function of the dynamical linear system is radially unbounded. If we can find the Reynolds number below which all the decoupled linear systems are always strictly passive, we will obtain a lower bound on the Reynolds number below which the flow is always stable.

Our results show that the linearized plane Poiseuille flow is always strictly passive when the Reynolds number is below $\mathrm{Re} = 49.6$, which is the same as the energy Reynolds

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number. It is to be expected that these two methods yield the same result even though the techniques employed to derive the results are completely different, since physically both approaches try to find a Reynolds number below which the perturbation energy of the flow does not increase with time. Essentially, the passivity approach uses the disturbance energy as the Lyapunov function of the system. Compared to the energy approach, the advantage of the passivity approach is the derivation is relatively simple and the passivity approach can be considered as an alternative tool to study the stability of flow systems like the energy approach, although it will inevitably be conservative.

The paper is organized as follows. Section II starts from the Navier-Stokes equations and develops the model of incompressible viscous plane Poiseuille flow. In section III, the linear part of the infinite-dimensional flow system described by PDEs is discretized and truncated, resulting in a number of decoupled linear time-invariant systems. The passivity of both nonlinear and linear components of the flow system is studied in section IV. In addition, definition of passive systems and passivity theorems are extended from Euclidean spaces to finite-dimensional Hilbert spaces. Section V concludes the paper.

II. Model of the System

In 3-dimensional Cartesian coordinates (x,y,z), the flow of an isothermal, incompressible viscous fluid is described by the Navier-Stokes equations

$$\frac{\partial \tilde{u}}{\partial t}+\tilde{u}\frac{\partial \tilde{u}}{\partial x}+\tilde{v}\frac{\partial \tilde{u}}{\partial y}+\tilde{w}\frac{\partial \tilde{u}}{\partial z}=-\frac{\partial \tilde{p}}{\partial x}+\frac{1}{\mathrm{Re}}\nabla^2\tilde{u}+\tilde{f}_x \quad (1)$$

$$\frac{\partial \tilde{v}}{\partial t}+\tilde{u}\frac{\partial \tilde{v}}{\partial x}+\tilde{v}\frac{\partial \tilde{v}}{\partial y}+\tilde{w}\frac{\partial \tilde{v}}{\partial z}=-\frac{\partial \tilde{p}}{\partial y}+\frac{1}{\mathrm{Re}}\nabla^2\tilde{v}+\tilde{f}_y \quad (2)$$

$$\frac{\partial \tilde{w}}{\partial t}+\tilde{u}\frac{\partial \tilde{w}}{\partial x}+\tilde{v}\frac{\partial \tilde{w}}{\partial y}+\tilde{w}\frac{\partial \tilde{w}}{\partial z}=-\frac{\partial \tilde{p}}{\partial z}+\frac{1}{\mathrm{Re}}\nabla^2\tilde{w}+\tilde{f}_z \quad (3)$$

and the continuity equation

$$\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{v}}{\partial y}+\frac{\partial \tilde{w}}{\partial z}=0 \quad (4)$$

where $\tilde{u}(x,y,z,t)$, $\tilde{v}(x,y,z,t)$ and $\tilde{w}(x,y,z,t)$ are the components of the velocity of the fluid in the x, y and z directions, respectively, $\tilde{f}_x(x,y,z,t)$, $\tilde{f}_y(x,y,z,t)$ and $\tilde{f}_z(x,y,z,t)$ are the components of the normalised external force in each of these directions, $\tilde{p}(x,y,z)$ is the pressure, and

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \quad (5)$$

Re is the Reynolds number, which for channel flow can be defined as

$$\mathrm{Re}=\frac{U_{\mathrm{CL}}h}{\nu} \quad (6)$$

with U_{CL} being the velocity of the steady-state flow at the centre line, h is the half width of the channel and ν is the kinematic viscosity.

Express the total flow as a deviation from a steady-state flow, (U,V,W),

$$(\tilde{u},\tilde{v},\tilde{w})=(U+u,V+v,W+w) \quad (7)$$

the components of the force as the deviation from constant forces

$$(\tilde{f}_x,\tilde{f}_y,\tilde{f}_z)=(F_x+f_x,F_y+f_y,F_z+f_z) \quad (8)$$

and the total pressure as a deviation from the steady-state pressure, P,

$$\tilde{p}=P+p \quad (9)$$

Consider flow between two parallel plates of infinite extent that are aligned so that y is the wall normal direction and the fluid flows in the region $y\in[-1,1]$. We seek a steady-state flow of the form $(U,0,0)$, so that x is the streamwise direction and there is no flow in the z direction and there are no constant external forces, so that $F_x=F_y=F_z=0$. Then it is straightforward to obtain $U(y)=1-y^2$ which gives the velocity profile in Fig. 1.

