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High-order Zames-Falb multiplier analysis using linear matrix inequalities

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Abstract— **This paper proposes an algorithm for stability analysis of systems containing slope-restricted nonlinearities using high-order Zames-Falb multipliers. The main innovation in this paper is the use of a new congruence transformation which enables multipliers of twice the order of the linear part of the system to be used in a linear-matrix-inequality (LMI) framework for stability analysis. Although the use of such high-order multipliers increases computational requirements, various numerical examples show that the resulting stability bounds are sometimes less conservative than using other similar approaches.**

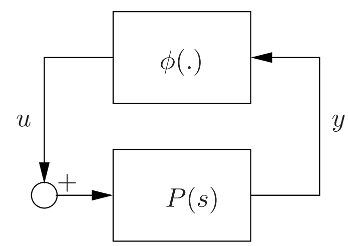

Fig. 1. System under consideration

I. INTRODUCTION

Many control problems can be interpreted as guaranteeing the stability, in some appropriate sense, of the system in Figure 1, where $P(s)$ is a finite dimensional linear time invariant (FDLTI) system. Depending on the information available about the nonlinear operator $\phi(.) : \mathcal{L}_{2e} \mapsto \mathcal{L}_{2e}$, various results can be applied to ascertain the stability of the closed-loop. When $\phi(.)$ is static and sector bounded, the Circle or Popov Criteria are popular choices for analysis. When $\phi(.)$ is static and *slope-restricted* a wider variety of techniques may be used for stability analysis; a selection of these may be found in [25], [9], [6], [21], [18], [16], [8].

Of all of these results, it appears that the most general and powerful is still that proposed by Zames and Falb [25] in the late 1960s. Indeed, it was recently proved [2] that various other stability results for slope-restricted nonlinearities can be interpreted in the framework of [25]. Zames and Falb characterised a class of multipliers, $\mathcal{M}_{ZF}$, such that every multiplier $M(s) \in \mathcal{M}_{ZF}$ preserved positivity of the inner product $\langle M, \phi \rangle \geq 0$. They further proved that, the stability of Figure 1 could be affirmed if, in the causal case, the transfer function $M(s)P(s)$ was passive (see [8]). Despite the appeal of Zames-Falb's multiplier approach, its use is not commonplace in either industry or academia.

The main problem with the results of [25] is that the search for *an appropriate* multiplier $M(s) \in \mathcal{M}_{ZF}$ is, in general, not trivial and may be viewed as an infinite-dimensional optimisation problem. Although developments in IQC analysis [7], [15], [1] and extensions to the multivariable case [19], [11], [12] have seen interest in Zames-Falb multipliers rekindled, none of these references addresses the search over Zames-Falb (or Zames-Falb-like) multipliers explicitly, although a useful "brute-force" approach is available in [15]. In a number of recent papers [23], [5], [4], the search for Zames-Falb multipliers has begun to be addressed and several papers have advocated the use of LMI-based methods in this search. In [23] an LMI-based method which searched over *causal, plant-order* Zames-Falb multipliers was introduced, and developed further in [22], [4], [3]. Examples given in these papers indicate how the proposed LMI-based searches were relatively efficient and in some cases able to return tight results.

Unfortunately, the restriction of the Zames-Falb multipliers to be of *plant order* may lead to results which are potentially conservative, meaning that the LMI-based results of [23], [22], [4], [3] may fail to establish stability in cases where stability actually holds. It is possible to construct examples where one may extend the search space to higher order multipliers in order to obtain less conservative results, but it then becomes more difficult to automate this search. The purpose of this paper is to show how the restriction of the search to *plant-order* Zames-Falb multipliers can be removed and how a high-order Zames-Falb stability analysis can be conducted using LMI's and a line search in a similar way to that proposed in [23], [22], [4], [3]. The novelty in this paper is a new congruence transformation, which can be viewed as an extended form of that proposed in [20], and which is used to cast the Zames-Falb multiplier stability analysis problem in the aforementioned form. This allows one to search efficiently for an appropriate Zames-Falb multiplier which has twice the order of the plant. It will be shown later that the increase in multiplier order allows less conservative results to be obtained.

The paper is structured as follows. The next section reviews some basics of the IQC approach ([16]) to Zames-Falb stability analysis which was developed in earlier work [23]. Following this, the structure of the high-order Zames-Falb multiplier is introduced and sufficient conditions for the stability of Figure 1 are derived. Similar to [22], [4], it

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is then shown how a Popov multiplier may be added to the high-order Zames-Falb multiplier in order to improve the results further. Numerical results then show the reduction in conservatism that is possible with high-order Zames-Falb multipliers and conclusions follow.

