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Stabilising Terminal Cost and Terminal Controller for ℓ_{asso}-MPC: Enhanced Optimality and Region of Attraction

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***Abstract*— In recent literature, ℓ_1-regularised MPC, or ℓ_{asso}-MPC, has been recommended for control tasks involving complex requirements on the control signals, for instance, the simultaneous solution of regulation and sharp control allocation for redundantly-actuated systems. This is due to the implicit thresholding ability of LASSO regression. In this paper, a stabilising terminal cost featuring a mixed ℓ_1/ℓ_2^2 penalty is presented. Then, a candidate terminal controller is computed, with the aim of enlarging the region of attraction.**

I. Introduction

The ℓ_1-regularised MPC (or ℓ_{asso}-MPC), a novel alternative to quadratic MPC, has been proposed in [12], and further refined in [11], [5], [6]. This technique employs LASSO, a powerful regression tool, which aims to induce sparseness in the solution [17]. The aim of ℓ_{asso}-MPC is to obtain spatially and/or temporally sparse control signals. As a result of it, most actuator commands (or their increments) are equal to zero. At the same time, the approach maintains the smooth state behaviour of quadratic MPC. This offers new possibilities for exploiting the structure of control problems with complex control signals requirements, for instance, control over communication networks [11], redundantly-actuated systems [5], requiring piecewise constant input signals [12], or handling minimum control magnitude constraints [4].

In this paper a deeper analysis of stabilising ℓ_{asso}-MPC is presented, for LTI systems. The approach is based on a terminal constraint set and a terminal cost. Two candidate terminal costs are presented, each of them featuring a stabilising scaling factor. Then, a candidate terminal controller is computed, that maximises the Domain of Attraction (DOA). The tradeoff between the DOA volume and the scaling factor is investigated, and an offline tuning procedure is proposed. The resulting ℓ_{asso}-MPC controller will solve, at each time step, a convex quadratic program. Differently from [5], in this paper no additional controllers will be applied to the plant. Four different strategies will be proposed, and demonstrated through simulation for a redundantly-actuated example. The proposed strategies will be then compared to the completion-based contractive-sets approach, from [6], and to an open-loop optimal control problem that approximates an infinite horizon problem.

The paper proceeds as follows. Section II introduces ℓ_{asso}-MPC. Section III computes the new stabilising terminal costs. In Section IV, a candidate terminal controller is computed, for maximal DOA and minimum conservativeness. Section V compares the proposed approach with the previous results of [6], while Section VI concludes the paper.

Selected proofs are provided in the Appendix.

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II. Predictive Control with ℓ_1-regularisation

This section introduces the use of ℓ_1-regularised least squares in MPC. Applications of the technique include the fields of machine learning, feature selection, and filtering [17], [12], [13], [14].

LASSO-based control can be seen as a deterministic problem, where the commands are required to have bounded magnitude and to be sparse in time as well as among the input channels. In particular, the use of LASSO-based control is suggested for simultaneous regulation and control allocation for over-actuated systems [5], [6].

Consider the Linear Time-Invariant (LTI) system

$$x(k+1) = Ax(k) + Bu(k). \tag{1}$$

System (1) is subject to hard constraints of the form

$$u \in \mathbb{U} \subset \mathbb{R}^m, \quad x \in \mathbb{X} \subset \mathbb{R}^n. \tag{2}$$

Predictions used by the MPC are denoted as $x_j = \hat{x}_{k+j|k}$.

Definition II.1. ($\boldsymbol{\ell_{asso}}$–MPC)
Consider the following finite-horizon constrained optimal control problem (FHCOCP)

$$\begin{aligned} V_N^o(x) = \min_{\underline{\mathrm{u}}} &\left\{ V_N(x, \underline{\mathrm{u}}) \triangleq F(x_N) + \sum_{j=0}^{N-1} \ell(x_j, u_j) \right\} \\ &s.t. \quad x_{j+1} = Ax_j + Bu_j, \\ &u_j \in \mathbb{U}, \quad x_j \in \mathbb{X}, \ j = 0, \ldots, N-1, \\ &x_0 = x, \ x_N \in \mathbb{X}_f, \end{aligned} \tag{3}$$

with stage cost

$$\ell(x_j, u_j) = x_j^T Q x_j + u_j^T R u_j + \|Su_j\|_1, \tag{4}$$

with $\underline{\mathrm{u}}^T = \begin{bmatrix} u_0^T & \cdots & u_{N-1}^T \end{bmatrix}$. At each iteration k, the ℓ_{asso}-MPC applies to the plant the first move of the optimal policy, $u(k) = u_0^\star$, obtained by online solution of (3-4), at the current state, $x = x(k)$. The generated implicit control law is referred to as $K_N(x) \equiv u_0^\star$.