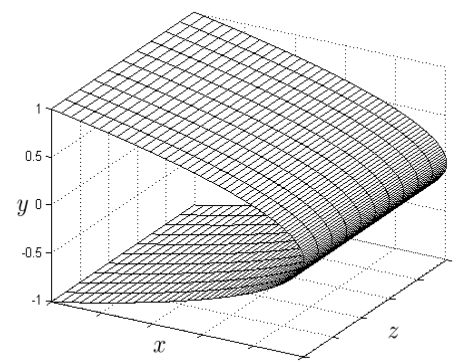

Fig. 1. Steady state velocity profile of plane Poiseuille flow

Both the steady state and the perturbed state satisfy the Navier-Stokes equations and continuity equation. Subtracting the equations for the steady state and perturbed state leads to the nonlinear disturbance equation

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+vU'+S_1=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\nabla^2 u+f_x \quad (10)$$

$$\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}+S_2=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\nabla^2 v+f_y \quad (11)$$

$$\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}+S_3=-\frac{\partial p}{\partial z}+\frac{1}{\mathrm{Re}}\nabla^2 w+f_z \quad (12)$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad (13)$$

where U' denotes $\frac{\mathrm{d}U}{\mathrm{d}y}$ and

$$S_1=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} \quad (14)$$

$$S_2=u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} \quad (15)$$

$$S_3=u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} \quad (16)$$

Non-slip boundary conditions are assumed at the walls

$$u(y=\pm1)=v(y=\pm1)=w(y=\pm1)=\frac{\partial v}{\partial y}(y=\pm1)=0 \quad (17)$$

Taking $\frac{\partial}{\partial x}$ of (10), $\frac{\partial}{\partial y}$ of (11) and $\frac{\partial}{\partial z}$ of (12) and using the continuity equation (13) gives an equation for perturbation pressure and then this resulting equation and (11) are used to eliminate p so that

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2 v - U''\frac{\partial v}{\partial x} - \frac{1}{\text{Re}}\nabla^4 v = \frac{\partial^2 S_1}{\partial y \partial x} - \left(\frac{\partial^2 S_2}{\partial x^2} + \frac{\partial^2 S_2}{\partial z^2}\right) + \frac{\partial^2 S_3}{\partial y \partial z} - \frac{\partial^2 f_x}{\partial y \partial x} + \left(\frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_y}{\partial z^2}\right) - \frac{\partial^2 f_z}{\partial y \partial z} \quad (18)$$

Define the vorticity as

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \quad (19)$$

and an expression for the evolution of the vorticity can be obtained by taking $\frac{\partial}{\partial z}$ of (10) and $\frac{\partial}{\partial x}$ of (12) and then subtracting the two resulting equations to give

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} + U'\frac{\partial v}{\partial z} - \frac{1}{\text{Re}}\nabla^2 \eta = \frac{\partial (f_x - S_1)}{\partial z} - \frac{\partial (f_z - S_2)}{\partial x} \quad (20)$$

The evolution of the flow is described in terms of v and η, so the description of the system is completed by expressions for u and w in terms of v and η. From (13) and (19), it yields

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) u = -\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial \eta}{\partial z} \quad (21)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) w = -\frac{\partial^2 v}{\partial z \partial y} - \frac{\partial \eta}{\partial x} \quad (22)$$

Equations (18) and (20)-(22) can be combined to give

$$\dot{\mathbf{x}} = \mathscr{A}\mathbf{x} + \mathscr{B}\mathbf{f} - \mathscr{B}\mathbf{S} \quad (23)$$

$$\mathbf{u} = \mathscr{C}\mathbf{x} \quad (24)$$

where the dot denotes time derivative, $\mathbf{x} = (v\ \ \eta)^{\mathrm{T}}$, $\mathbf{f} = (f_x\ f_y\ f_z)^{\mathrm{T}}$, $\mathbf{S} = (S_1\ S_2\ S_3)^{\mathrm{T}}$, $\mathbf{u} = (u\ v\ w)^{\mathrm{T}}$ and

$$\mathscr{A} = \mathscr{E}^{-1}\begin{bmatrix} -U\frac{\partial}{\partial x}\nabla^2 + U''\frac{\partial}{\partial x} + \frac{1}{\text{Re}}\nabla^4 & 0 \\ -U'\frac{\partial}{\partial z} & -U\frac{\partial}{\partial x} + \frac{1}{\text{Re}}\nabla^2 \end{bmatrix} \quad (25)$$

$$\mathscr{B} = \mathscr{E}^{-1}\begin{bmatrix} -\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} & -\frac{\partial^2}{\partial y \partial z} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \end{bmatrix} \quad (26)$$

$$\mathscr{E} = \begin{bmatrix} \nabla^2 & 0 \\ 0 & 1 \end{bmatrix} \quad (27)$$

$$\mathscr{C} = \begin{bmatrix} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{\partial^2}{\partial x \partial y} & \frac{\partial}{\partial z} \\ 1 & 0 \\ -\frac{\partial^2}{\partial z \partial y} & -\frac{\partial}{\partial x} \end{bmatrix} \quad (28)$$