Notation. Notation is standard throughout. The $\mathcal{L}_2$ norm of a function $x(t)$ is defined as $\|x\|_2 := \sqrt{\int_0^\infty \|x(t)\|^2 dt}$ where $\|x\|$ denotes a vector's Euclidean norm. The space of functions with finite $\mathcal{L}_2$ norm is denoted $\mathcal{L}_2$ and the extended space $\mathcal{L}_{2e}$. Similarly, the $\mathcal{L}_1$ norm of a function $\delta(t)$ is defined $\|\delta\|_1 := \int_0^\infty \|\delta(t)\|dt$ and the space where this norm is finite is denoted $\mathcal{L}_1$. With some abuse of notation we say that a transfer function $H(s) \in \mathcal{L}_1$ if the impulse response of its time-domain linear operator, $h(t)$, is in $\mathcal{L}_1$. The space of real rational transfer function matrices, bounded on the imaginary axis is denoted by $\mathcal{RL}_\infty$; the subspace which is analytically continuous in the right half complex plane is denoted $\mathcal{RH}_\infty$. The order (or MacMillan degree) of a transfer function matrix $M(s)$ is denoted $\text{ord}[M(s)]$. An operator H is described as bounded if $\|H(u)\| \leq \gamma\|u\|$ for all $u \in \mathcal{L}_{2\text{e}}$ and some $\gamma > 0$. A function $\phi(.) : \mathbb{R} \mapsto \mathbb{R}$ is said to have a slope restriction [1] $[0, \alpha]$ if

$$0 \leq \frac{\phi(x) - \phi(y)}{x - y} \leq \alpha \quad \forall x, y \in \mathbb{R}, \quad \alpha > 0$$

We use the shorthand notation $\phi \in \partial[0, \alpha]$ to indicate that a function has this property. Similarly, a function $\phi(.)$ is said to have a sector bound $[0, \beta]$ if

$$0 \leq \phi(x)x \leq \beta x^2 \quad \forall x \in \mathbb{R}, \quad \beta > 0$$

If this property holds, we write $\phi \in \text{Sector}[0, \beta]$. If $\phi(0) = 0$ the slope restriction implies sector-boundedness with $\alpha = \beta$ (see [17], [22]) but the converse is not true.

II. IQC analysis with Zames-Falb multipliers

Consider again Figure 1 in which $P(s)$ is the FDLTI part of the system with state-space realisation

$$P(s) \sim \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array} \right] \qquad (1)$$

where $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times 1}$, $C_p \in \mathbb{R}^{1 \times n_p}$, $D_p \in \mathbb{R}$. In this paper we assume that $\phi(.) : \mathbb{R} \mapsto \mathbb{R}$ is a static nonlinearity satisfying the following assumption.

Assumption 1: $\phi(.) : \mathbb{R} \mapsto \mathbb{R}$ satisfies the following properties:

i) It is bounded, odd and $\phi(0) = 0$
ii) It has slope restriction $\partial\phi \in [0, \alpha]$

A $\phi(.)$ satisfying these properties is said to belong to $\mathcal{N}_\alpha^S$

Without loss of generality, the lower gradient of the slope is assumed to be zero; if this is not the case, loop-shifting can be used to pose an equivalent problem where the "loop-shifted" nonlinearity, $\tilde{\phi}$, is such that $\partial\tilde{\phi} \in [0, \tilde{\alpha}]$. If $\phi(.) \in \mathcal{N}_\alpha^S$, it satisfies the IQC ([15], [16], [7]) defined by

$$\int_{-\infty}^{\infty} \left[\begin{array}{c} \hat{y}(j\omega) \\ \hat{u}(j\omega) \end{array} \right]^\star \Pi(j\omega) \left[\begin{array}{c} \hat{y}(j\omega) \\ \hat{u}(j\omega) \end{array} \right] d\omega \geq 0 \qquad (2)$$

[1]By this, we mean that the sub-differential of $\phi(x)$ belongs to the interval $[0, \alpha]$ for all $x \in \mathbb{R}$

where $\hat{u}(j\omega)$ and $\hat{y}(j\omega)$ are the Fourier Transforms of $u(t)$ and $y(t)$ respectively, and $\Pi(j\omega)$ is given by

$$\Pi(j\omega) = \left[\begin{array}{cc} 0 & \alpha M^\star(j\omega) \\ \alpha M(j\omega) & -M^\star(j\omega) - M(j\omega) \end{array} \right] \qquad (3)$$

The transfer function $M(s)$ is the so-called "Zames-Falb multiplier" ([25]); this class of multipliers is defined below.

Definition 1: A transfer function $M(s) := H_0 - H(s) \in \mathcal{RL}_\infty$ is said to belong to the set $\mathcal{M}_{ZF}$ if $H_0 > 0$ and $H(s) \in \mathcal{L}_1$ is such that $\|H(s)\|_1 \leq H_0$.

When $M(s) \in \mathcal{M}_{ZF}$, the IQC (2)-(3) captures the largest class of "multipliers" for $\phi(.) \in \mathcal{N}_\alpha^S$. For the more general case when $\phi(.)$ is vector valued, the reader is referred to [7], [13], but for our work $M(s) \in \mathcal{M}_{ZF}$ will be sufficient. The basic stability result (stated in an IQC context) for the system in Figure 1 can therefore be stated by re-writing the results in [16], [7] as the following theorem.