Throughout the paper, the following is assumed.

Assumption 1. For the LTI system (1):
- **(H0)** (A, B) is stabilizable, $(Q^{1/2}, A)$ is detectable,
- **(H1)** $Q \succeq 0$, $R \succ 0$, $S \neq 0$,
- **(H2)** $\mathbb{X}$, $\mathbb{X}_f$, and $\mathbb{U}$ are convex and compact (C-sets), containing the origin,
- **(H3)** $x(0) \in \mathbb{X}_N = \{x \in \mathbb{X} : \exists \underline{\mathrm{u}} \in \mathbb{U}^N, \ V_N(x, \underline{\mathrm{u}}) < \infty\}$.

III. ℓ_{asso}-MPC WITH ℓ_1/ℓ_2^2-TERMINAL COST

A. Unconstrained solution & input thresholding

Assume the terminal cost to be

$$F(x) = \|Px\|_1 + x^T Z x, \text{ with } Z \succeq 0. \tag{5}$$

Under Assumption 1, problem (3–4) is convex, but non-differentiable at the origin. The ℓ_1-norm input penalty is used to force most of the solution's elements to be null. This behaviour is referred to as *sparseness*. On the other hand, classic quadratic costs, cause the decision variables (control inputs) to assume non-zero values for most of the time.

The key feature of ℓ_{asso}-MPC is an input thresholding capability. In particular, when the terminal cost is quadratic, some inputs will be zero, conditional the state being in a certain region [6]. In the following, it is shown that the thresholding capability is inherited by ℓ_{asso}-MPC with the proposed terminal cost (5).

Define

$$\Psi = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \Theta = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}, \tag{6}$$

$$\begin{array}{ll} \mathbf{Q} = \text{BlkDiag}\{I_{N\times N}\otimes Q, Z\}, & \mathbf{R} = I_{N\times N}\otimes R, \\ \phi = [A^{N-1}B, A^{N-2}B, \cdots, B], & W = I_{N\times N}\otimes S, \\ H = (\Theta^T\mathbf{Q}\Theta + \mathbf{R}),\ \Phi = P\phi, & \tilde{W}^T = [W^T \ \Phi^T], \end{array} \tag{7}$$

where BlkDiag$\{A, B\}$ denotes a block-diagonal matrix, and $\otimes$ denotes the Kronecker product. Define also $\underline{\mathrm{e}}^T = [\underline{\mathrm{u}}^T \ (Px_N)^T]$. Problem (3–4–5) has the following Karush-Kuhn-Tucker (KKT) optimality conditions, for the unconstrained case[1]

$$-2(\Theta^T\mathbf{Q}\Psi x + H\underline{\mathrm{u}}^\star) \in \tilde{W}^T \ \partial\|\tilde{W}\underline{\mathrm{e}}^\star\|_1. \tag{8}$$

Theorem III.1. The unconstrained ℓ_{asso}-MPC, with $F(x)$ as in (5), has an implicit input thresholding capability. In particular, assume $W \in \mathcal{D}_{++}$. Then if $\underline{\mathrm{u}}_j^\star \neq 0,\ \forall j \neq i$, it follows that

$$2\Big|\Gamma_i \ x(k) + \sum_{j\neq i} H_{ij}\underline{\mathrm{u}}_j^\star\Big| + \sum_j |\Phi_{ji}| < W_{ii} \Rightarrow \underline{\mathrm{u}}_i^\star = 0. \tag{9}$$

Proof. Proof is omitted for brevity and can be found in [7]. ■

B. Asymptotic stability

The following assumption is instrumental for investigating stability of predictive controllers

Assumption 2. $\mathbb{X}_f$ is positively invariant, under an admissible controller, $u = Kx$, with $\rho(A + BK) < 1$. Moreover, $F((A + BK)x) - F(x) \leq -\ell(x, u)$, for all $x \in \mathbb{X}_f$.

A very well known result in MPC is the following

Theorem III.2 (Rawlings & Mayne [16])**.** If Assumptions 1, 2, then, the closed loop system under the MPC controller is Locally Exponentially Stable (LES), with domain of attraction $\mathbb{X}_N$.

[1]The term $\partial\|\underline{\mathrm{u}}^\star\|_1$ is the 1-norm subdifferential [18] evaluated at $\underline{\mathrm{u}}^\star$.