The mapping from $(u\ v\ w)^{\mathrm{T}}$ to $(S_1\ S_2\ S_3)^{\mathrm{T}}$ can be viewed as a memoryless nonlinearity of the form $\mathbf{S} = \Delta(\mathbf{u})$. As shown in Fig. 2, the system can be regarded as the feedback connection between the linear time-invariant system $(\mathscr{A}, \mathscr{B}, \mathscr{C})$, and a memoryless nonlinearity $\Delta(.)$, with $(f_x\ f_y\ f_z)^{\mathrm{T}}$ acting as an external input to the system.

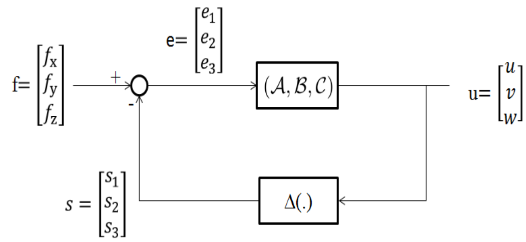

Fig. 2. Block diagram of the system

III. Semi-Discrete Formulation

Assume that the flow (together with the external force) is periodic in the x and z directions, with periods L_x and L_z respectively, then $v(x,y,z,t)$ can be expressed in the form

$$v(x,y,z,t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \tilde{v}_{mn}(y,t)\,\mathrm{e}^{\mathrm{i}\alpha_m x}\mathrm{e}^{\mathrm{i}\beta_n z} \quad (29)$$

where $\alpha_m = \frac{2\pi m}{L_x}$ and $\beta_n = \frac{2\pi n}{L_z}$ are the streamwise and spanwise wavenumbers. Similarly, $\eta(x,y,z,t)$, $u(x,y,z,t)$, $w(x,y,z,t)$, $f_x(x,y,z,t)$, $f_y(x,y,z,t)$ and $f_z(x,y,z,t)$ can also be written in the Fourier series form. It follows from (14) to (16) that the nonlinearity $S_1(x,y,z,t)$, $S_2(x,y,z,t)$ and $S_3(x,y,z,t)$ are also periodic in the x and z directions and can be expressed in terms of Fourier series.

The linear system can now be reduced to a set of decoupled systems

$$\dot{\tilde{\mathbf{x}}}_{mn} = \mathscr{A}_{mn}\tilde{\mathbf{x}}_{mn} + \mathscr{B}_{mn}\tilde{\mathbf{e}}_{mn} \quad (30)$$

$$\tilde{\mathbf{u}}_{mn} = \mathscr{C}_{mn}\tilde{\mathbf{x}}_{mn} \quad (31)$$

with states $\tilde{\mathbf{x}}_{mn} = (\tilde{v}_{mn}\ \tilde{\eta}_{mn})^{\mathrm{T}}$, inputs $\tilde{\mathbf{e}}_{mn} = (\tilde{e}_{mn}^{(1)}\ \tilde{e}_{mn}^{(2)}\ \tilde{e}_{mn}^{(3)})^{\mathrm{T}}$, outputs $\tilde{\mathbf{u}}_{mn} = (\tilde{u}_{mn}\ \tilde{v}_{mn}\ \tilde{w}_{mn})^{\mathrm{T}}$ and

$$\mathscr{A}_{mn} = \begin{bmatrix} \mathscr{D}^2 - k_{mn}^2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathscr{L}_{OS} & 0 \\ -\mathrm{i}\beta_n U' & \mathscr{L}_{SQ} \end{bmatrix} \quad (32)$$

$$\mathscr{B}_{mn} = \begin{bmatrix} \mathscr{D}^2 - k_{mn}^2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\mathrm{i}\alpha_m \mathscr{D} & -k_{mn}^2 & -\mathrm{i}\beta_n \mathscr{D} \\ \mathrm{i}\beta_n & 0 & -\mathrm{i}\alpha_m \end{bmatrix} \quad (33)$$

$$\mathscr{C}_{mn} = \begin{bmatrix} -k_{mn}^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -k_{mn}^2 \end{bmatrix}^{-1} \begin{bmatrix} -\mathrm{i}\alpha_m \mathscr{D} & \mathrm{i}\beta_n \\ 1 & 0 \\ -\mathrm{i}\beta_n \mathscr{D} & -\mathrm{i}\alpha_m \end{bmatrix} \quad (34)$$

where $k_{mn}^2 = \alpha_m^2 + \beta_n^2$, $\mathscr{D}$ denotes differentiation in the y direction, and

$$\mathscr{L}_{OS} = -\mathrm{i}\alpha_m U(\mathscr{D}^2 - k_{mn}^2) + \mathrm{i}\alpha_m U'' + \frac{1}{\text{Re}}(\mathscr{D}^2 - k_{mn}^2)^2 \quad (35)$$

$$\mathscr{L}_{SQ} = -\mathrm{i}\alpha_m U + \frac{1}{\text{Re}}(\mathscr{D}^2 - k_{mn}^2) \quad (36)$$

These expressions are not valid when $m = n = 0$. In this mode the state is defined as $\tilde{\mathbf{x}}_{00} = (\tilde{u}_{00}\ \tilde{w}_{00})^{\mathrm{T}}$ and the state space model can be obtained accordingly [5].