Theorem 1: Consider Figure 1 where $P(s) \in \mathcal{RH}_\infty$ and $\phi \in \mathcal{N}_S^\alpha$ satisfies the IQC defined by (2) and (3) where $M(s) \in \mathcal{M}_{ZF}$. Assume that the closed loop system is well-posed. Then the system is asymptotically stable if

$$\left[\begin{array}{c} P(j\omega) \\ I \end{array} \right]^\star \Pi(j\omega) \left[\begin{array}{c} P(j\omega) \\ I \end{array} \right] < 0 \quad \forall \omega \in \mathbb{R} \qquad (4)$$

Thus stability of the system essentially reduces to finding suitable $H_0 > 0$ and $H(s) \in \mathcal{L}_1$ such that inequality (4) holds.

Remark 1: In general, this result is quite difficult to apply because it requires one to determine $M(s)$ of a very general structure: the order can be arbitrarily high and $M(s)$ can be non-causal. In [15], [7], [14] the approach advocated has been to fix a structure of $M(s) \in \mathcal{M}_{ZF}$ and then to check inequality (4); no real guidance on how to choose $M(s)$ was given and [7] essentially proposes the use of simple *ad hoc* structures of $M(s)$ followed by stability checks; although useful for some systems, this approach is rather unsatisfactory in general. In [23], [22], it was shown how, by restricting $M(s)$ to be causal and of order equal to that of $P(s)$, it was possible to choose $M(s)$ more systematically using LMIs and a line search; [4] then extended this search to anti-causal multipliers. □

III. High-order Zames-Falb multiplier analysis

The aim of this section is to provide a tractable way of determining whether the system in Figure 1 is stable when $\phi \in \mathcal{N}_\alpha^S$ but instead of restricting the order of the multiplier to be $\text{ord}[M(s)] = n_p$ (as in [23], [3]) the order is allowed to be such that $\text{ord}[M(s)] = 2n_p$.

A. Preliminary results

The preliminary results required are summarised below.

Lemma 1 ([20]): Consider a transfer function $H(s) \sim (A_H, B_H, C_H, D_H) \in \mathcal{RH}_\infty$. Then $\|H\|_1 \leq \xi$ if there exists a positive definite matrix Y and scalars $\mu > 0$ and $\lambda > 0$

$$\begin{bmatrix} \mathbf{S_{11}}A_p+A_p'\mathbf{S_{11}} & \mathbf{S_{11}}A_p+A_p'\mathbf{P_{11}}-\alpha C_p'(\mathbf{B_{H1}}+\mathbf{B_{H2}})'+\mathbf{A_{H1}}'-\mathbf{A_{H2}}' & A_p'\bar{\mathbf{N}}-\alpha C_p'\mathbf{B_{H2}}'-\mathbf{A_{H2}}' & \mathbf{S_{11}}B_p+\alpha C_p'(I-\mathbf{D_H})'+\mathbf{C_{H1}}'-\mathbf{C_{H2}}' \\ \star & A_p'\mathbf{P_{11}}+\mathbf{P_{11}}A_p-\alpha[(\mathbf{B_{H1}}+\mathbf{B_{H2}})C_p-C_p'(\mathbf{B_{H1}}+\mathbf{B_{H2}})'] & A_p'\bar{\mathbf{N}}-\alpha C_p'\mathbf{B_{H2}}'+\mathbf{A_{H2}}' & \mathbf{P_{11}}B_p+\alpha C_p'(I-\mathbf{D_H})'+(\mathbf{B_{H1}}+\mathbf{B_{H2}})X \\ \star & \star & \mathbf{A_{H2}}+\mathbf{A_{H2}}' & \bar{\mathbf{N}}B_p+\mathbf{B_{H2}}X+\mathbf{C_{H2}}' \\ \star & \star & \star & -(I-\mathbf{D_H})X-X'(I-\mathbf{D_H})' \end{bmatrix} < 0 \quad (13)$$

$$\begin{bmatrix} -\mathbf{A_{H1}}-\mathbf{A_{H1}}'+\lambda(\mathbf{P_{11}}-\mathbf{S_{11}}-\bar{\mathbf{N}}) & 0 & \mathbf{B_{H1}} \\ \star & \mathbf{A_{H2}}+\mathbf{A}'_{\mathbf{H2}}+\lambda\bar{\mathbf{N}} & \mathbf{B_{H2}} \\ \star & \star & -\mu \end{bmatrix} < 0 \quad (14)$$

$$\begin{bmatrix} \lambda(\mathbf{P_{11}}-\mathbf{S_{11}}-\bar{\mathbf{N}}) & 0 & 0 & \mathbf{C_{H1}}' \\ \star & \lambda\bar{\mathbf{N}} & 0 & \mathbf{C_{H2}}' \\ \star & \star & 1-\mu & \mathbf{D_H} \\ \star & \star & \star & 1 \end{bmatrix} > 0 \quad (15)$$

such that the following matrix inequalities hold:

$$\begin{bmatrix} A_H'Y+YA_H+\lambda Y & YB_H \\ \star & -\mu I \end{bmatrix} < 0 \quad (5)$$

$$\begin{bmatrix} \lambda Y & 0 & C_H' \\ \star & (\xi-\mu)I & D_H' \\ \star & \star & \xi \end{bmatrix} \geq 0 \quad (6)$$

The following result from [23] is an application of the KYP Lemma which allows the frequency domain IQC in Theorem 1 to be interpreted as a matrix inequality.