C. Computation of the terminal cost

The aim of this section is to construct a terminal penalty, $F(x)$, a candidate terminal controller, $u = Kx$, and a terminal set, $\mathbb{X}_f$, such that Assumption 2 is satisfied. The computation of stabilising terminal costs, in the following, is partially based on the concepts in [10], [15].

Consider a given stabilising K.

1) ℓ_1-terminal cost: Assume $Z = 0$. Define $\tilde{Q} = Q^{1/2}$, $\tilde{R} = R^{1/2}$. For Assumption 2 to hold, we need

$$\begin{array}{l} \|P(A+BK)x\|_1 - \|Px\|_1 + \|\tilde{Q}x\|_2^2 \\ +\|\tilde{R}Kx\|_2^2 + \|SKx\|_1 \leq 0,\ \forall x \in \mathbb{X}_f. \end{array} \tag{10}$$

Assume $P = \alpha\hat{P}$, where $\alpha > 0$ and $\hat{P} \succ 0$, and define

$$\begin{aligned} \beta &= 1 - \|\hat{P}(A+BK)\hat{P}^{-1}\|_1 \\ \sigma &= \|\tilde{Q}\hat{P}^{-1}\|_2^2 + \|\tilde{R}K\hat{P}^{-1}\|_2^2, \\ \zeta &= \|SK\hat{P}^{-1}\|_1. \end{aligned} \tag{11}$$

The following result is obtained

Lemma III.3. Consider a stabilising K. Then, condition (10) is satisfied $\forall x \in \mathbb{X}_f$, and Assumption 2 is also satisfied, if $\beta > 0$ and

$$\alpha \geq \frac{1}{\beta}\left(\max_{x\in\mathbb{X}_f} \|\hat{P}x\|_1\sigma + \zeta\right). \tag{12}$$

Proof. See Appendix. ■

2) Mixed ℓ_1/ℓ_2^2-terminal cost: Assume now $P = \alpha\hat{P} \succ 0$ and $Z \succ 0$.

Lemma III.4. Consider a stabilising K. Then, Assumption 2 is satisfied, if Z solves the Lyapunov equation

$$(A+BK)^T Z(A+BK) - Z = -(Q + K^T RK), \tag{13}$$

and if

$$\alpha \geq \frac{\zeta}{\beta},\ \beta > 0. \tag{14}$$

Proof. See Appendix. ■

The following algorithm is proposed for computing α

Algorithm 1 Computation of α

```
Require: K : ρ(A+BK) < 1, meth = {L1, mix}, P̂ ≻ 0
  if β ≤ 0 then
    return -1
  end if
  if meth == L1 then
    α ← (1/β) (max_{x∈X_f} ‖P̂x‖_1 σ + ζ)
  else
    if meth == mix then
      α ← ζ/β
    end if
  end if
  return α.
```

IV. Computation of K and region of attraction

The results of the previous section assume an arbitrary stabilising K. The aim of this section is to select a K that maximises the Domain of Attraction (DOA), corresponding to the feasible region, $\mathbb{X}_N$. The tradeoff between the magnitude of α and the size of the DOA is also considered.

Define $\mathcal{O}_\infty$ as the *maximal admissible set* [3], [8]

$$\begin{aligned}\mathcal{O}_\infty = \{x \in \mathbb{R}^n \mid (A+BK)^k x \in \bar{X} \ \forall k \geq 0\},\\ \bar{X} = \{x \in \mathbb{X} \mid Kx \in \mathbb{U}\}.\end{aligned} \tag{15}$$

Define $f^i_{\underline{\mathbf{u}}}(x)$, as the i-step evolution of (1), under a policy $\underline{\mathbf{u}}$. Since, by definition, $\mathbb{X}_N = \mathcal{K}_N(\mathbb{X}, \mathcal{O}_\infty)$, that is

$$\mathbb{X}_N = \{x \in \mathbb{X} : \ \exists \underline{\mathbf{u}} \in \mathbb{U}^N | f^i_{\underline{\mathbf{u}}}(x) \in \mathbb{X}, \ f^N_{\underline{\mathbf{u}}}(x) \in \mathcal{O}_\infty\}, \tag{16}$$

where $i = 0, \dots, N$, then it follows that

$$\mathbb{X} \supseteq \mathbb{X}_N \supseteq \mathcal{O}_\infty. \tag{17}$$

Therefore, maximising $\mathcal{O}_\infty$ may be beneficial to the DOA. In particular, if $\mathbb{X}$ is positively invariant under an admissible K, we have $\mathbb{X} \equiv \mathcal{O}_\infty$. In other words, if such K can be constructed, the DOA will be maximised.