Each of these decoupled systems varies continuously in the x, y and z directions and as a result, has infinite dimensions. A finite dimensional approximation can be obtained by first replacing the Fourier series with finite sums by choosing $(mn) \in [-M,\dots,0,\dots,M] \times [-N,\dots,0,\dots,N]$, for sufficiently large M and N and then using a Chebyshev discretisation of $K+1$ points in the y direction, and replacing $\mathscr{D}$ by the Chebyshev spectral differentiation matrix $\mathbf{D}_K \in \mathbb{R}^{(K-1)\times(K-1)}$, on which the boundary conditions are implicitly imposed [12]. This gives a finite set of $(2M+1)(2N+1)$ decoupled state models

$$\dot{\check{\mathbf{x}}}_{mn} = \mathbf{A}_{mn}\check{\mathbf{x}}_{mn} + \mathbf{B}_{mn}\check{\mathbf{e}}_{mn} \tag{37}$$

$$\check{\mathbf{u}}_{mn} = \mathbf{C}_{mn}\check{\mathbf{x}}_{mn} \tag{38}$$

where $\check{\mathbf{x}}_{mn} \in \mathbb{C}^{2(K-1)}$, $\check{\mathbf{e}}_{mn} \in \mathbb{C}^{3(K-1)}$, $\check{\mathbf{u}}_{mn} \in \mathbb{C}^{3(K-1)}$ are vectors which stack the values of $\tilde{\mathbf{x}}_{mn}, \tilde{\mathbf{e}}_{mn}$ and $\tilde{\mathbf{u}}_{mn}$ at all the $K-1$ sample points in the y direction, $\mathbf{A}_{mn}, \mathbf{B}_{mn}$ and $\mathbf{C}_{mn}$ are discretized versions of $\mathscr{A}_{mn}, \mathscr{B}_{mn}$ and $\mathscr{C}_{mn}$, respectively

$$\mathbf{A}_{mn} = \begin{bmatrix} \mathbf{D}_K^2 - k_{mn}^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{L}_{OS} & \mathbf{0} \\ -\mathrm{i}\beta_n\mathbf{U}' & \mathbf{L}_{SQ} \end{bmatrix} \tag{39}$$

$$\mathbf{B}_{mn} = \begin{bmatrix} \mathbf{D}_K^2 - k_{mn}^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} -\mathrm{i}\alpha_m\mathbf{D}_K & -k_{mn}^2\mathbf{I} & -\mathrm{i}\beta_n\mathbf{D}_K \\ \mathrm{i}\beta_n\mathbf{I} & \mathbf{0} & -\mathrm{i}\alpha_m\mathbf{I} \end{bmatrix} \tag{40}$$

$$\mathbf{C}_{mn} = \begin{bmatrix} -k_{mn}^2\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -k_{mn}^2\mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} -\mathrm{i}\alpha_m\mathbf{D}_K & \mathrm{i}\beta_n\mathbf{I} \\ \mathbf{I} & \mathbf{0} \\ -\mathrm{i}\beta_n\mathbf{D}_K & -\mathrm{i}\alpha_m\mathbf{I} \end{bmatrix} \tag{41}$$

where $\mathbf{U} \in \mathbb{C}^{(K-1)\times(K-1)}$, $\mathbf{U}' \in \mathbb{C}^{(K-1)\times(K-1)}$ and $\mathbf{U}'' \in \mathbb{C}^{(K-1)\times(K-1)}$ are diagonal matrices formed from the values of the steady state velocity profile and its derivatives with respect to y, at each of the sample points,

$$\mathbf{L}_{OS} = -\mathrm{i}\alpha_m\mathbf{U}(\mathbf{D}_K^2 - k_{mn}^2\mathbf{I}) + \mathrm{i}\alpha_m\mathbf{U}'' + \frac{1}{\mathrm{Re}}(\mathbf{D}_K^2 - k_{mn}^2\mathbf{I})^2 \tag{42}$$

and

$$\mathbf{L}_{SQ} = -\mathrm{i}\alpha_m\mathbf{U} + \frac{1}{\mathrm{Re}}(\mathbf{D}_K^2 - k_{mn}^2\mathbf{I}) \tag{43}$$

IV. Passivity and Stability

Originating from circuit theory, passivity is a concept closely related to energy [13]. Roughly speaking, a passive system is a system which does not generate energy.