Proposition 1 ([23]): The system depicted in Figure 1 is globally asymptotically stable if there exists a real symmetric matrix $P = P'$ and a transfer function

$$H(s) \sim \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right] \quad (7)$$

where $\|H(s)\|_1 \leq 1$, such that the following matrix inequality is satisfied.

$$\begin{bmatrix} A_I'P+PA_I & PB_I+C_I' \\ \star & -D_I-D_I' \end{bmatrix} < 0 \quad (8)$$

where the matrices A_I, B_I, C_I, D_I are given by

$$\left[\begin{array}{c|c} A_I & B_I \\ \hline C_I & D_I \end{array}\right] = \left[\begin{array}{cc|c} A_p & 0 & B_p \\ -B_H\alpha C_p & A_H & B_H(I-\alpha D_p) \\ \hline \alpha C_p - D_H\alpha C_p & C_H & (I-D_H)(I-\alpha D_p) \end{array}\right]$$

B. Main result

Proposition 1 states that the system in Figure 1 is stable if there exists a transfer function $H(s)$ with state-space realisation (7) such that $\|H(s)\|_1 \leq 1$ and such that inequality (8) is satisfied. In previous papers it has been assumed that $\text{ord}[M(s)] = \text{ord}[H(s)] = n_p$. This allowed the use of an adapted version of Scherer's change of variables ([20]) to transform the inequality (8) into LMI form. In this paper, the goal is to transform Proposition 1 into a more tractable form but under the assumption that $\text{ord}[M(s)] = \text{ord}[H(s)] = 2n_p$, which requires a change of variables somewhat different to that proposed in [20]. Without loss of generality, the following state-space realisation of $H(s)$ is assumed:

$$\left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right] = \left[\begin{array}{cc|c} A_{H1} & 0 & B_{H1} \\ 0 & A_{H2} & B_{H2} \\ \hline C_{H1} & C_{H2} & D_H \end{array}\right] \quad (9)$$

where $A_{H1}, A_{H2} \in \mathbb{R}^{n_p \times n_p}$, $B_{H1}, B_{H2} \in \mathbb{R}^{n_p \times 1}$, $C_{H1}, C_{H2} \in \mathbb{R}^{1 \times n_p}$ and $D_H \in \mathbb{R}$. Using this, the matrices A_I, B_I, C_I and D_I become:

$$A_I = \begin{bmatrix} A_p & 0 & 0 \\ \alpha B_{H1}C_p & A_{H1} & 0 \\ \alpha B_{H2}C_p & 0 & A_{H2} \end{bmatrix} \quad (10)$$

$$B_I = \begin{bmatrix} B_p \\ B_{H1}(I-\alpha D_p) \\ B_{H2}(I-\alpha D_p) \end{bmatrix} \quad C_I' = \begin{bmatrix} C_p'(I-\alpha D_p)' \\ C_{H1}' \\ C_{H2}' \end{bmatrix}' \quad (11)$$

$$D_I = (I-D_H)(I-\alpha D_p) \quad (12)$$

Stability will then be ensured providing inequality (8) holds and providing that $\|H\|_1 \leq 1$. The following is the main result of the paper.

Proposition 2: Assume $P(s) \in \mathcal{RH}_\infty$ and that $\phi(.) \in \mathcal{N}_\alpha^{\mathrm{S}}$. Then the system in Figure 1 is globally asymptotically stable if there exist positive definite matrices $\mathbf{S_{11}}, \mathbf{P_{11}}$ and $\bar{\mathbf{N}}$, positive scalars λ, μ and $\mathbf{D_H}$, and unstructured matrices $\mathbf{A_{H1}}$, $\mathbf{B_{H1}}$, $\mathbf{C_{H1}}$, $\mathbf{A_{H2}}$, $\mathbf{B_{H2}}$ and $\mathbf{C_{H2}}$ such that inequalities (13), (14) and (15) [top of page] hold, where $X := I - \alpha D_p$.

Remark 2: For fixed $\alpha > 0$ and $\lambda > 0$ the conditions (13)-(15) form a system of linear matrix inequalities. Thus for a given slope restriction, the search for $M(s)$ of twice the order of the nominal plant can be solved as an LMI-problem plus a line search over λ. □

Proof of Proposition 2: The proof of Proposition 2 requires the translation of inequality (8) and the $\mathcal{L}_1$ constraint on $H(s)$ in Proposition 1, into the matrix inequalities (13), (14) and (15). There are three distinct parts to the proof which broadly parallel the proof in [23]. Due to space constraints, much of the detail is left to the reader.