A. Passivity of the Nonlinearity

The nonlinear part of the plane Poiseuille flow does not produce or consume any perturbation energy if the kinetic energy of disturbance is defined as

$$E(t) = \frac{1}{2}\int_\Omega (u^2 + v^2 + w^2)\,\mathrm{d}\Omega \tag{44}$$

where Ω is the volume in which the fluid flow is evolving [5]. With respect to this definition, the nonlinear terms of the flow preserve energy since

$$\int_\Omega (uS_1 + vS_2 + wS_3)\,\mathrm{d}\Omega = 0 \tag{45}$$

This property can be readily proved by using integration by parts together with the divergence free condition, the periodic conditions and boundary conditions. This feature of the nonlinearity is extensively used in many energy and energy-like approaches [7], [14], [15].

Replacing all the vectors in (45) by the corresponding Fourier series, the memoryless nonlinear system $\Delta(.)$ can be said to be passive in the sense that

$$\langle \tilde{\mathfrak{u}}, \tilde{\mathfrak{s}} \rangle = \int_{-1}^{1} \tilde{\mathfrak{u}}^*\tilde{\mathfrak{s}}\mathrm{d}y = 0 \tag{46}$$

where the asterisk denotes conjugate transpose, and

$$\tilde{\mathfrak{u}} = (\dots \tilde{u}_{mn}\ \tilde{v}_{mn}\ \tilde{w}_{mn} \dots)^T \tag{47}$$

$$\tilde{\mathfrak{s}} = (\dots \tilde{S}_{mn}^{(1)}\ \tilde{S}_{mn}^{(2)}\ \tilde{S}_{mn}^{(3)} \dots)^T \tag{48}$$

B. Passivity of Linearized Flow

Since the nonlinear system is memoryless, it is relatively easy to study its passivity within the framework of infinite-dimensional theory. For dynamical systems, the situation becomes far more involved because of the requirement to show the well-posedness of the systems [16]. For this reason, we choose to reduce all the infinite-dimensional decoupled linear systems (30) - (31) to finite-dimensional ones (37) - (38) by discretization.

Given that the nonlinear part of the flow system is passive, then using ideas from finite-dimensional systems theory, it is natural to expect the closed-loop flow system without external forcing is stable when the dynamical linear system $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is strictly passive. Furthermore, the linear system is expected to be strictly passive if all the linearized decoupled systems (37) - (38) are strictly passive.

It now remains to check the passivity of these systems. The standard definition of passive dynamical systems is given in, for example, [11]. However, it does not apply to the flow system under consideration because the nonlinearity is lossless with respect to the inner product (46) rather than the standard scalar product. Consequently, there is a need to extend the definition.

Let $\mathscr{H}_1 \subset \mathbb{C}^n$ and $\mathscr{H}_2 \subset \mathbb{C}^p$ be two Hilbert spaces with inner product defined as $\langle\cdot,\cdot\rangle_{\mathscr{H}_1} : \mathscr{H}_1 \times \mathscr{H}_1 \to \mathbb{C}$ and $\langle\cdot,\cdot\rangle_{\mathscr{H}_2} : \mathscr{H}_2 \times \mathscr{H}_2 \to \mathbb{C}$, respectively. We denote a (possibly nonlinear) operator F from $\mathscr{H}_1$ to $\mathscr{H}_2$ as $F : \mathscr{H}_1 \to \mathscr{H}_2$.

Definition 1: For a linear operator $F : \mathscr{H}_1 \to \mathscr{H}_2$, the unique linear operator $F^\dagger : \mathscr{H}_2 \to \mathscr{H}_1$ satisfying

$$\langle Fx, y\rangle_{\mathscr{H}_2} = \langle x, F^\dagger y\rangle_{\mathscr{H}_1}, \quad \forall\, x \in \mathscr{H}_1, y \in \mathscr{H}_2$$

is called the adjoint of F.

In the case that $\mathscr{H}_1 = \mathscr{H}_2$, F is said to be a linear operator on the Hilbert space $\mathscr{H}_1$ or $\mathscr{H}_2$. F is called self-adjoint if $F = F^\dagger$. It is called strictly positive and denoted as $F > 0$ if it is self-adjoint and

$$\langle Fx, x\rangle_{\mathscr{H}_1} > 0, \quad \forall\, x \in \mathscr{H}_1 - \{0\}$$

Since both $\mathscr{H}_1$ and $\mathscr{H}_2$ are finite-dimensional, all linear operators are matrices. It is easy to see that positive definite matrices are special cases of strictly positive matrices.