Part 1: Main inequality. Consider inequality (8) where the state-space matrices (A_I, B_I, C_I, D_I) are given in equations (10)-(12). Because $\text{ord}[H(s)] = 2n_p$, it follows that the nonsingular matrix $P \in \mathbb{R}^{3n_p \times 3n_p}$. Under the additional assumption that $P > 0$, let P and $Q := P^{-1}$ be partitioned into $n_p \times n_p$ sub-matrices:

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q'_{12} & Q_{22} & Q_{23} \\ Q'_{13} & Q'_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} \mathbf{P_{11}} & P_{12} & P_{13} \\ P'_{12} & P_{22} & P_{23} \\ P'_{13} & P_{23} & P_{33} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{16}$$

where, P_{12}, P_{13}, Q_{12} and Q_{13} are full rank matrices. It is assumed, without loss of generality that $P_{33} := P'_{13} N P_{13}$, where $N > 0$, and also (possibly with some loss of generality) that $P_{23} = 0$. Thus, introducing the matrices

$$\Pi_1 := \begin{bmatrix} Q_{11} & I & 0 \\ Q'_{12} & 0 & 0 \\ Q'_{13} & 0 & I \end{bmatrix} \quad \Pi_2 := \begin{bmatrix} I & 0 & 0 \\ \mathbf{P_{11}} & P_{12} & P_{13} \\ P'_{13} & 0 & P_{33} \end{bmatrix} \tag{17}$$

it follows that $\Pi'_1 P = \Pi_2$. Thus, applying, the congruence transformation

$$\mathrm{diag}(\Pi'_1, I) \tag{18}$$

to inequality (8) and carrying out algebra similar to [23], yields the matrix inequality (13), where the following matrix variables have been defined:

$$\mathbf{A_{H1}} := P_{12} A_{H1} Q'_{12} \mathbf{S_{11}} \tag{19}$$
$$\mathbf{B_{H1}} := P_{12} B_{H1} \tag{20}$$
$$\mathbf{C_{H1}} := C_{H1} Q'_{12} \mathbf{S_{11}} \tag{21}$$
$$\mathbf{A_{H2}} := -P_{13} A_{H2} Q'_{13} \mathbf{S_{11}} \tag{22}$$
$$\mathbf{B_{H2}} := P_{13} B_{H2} \tag{23}$$
$$\mathbf{C_{H2}} := C_{H2} P_{13}^{-1'} \bar{\mathbf{N}} \tag{24}$$
$$\mathbf{D_H} := D_H \tag{25}$$

and $\mathbf{S_{11}} := Q_{11}^{-1}$ and $\bar{\mathbf{N}} := N^{-1}$. In the above derivation it is useful to note, using the identity (16), that

$$Q_{13} = -Q_{11} P_{13} P_{33}^{-1} \tag{26}$$
$$\mathbf{A_{H2}} = P_{13} A_{H2} P_{13}^{-1} N^{-1} \tag{27}$$

Part 2: The $\mathcal{L}_1$ conditions. From Lemma 1, a sufficient condition for $\|H\|_1 \leq 1$ is for there to exist a positive definite matrix $Y \in \mathbb{R}^{2n_p \times 2n_p}$, and scalars $\lambda > 0, \mu > 0$ such that matrix inequalities (5)-(6) are satisfied with $\xi = 1$. Similar to [23], the objective is to convert inequalities (5)-(6) to inequalities (14)-(15) but using the multiplier (9) and the higher-order P-matrix given in equation (16). Choosing

$$Y = \begin{bmatrix} P_{22} & 0 \\ 0 & P_{33} \end{bmatrix} \in \mathbb{R}^{2n_p \times 2n_p} \tag{28}$$

inequality (5) becomes

$$\begin{bmatrix} A'_{H1} P_{22} + P_{22} A_{H1} + \lambda P_{22} & 0 & P_{22} B_{H1} \\ \star & A'_{H2} P_{33} + P_{33} A_{H2} + \lambda P_{33} & P_{33} B_{H2} \\ \star & \star & -\mu \end{bmatrix} < 0 \tag{29}$$

Applying the congruence transformation

$$\mathrm{diag}(Q_{11}^{-1} Q_{12}, (N P_{13})^{-1}, I)$$

to the above inequality leads, after some algebra, to inequality (14). In the derivation, equation (26) has been used along with the equality below which follows from the identity (16).

$$P_{12} Q'_{12} = I - P_{11} Q_{11} - P'_{13} Q'_{13}$$

The second $\mathcal{L}_1$ inequality is derived in a similar way to the first: in inequality (6), Y is given the structure (28) and then the congruence transformation below is applied:

$$\mathrm{diag}(Q_{12}, (N P_{13})^{-1}, I, I)$$

After some algebra, inequality (15) follows.