Consider a finite-dimensional dynamical system

$$\dot{q} = f(q, v) \tag{49}$$

$$u = h(q, v) \tag{50}$$

where $f : \mathscr{H}_1 \times \mathscr{H}_2 \to \mathscr{H}_1$ is locally Lipschitz, $h : \mathscr{H}_1 \times \mathscr{H}_2 \to \mathscr{H}_2$ is continuous, $f(0,0) = 0$ and $h(0,0) = 0$.

Definition 2: The system (49)-(50) is called passive if there exists a continuously differentiable positive semidefinite storage function $V(q)$ such that

$$\Re e \langle u,v\rangle_{\mathscr{H}_2} \geq \dot{V} = \frac{\partial V}{\partial q} f(q,v), \qquad \forall\ (q,v) \in \mathscr{H}_1 \times \mathscr{H}_2 \quad (51)$$

where $\Re e(.)$ denotes the real part. The system is called strictly passive if

$$\Re e \langle u,v\rangle_{\mathscr{H}_2} \geq \dot{V} + \phi(q), \qquad \forall\ (q,v) \in \mathscr{H}_1 \times \mathscr{H}_2 \qquad (52)$$

where $\phi(q)$ is a positive definite function. In the case that the system is memoryless, the convention that $V = 0$ is adopted.

If $\mathscr{H}_2 = \mathbb{C}^p$ and the inner product is defined as the standard scalar product $\langle u,v\rangle = \sum_{i=1}^{p} u_i \bar{v}_i$, then a finite-dimensional linear time-invariant system G is passive if and only if its transfer function $G(s)$ is positive real [17]. The test for positive real transfer functions is given by the positive real lemma [18]. Moreover, the system G is strictly passive if its transfer function $G(s)$ is strictly positive real and the Kalman-Yakubovich-Popov (KYP) lemma gives a test of strictly positive real transfer functions.

If $\mathscr{H}_2$ is not a Euclidean space, the positive real lemma and KYP lemma can not be used directly because in $\mathscr{H}_2$ the adjoint of a matrix is not its conjugate transpose.

Similarly, we are able to extend the positive real lemma and KYP lemma to general finite-dimensional Hilbert spaces.

Proposition 1: Consider the finite-dimensional linear time-invariant system

$$\dot{q} = Aq + Bv \qquad (53)$$
$$u = Cq \qquad (54)$$

where $q \in \mathscr{H}_1$,$v,u \in \mathscr{H}_2$, $A : \mathscr{H}_1 \to \mathscr{H}_1$, $B : \mathscr{H}_2 \to \mathscr{H}_1$ and $C : \mathscr{H}_1 \to \mathscr{H}_2$. It is passive in the sense of Definition 2 if there exists a strictly positive $P : \mathscr{H}_1 \to \mathscr{H}_1$ such that

$$PA + (PA)^{\dagger} \leq 0 \qquad (55)$$
$$PB = C^{\dagger} \qquad (56)$$

The system is strictly passive in the sense of Definition 2 if there exist a strictly positive $P : \mathscr{H}_1 \to \mathscr{H}_1$ and a positive constant ε such that

$$PA + (PA)^{\dagger} \leq -\varepsilon P \qquad (57)$$
$$PB = C^{\dagger} \qquad (58)$$

Proof: Use $V(q) = \frac{1}{2}\langle q,Pq\rangle_{\mathscr{H}_1}$ as the storage function,

$$\begin{aligned}
&\Re e \langle v,u\rangle_{\mathscr{H}_2} - \dot{V} \\
&= \Re e \langle v,Cq\rangle_{\mathscr{H}_2} - \frac{1}{2}\frac{\mathrm{d}\langle q,Pq\rangle_{\mathscr{H}_1}}{\mathrm{d}t} \\
&= \Re e \langle v,Cq\rangle_{\mathscr{H}_2} - \frac{1}{2}(\langle \dot{q},Pq\rangle_{\mathscr{H}_1} + \langle q,P\dot{q}\rangle_{\mathscr{H}_1}) \\
&= \Re e \left(\langle q,C^{\dagger}v\rangle_{\mathscr{H}_1} - \langle q,P(Aq+Bv)\rangle_{\mathscr{H}_1}\right) \\
&= \Re e \left(\langle q,(C^{\dagger} - PB)v\rangle_{\mathscr{H}_1} - \langle q,PAq\rangle_{\mathscr{H}_1}\right) \\
&\geq \frac{1}{2}\varepsilon \langle q,Pq\rangle_{\mathscr{H}_1}
\end{aligned}$$

In the case of $\varepsilon = 0$, the system is passive, but when $\varepsilon > 0$, the system is strictly passive. ■

The extended passivity theorem (Theorem 6.4 in [11]) can be proved in a similar manner.

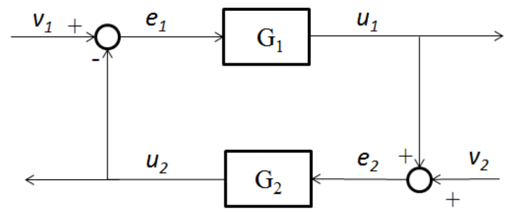

Fig. 3. Feedback connection

Proposition 2: Consider the feedback connection of Fig. 3. G_1 is a linear time-invariant system described by

$$\dot{q}_1 = Aq_1 + Be_1 \qquad (59)$$
$$u_1 = Cq_1 \qquad (60)$$

and G_2 is a memoryless system represented by

$$u_2 = G_2(e_2) \qquad (61)$$

where $q_1 \in \mathscr{H}_1$, $e_1,e_2,u_1,u_2 \in \mathscr{H}_2$, $A : \mathscr{H}_1 \to \mathscr{H}_1$, $B : \mathscr{H}_2 \to \mathscr{H}_1$ and $C : \mathscr{H}_1 \to \mathscr{H}_2$.

Suppose that the memoryless system is passive, that is

$$\Re e \langle e_2,u_2\rangle_{\mathscr{H}_2} > 0 \qquad (62)$$

then the origin of the closed-looped system (when $v = 0$) is globally uniformly asymptotically stable if the dynamical system satisfies the conditions (57) to (58).

Proof: Choose $V = V_1(q_1) = \frac{1}{2}\langle q_1,Pq_1\rangle_{\mathscr{H}_1}$ as the Lyapunov function candidate. $e_1 = -u_2$ and $e_2 = u_1$ since $v_1 = v_2 = 0$, then as shown previously,

$$\begin{aligned}
\dot{V} &= \dot{V}_1 \leq \Re e \langle e_1,u_1\rangle_{\mathscr{H}_2} - \frac{1}{2}\varepsilon \langle q,Pq\rangle_{\mathscr{H}_1} \\
&= -\Re e \langle u_2,e_2\rangle_{\mathscr{H}_2} - \frac{1}{2}\varepsilon \langle q,Pq\rangle_{\mathscr{H}_1} \\
&< -\frac{1}{2}\varepsilon \langle q,Pq\rangle_{\mathscr{H}_1}
\end{aligned}$$

Thus it is evident that V is positive definite and $\dot{V}$ is negative definite. Moreover, V is radially unbounded. Then the conclusion follows. ■

Applying this result to Definition 2, the system described by (37) - (38) is said to be strictly passive if there exists a positive semidefinite storage function $V(\check{\mathbf{x}}_{mn})$ such that

$$\Re e \langle \check{\mathbf{u}}_{mn},\check{\mathbf{e}}_{mn}\rangle \geq \dot{V} + \phi(\check{\mathbf{x}}_{mn}) \qquad (63)$$

$\forall\ (\check{\mathbf{x}}_{mn},\check{\mathbf{e}}_{mn}) \in \mathbb{C}^{2(K-1)} \times \mathbb{C}^{3(K-1)}$ where $\phi(\check{\mathbf{x}}_{mn})$ is a positive definite function and

$$\langle \check{\mathbf{u}}_{mn},\check{\mathbf{e}}_{mn}\rangle = \int_{-1}^{1} \tilde{\mathbf{u}}_{mn}^{*}\tilde{\mathbf{e}}_{mn}\mathrm{d}y \qquad (64)$$

In accordance with Proposition 1, system (37) - (38) is strictly passive if there exist a $\mathbf{P}_{mn} = \mathbf{P}_{mn}^{\dagger} > 0$ such that

$$\mathbf{P}_{mn}\mathbf{A}_{mn} + (\mathbf{P}_{mn}\mathbf{A}_{mn})^{\dagger} < 0 \qquad (65)$$
$$\mathbf{P}_{mn}\mathbf{B}_{mn} = \mathbf{C}_{mn}^{\dagger} \qquad (66)$$

With respect to the inner product $\langle \gamma, \xi \rangle = \int_{-1}^{1} \gamma^* \xi \mathrm{d}y$, it can be proved that [19] $\mathscr{D}^\dagger = -\mathscr{D}$ when the boundary condition

$$\gamma(y = \pm 1) = \xi(y = \pm 1) = 0 \tag{67}$$

is assumed, thus the adjoint of $\mathscr{C}_{mn}$ given in (34) is

$$\mathscr{C}_{mn}^\dagger = \frac{1}{k_{mn}^2} \begin{bmatrix} \mathrm{i}\alpha_m \mathscr{D} & k_{mn}^2 & \mathrm{i}\beta_n \mathscr{D} \\ \mathrm{i}\beta_n & 0 & -\mathrm{i}\alpha_m \end{bmatrix} \tag{68}$$