Part 3: Positive Definiteness of P. In Part 1 of the proof it was assumed that $P > 0$ and indeed this was necessary for the subsequent development of the results. Here, we show that satisfaction of the LMI's guarantees that this is the case. Note that $P > 0$ is equivalent to $\Pi'_1 P \Pi_1 = \Pi_2 \Pi_1 > 0$, which can be written as

$$\Pi_2 \Pi_1 = \begin{bmatrix} Q_{11} & I & 0 \\ I & \mathbf{P_{11}} & P_{13} \\ 0 & P'_{13} & P'_{13} N P_{13} \end{bmatrix} > 0 \tag{30}$$

By congruence, this is equivalent to

$$\begin{bmatrix} Q_{11} & I & 0 \\ I & \mathbf{P_{11}} & I \\ 0 & I & N \end{bmatrix} > 0 \tag{31}$$

By the Schur Complement this is equivalent to

$$\begin{bmatrix} Q_{11} & I \\ I & \mathbf{P_{11}} - \bar{\mathbf{N}} \end{bmatrix} > 0 \tag{32}$$

$$\Leftrightarrow \quad \mathbf{P_{11}} - \mathbf{S_{11}} - \bar{\mathbf{N}} > 0 \tag{33}$$

which is implied by inequality (15). □□

C. Multiplier reconstruction

The high-order multiplier may be reconstructed in a similar way to that proposed in [3]. Note that the state-space matrices of the multiplier are recovered from the LMI variables through the equations

$$\begin{bmatrix} \mathbf{A_{H1}} & 0 \\ 0 & \mathbf{A_{H2}} \end{bmatrix} = \begin{bmatrix} P_{12} & 0 \\ 0 & P_{13} \end{bmatrix} \begin{bmatrix} A_{H1} & 0 \\ 0 & A_{H2} \end{bmatrix} \begin{bmatrix} Q'_{12}\mathbf{S_{11}} & 0 \\ 0 & P_{13}^{-1} N^{-1} \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} \mathbf{B_{H1}} \\ \mathbf{B_{H2}} \end{bmatrix} = \begin{bmatrix} P_{12} & 0 \\ 0 & P_{13} \end{bmatrix} \begin{bmatrix} B_{H1} \\ B_{H2} \end{bmatrix} \tag{35}$$

$$\begin{bmatrix} \mathbf{C_{H1}} & \mathbf{C_{H2}} \end{bmatrix} = \begin{bmatrix} C_{H1} & C_{H2} \end{bmatrix} \begin{bmatrix} Q'_{12}\mathbf{S_{11}} & 0 \\ 0 & P_{13}^{-1} N^{-1} \end{bmatrix} \tag{36}$$

which imply

$$\begin{bmatrix} A_{H1} & 0 \\ 0 & A_{H2} \end{bmatrix} = \begin{bmatrix} P_{12} & 0 \\ 0 & P_{13} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A_{H1}} & 0 \\ 0 & \mathbf{A_{H2}} \end{bmatrix} \begin{bmatrix} Q'_{12}\mathbf{S_{11}} & 0 \\ 0 & P_{13}^{-1} N^{-1} \end{bmatrix}^{-1} \tag{37}$$

$$\begin{bmatrix} B_{H1} \\ B_{H2} \end{bmatrix} = \begin{bmatrix} P_{12} & 0 \\ 0 & P_{13} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B_{H1}} \\ \mathbf{B_{H2}} \end{bmatrix} \tag{38}$$

$$\begin{bmatrix} C_{H1} & C_{H2} \end{bmatrix} = \begin{bmatrix} \mathbf{C_{H1}} & \mathbf{C_{H2}} \end{bmatrix} \begin{bmatrix} Q'_{12}\mathbf{S_{11}} & 0 \\ 0 & P_{13}^{-1} N^{-1} \end{bmatrix}^{-1} \tag{39}$$

The above expressions imply that the matrices P_{12} and P_{13} must be found in order for the multiplier $M(s)$ to be reconstructed. While it is guaranteed that P_{12} and P_{13} exist, their computation can be avoided, in a similar way to

$$\begin{bmatrix} \mathbf{S_{11}}A_p+A_p'\mathbf{S_{11}} & \mathbf{S_{11}}A_p+A_p'\mathbf{P_{11}}-\alpha C_p'(\mathbf{B_{H1}}+\mathbf{B_{H2}})'+\mathbf{A_{H1}}'-\mathbf{A_{H2}}' & A_p'\bar{\mathbf{N}}-\alpha C_p'\mathbf{B_{H2}}'-\mathbf{A_{H2}}' & \mathbf{S_{11}}B_p+\alpha C_p'(I-\mathbf{D_H})'+\mathbf{C_{H1}}'-\mathbf{C_{H2}}' \\ \star & A_p'\mathbf{P_{11}}+\mathbf{P_{11}}A_p-\alpha[(\mathbf{B_{H1}}+\mathbf{B_{H2}})C_p-C_p'(\mathbf{B_{H1}}+\mathbf{B_{H2}})'] & A_p'\bar{\mathbf{N}}-\alpha C_p'\mathbf{B_{H2}}'+\mathbf{A_{H2}}' & \mathbf{P_{11}}B_p+\alpha C_p'(I-\mathbf{D_H})'+(\mathbf{B_{H1}}+\mathbf{B_{H2}})X \\ \star & \star & \mathbf{A_{H2}}+\mathbf{A_{H2}}' & \bar{\mathbf{N}}B_p+\mathbf{B_{H2}}X+\mathbf{C_{H2}}' \\ \star & \star & \star & -(I-\mathbf{D_H})X-X'(I-\mathbf{D_H})' \end{bmatrix}$$
$$+\begin{bmatrix} 0 & 0 & 0 & (\alpha\nu I+\eta A_p')C_p' \\ \star & 0 & 0 & (\alpha\nu I+\eta A_p')C_p' \\ \star & \star & 0 & 0 \\ \star & \star & \star & \eta C_pB_p+\eta B_p'C_p' \end{bmatrix} < 0 \quad (49)$$

that proposed in [3], by considering a different set of coordinates. Thus, observe that the identity (16) implies:

$$Q_{11}\mathbf{P_{11}}+Q_{12}'P_{12}'+Q_{13}'P_{13}=0 \quad (40)$$

$$Q_{11}P_{12}+Q_{12}P_{22}=0 \quad (41)$$

Choosing, $P_{22}=I$, and noting that $\bar{\mathbf{N}}=-\mathbf{S_{11}}Q_{13}P_{13}'$, we have, from the above,

$$P_{12}P_{12}'=\mathbf{P_{11}}-\mathbf{S_{11}}-\bar{\mathbf{N}} \quad (42)$$

Thus, applying the similarity transformation

$$T=\begin{bmatrix} -P_{12}' & 0 \\ 0 & P_{13}^{-1}N^{-1} \end{bmatrix}^{-1} \quad (43)$$

to the multiplier realisation equations (37)-(39), it follows that an alternative state-space realisation of the multiplier is given by

$$\tilde{A}_H=\begin{bmatrix} \tilde{A}_{H1} & 0 \\ 0 & \tilde{A}_{H2} \end{bmatrix}=\begin{bmatrix} -(\mathbf{P_{11}}-\mathbf{S_{11}}-\bar{\mathbf{N}})^{-1} & 0 \\ 0 & \bar{\mathbf{N}}^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{A_{H1}} & 0 \\ 0 & \mathbf{A_{H2}} \end{bmatrix} \quad (44)$$

$$\tilde{B}_H=\begin{bmatrix} \tilde{B}_{H1} \\ \tilde{B}_{H2} \end{bmatrix}=\begin{bmatrix} -(\mathbf{P_{11}}-\mathbf{S_{11}}-\bar{\mathbf{N}})^{-1} & 0 \\ 0 & \bar{\mathbf{N}}^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{B_{H1}} \\ \mathbf{B_{H2}} \end{bmatrix} \quad (45)$$

$$\tilde{C}_H=\begin{bmatrix} \tilde{C}_{H1} & \tilde{C}_{H2} \end{bmatrix}=\begin{bmatrix} \mathbf{C_{H1}} & \mathbf{C_{H2}} \end{bmatrix} \quad (46)$$

Hence the multiplier is given by $M(s)=H(s)-1$ with

$$H(s)\sim(\tilde{A}_H,\tilde{B}_H,\tilde{C}_H,\mathbf{D_H}) \quad (47)$$

IV. Adding a Popov multiplier

It was noted earlier in the paper that, when $\phi(0)=0$, any slope-restricted nonlinearity is also sector bounded, viz:

$$\phi\in\mathcal{N}_\alpha^S \Rightarrow \phi\in\text{Sector}[0,\alpha]$$

In this case it is possible to augment the Zames-Falb multiplier with a Popov multiplier in order to reduce the conservatism of the results [22], [24] further. Although it has recently been proved [2] that stability analysis using Popov multipliers is equivalent to stability analysis using a special case of Zames-Falb multipliers, it appears there is still some numerical advantage by conducting a stability analysis where each of the multipliers appears distinctly. Thus, with a Popov multiplier, the IQC satisfied by $\phi(.)$ becomes that given in equation (3) but with $\Pi(j\omega)$ given by [10], [22]:

$$\Pi(j\omega)=\begin{bmatrix} 0 & (\nu\alpha+\eta j\omega+\alpha M(j\omega))^* \\ (\nu\alpha+\eta j\omega+\alpha M(j\omega)) & -2\nu-M^*(j\omega)-M(j\omega) \end{bmatrix},$$
$$\nu\in\mathbb{R}_+,\quad \eta\in\mathbb{R},\quad M(s)\in\mathcal{M}_{ZF} \quad (48)$$

This leads to the following result.