It follows that $\mathbf{D}_K^\dagger = -\mathbf{D}_K$ and naturally $\mathbf{C}_{mn}^\dagger$ can be obtained by discretizing $\mathscr{C}_{mn}^\dagger$ in the y direction

$$\mathbf{C}_{mn}^\dagger = \frac{1}{k_{mn}^2} \begin{bmatrix} \mathrm{i}\alpha_m \mathbf{D}_K & k_{mn}^2 \mathbf{I} & \mathrm{i}\beta_n \mathbf{D}_K \\ \mathrm{i}\beta_n \mathbf{I} & \mathbf{0} & -\mathrm{i}\alpha_m \mathbf{I} \end{bmatrix} \tag{69}$$

We therefore find that the only solution to (66) is

$$\mathbf{P}_{mn} = \frac{1}{k_{mn}^2} \begin{bmatrix} k_{mn}^2 \mathbf{I} - \mathbf{D}_K^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{70}$$

which is self-adjoint and positive definite.

Similarly, using integration by parts, it can be shown [5]

$$\mathscr{L}_{OS}^\dagger = \mathrm{i}\alpha_m U(\mathscr{D}^2 - k_{mn}^2) + 2\mathrm{i}\alpha_m U' \mathscr{D} + \frac{1}{\mathrm{Re}}(\mathscr{D}^2 - k_{mn}^2)^2 \tag{71}$$

$$\mathscr{L}_{SQ}^\dagger = \mathrm{i}\alpha_m U + \frac{1}{\mathrm{Re}}(\mathscr{D}^2 - k_{mn}^2) \tag{72}$$

discretization in the y direction yields

$$\mathbf{L}_{OS}^\dagger = \mathrm{i}\alpha_m \mathbf{U}(\mathbf{D}_K^2 - k_{mn}^2 \mathbf{I}) + 2\mathrm{i}\alpha_m \mathbf{U}' \mathbf{D}_K + \frac{1}{\mathrm{Re}}(\mathbf{D}_K^2 - k_{mn}^2 \mathbf{I})^2 \tag{73}$$

$$\mathbf{L}_{SQ}^\dagger = \mathrm{i}\alpha_m \mathbf{U} + \frac{1}{\mathrm{Re}}(\mathbf{D}_K^2 - k_{mn}^2 \mathbf{I}) \tag{74}$$

Then we have

$$\mathbf{P}_{mn}\mathbf{A}_{mn} + (\mathbf{P}_{mn}\mathbf{A}_{mn})^\dagger = \frac{1}{k_{mn}^2} \begin{bmatrix} -\mathbf{L}_{OS} - \mathbf{L}_{OS}^\dagger & \mathrm{i}\beta_n \mathbf{U}' \\ -\mathrm{i}\beta_n \mathbf{U}' & \mathbf{L}_{SQ} + \mathbf{L}_{SQ}^\dagger \end{bmatrix} \tag{75}$$

and when it is negative definite for all wavenumber pairs (m,n), the linear part of the flow system is strictly passive and moreover, the entire flow system without external forcing is globally uniformly asymptotically stable [11].

Numerical computation shows that $\mathbf{P}_{mn}\mathbf{A}_{mn} + (\mathbf{P}_{mn}\mathbf{A}_{mn})^\dagger < 0$ for all wavenumber pairs (m,n) when the Reynolds number does not exceed $\mathrm{Re} = 49.6$, which is the same as the energy Reynolds number derived by the classical energy approach [7]. The lowest subcritical Reynolds number is obtained at $\alpha = 0, \beta \approx 2.05$. So far the plane Poiseuille flow has been considered is 3-dimensional. The result for 2-dimensional flow can also be obtained by using the same approach. Our result shows that in the case of 2-dimensional plane Poiseuille flow, the subscritical Reynolds number is $\mathrm{Re} = 87.7$, which is equal to the energy Reynolds number obtained by Orr [20]. In addition, the passivity approach can be readily applied to plane Couette flow. Our results agree with the energy Reynolds numbers derived by energy approach for both 2-dimensional and 3-dimensional cases.

V. Concluding Remarks

In this paper, we have shown that the passivity approach can be used to study the stability of incompressible flow systems. The Reynolds numbers below which the plane Poiseuille flow is guaranteed to be stable are $\mathrm{Re} = 49.6$ and $\mathrm{Re} = 87.7$ for 3-dimensional and 2-dimensional cases, respectively. The equivalence of passivity approach and energy approach for plane Poiseuille flow is largely due to the fact that the nonlinearity does not consume or create any perturbation energy. In the process of finding these subcritical Reynolds numbers, we have extended the definition of passive system and KYP lemma, as well as the passivity theorem, from finite-dimensional Euclidean spaces to finite-dimensional Hilbert spaces. The extension to the infinite-dimensional Hilbert spaces is significantly more complicated and is beyond the scope of this paper.

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