Proposition 3: Assume $P(s)\in\mathcal{RH}_\infty$ and that $\phi(.)\in\mathcal{N}_\alpha^{\text{S}}$. Then the system in Figure 1 is globally asymptotically stable if there exist positive definite matrices $\mathbf{S_{11}},\mathbf{P_{11}}$ and $\bar{\mathbf{N}}$, positive scalars λ, μ, ν, $\mathbf{D_H}$, unstructured matrices $\mathbf{A_{H1}}$, $\mathbf{B_{H1}}$, $\mathbf{C_{H1}}$, $\mathbf{A_{H2}}$, $\mathbf{B_{H2}}$ and $\mathbf{C_{H2}}$, and an indefinite scalar η, such that inequalities (14) and (15) from Prop. 2 hold and in addition inequality (49) also holds [top of page].

Proof: Omitted, but follows by combining the results of [22] with those in Section III. □□

V. Examples

This section compares the new high-order multiplier results derived here with previous results: namely a Lyapunov-based approach given [18], (which have recently been shown to be equivalent to a class of Zames-Falb multiplier - see [2]); previous plant-order causal multipliers, both with and without Popov extensions ([23], [22]); and recent plant-order anti-causal results, [4]. These criteria are amongst the best available in the literature for computing Zames-Falb multipliers so comparisons made are against the state-of-the-art. The plants used for the comparison are listed in Table I: the examples are identical to those used in [24] (see also [23], [3], [4]), but they have all been used elsewhere in the literature [23], [21], [18], [3], [5].

Example	P(s)	Source
1	$P_1(s)=-\frac{s^2-0.2s-0.1}{s^3+2s^2+s+1}$	[9]
2	$P_2(s)=-P_1(s)$	[18]
3	$P_3(s)=-\frac{s^2}{s^4+0.2s^3+6s^2+0.1s+1}$	[23]
4	$P_4(s)=-P_3(s)$	[23]
5	$P_5(s)=-\frac{s^2}{s^4+0.0003s^3+10s^2+0.0021s+9}$	[6]
6	$P_6(s)=-P_5(s)$	[23]

TABLE I
Table of transfer functions $P(s)$

Table II shows a comparison of the maximum sector/slope sizes obtained using the results of [18], [23], [22] and [4]. Also shown is the linear upper bound which is computed from the gain margin of the system (called the Nyquist value in [4]) which gives an idea of the potential conservatism of the methods. The highest value of α for which stability is guaranteed is highlighted in the table. Observe that, while the approaches of [18] and [4] produce the least conservative estimates of stability in some cases, the new high-order Zames-Falb multiplier approach appears quite competitive

Method	Example					
	1	2	3	4	5	6
Park [18]	4.5866	1.0891	0.7883	0.7083	0.0018335	0.0018325
Zames-Falb (causal plant order) [23]	2.4283	1.0891	0.7061	0.8526	0.0018112	0.0009505
Zames-Falb (causal plant order) + Popov [24]	3.5072	1.0891	0.7791	1.0864	0.0033046	0.0031513
Zames-Falb (anti-causal plant order) [4]	4.5866	1.0745	0.9846	0.6135	0.00095	0.00182
Zames-Falb (anti-causal plant order) + Popov [4]	4.5866	1.0745	1.4513	0.7222	0.00319	0.00333
Zames-Falb (causal, twice plant order) Prop.2	2.6142	1.0891	1.4593	0.99814	0.0017765	0.0021600
Zames-Falb (causal, twice order) + Popov Prop.3	3.7803	1.0891	1.5222	1.1934	0.0034141	0.0021600
Linear upper bound	4.5866	1.0891	∞	3.5000	∞	1.7142

TABLE II

MAXIMUM SLOPE SIZES, α, FOR WHICH STABILITY IS GUARANTEED

in many cases, especially when combined with the Popov multiplier.

Remark 3: A simple way to obtain a high order multiplier is to introduce uncontrollable/unobservable modes into $P(s)$ and use existing analysis results ([23], [4]). Assuming that the order of the minimal realisation of $P(s)$ is n_p, the introduction of $n_{uc/o}$ uncontrollable/unobservable modes would lead to a multiplier of order $\text{ord}[M(s)] = n_p + n_{uc/o}$. Although the integer $n_{uc/o}$ is arbitrary, the dynamics associated with these additional modes influences the analysis, and the arising results may be better or worse than the standard $\text{ord}[M(s)] = n_p$ case. The appealing feature about Proposition 3 is it is completely systematic. Some numerical experiments carried out with assigning uncontrollable/unobservable modes to $P(s)$ led to unpredictable results. □

VI. CONCLUSION

This paper has proposed two algorithms for stability analysis using high-order Zames-Falb multipliers. The basic approach is similar to existing LMI-based methods but the derivation of the main propositions which form the basis of the algorithms is different and requires the use of a new congruence transformation. A selection of examples from the literature have illustrated the potential of the results. An obvious future step would be to allow the use of non-causal multipliers which comprise an n_p'th order causal part and an n_p'th order anti-causal part.

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