Definition IV.1. A C-set $\mathcal{S}$ is λ-contractive for $x^+ = f(x, u)$ if there exists a $\lambda \in [0, 1)$, and u such that $x \in \mathcal{S} \Rightarrow f(x, u) \in \lambda\mathcal{S}$.[2]

Define the Minkowski function of a C-set $\mathcal{S}$ as

$$\psi_{\mathcal{S}}(x) = \min_\gamma \{\gamma \mid x \in \gamma\mathcal{S}\}. \tag{18}$$

Theorem IV.1 (Blanchini [1], Th. 3.3)**.** The following statements are equivalent:

- The C-set $\mathcal{S}$ is λ-contractive, for (1),
- The Minkowski function of $\mathcal{S}$, $\psi_{\mathcal{S}}(x)$, is a (local) Control Lyapunov Function (CLF) for the system, and $\exists u(x)$ such that the system is (locally) Asymptotically Stable (AS or LAS).

Consider the following

Theorem IV.2 (Blanchini & Miani [2], Cor. 4.43)**.** A polyhedral C-set $\mathcal{S}$ is λ-contractive if and only if for each of its vertices v, there exists a control u such that $Av + Bu \in \lambda\mathcal{S}$, where $\lambda \in [0, 1)$.

Assume that $\mathbb{X} = \{x \,\|Lx \leq \underline{1}\}$, with vertices $v \in \mathcal{V}(\mathbb{X})$, and $\mathbb{U} = \{u \mid Eu \leq \underline{1}\}$, where $\underline{1}$ is a vector of ones with compatible length. Then, $\psi_{\mathbb{X}}(x) \equiv \max_i L_i x$, and the terminal controller K can be chosen as the solution of

$$\begin{array}{lll}\max_\gamma & \min_K & \|\hat{P}(A+BK)\hat{P}^{-1}\|_1 + \nu\|K\|_1\\ & \text{s.t.:} & L(A+BK)v \leq \lambda\underline{1}\\ & & EKv \leq \underline{1}, \ \forall v \in \mathcal{V}(\gamma\mathbb{X}),\\ & & \|\hat{P}(A+BK)\hat{P}^{-1}\|_1 < 1, \ \nu \geq 0,\\ & & \lambda \in (0,1), \ \gamma \in (0,1].\end{array} \tag{19}$$

Remark 1. In problem (19), if $\nu = 0$ then β is made as close as possible to 1, potentially reducing α.

Theorem IV.3. Given $(K^\star, \gamma^\star)$, solving problem (19), then $\gamma^\star\mathbb{X}$ is λ-contractive for (1) under $u = K^\star x$, and $\rho(A + BK^\star) < 1$.

Proof. See Appendix. ■

Lemma IV.4. Given $(K^\star, \gamma^\star)$, solving problem (19), then $\mathcal{O}_\infty \supseteq \gamma^\star\mathbb{X}$.

Proof. See Appendix. ■

The values $(K^\star, \gamma^\star)$, solving problem (19), maximise the inner approximation of the DOA,

$$\tilde{\mathbb{X}}_N = \mathcal{K}_N(\mathbb{X}, \gamma^\star\mathbb{X}) \subseteq \mathbb{X}_N. \tag{20}$$

Remark 2. If $\gamma^\star = 1$, then $\mathcal{O}_\infty = \mathbb{X}$, and $\mathbb{X}_N = \mathbb{X}$, $\forall N > 0$.

Hence, if $\gamma^\star = 1$, the DOA is maximal independently from N.

Problem (19) is non-convex in γ. However, the problem can be addressed by considering a discrete grid of possible γs. Then, minimising over K for all γs in the grid provides a tractable approximation of (19), consisting of a set of (convex) Linear Programmes (LPs).

A. The $\alpha|\gamma$ tradeoff

From Problem (19) a tradeoff appears between the reduction of α (by reducing $1 - \beta$) and the maximisation of the terminal set volume. In particular, it is expectable to obtain a smallest value of α, from a smaller γ. The converse, on the other hand, is not necessarily true. On the other hand, the parameter ν can be used to regulate the tradeoff between α and the sparsity of K, potentially leading to further sparsity in the MPC solution. In particular, from (11) it can be noticed that a sparse K can make the terminal cost independent from certain entries of S. This could provide some degree of freedom for tuning $\ell(x, u)$ without re-computing α.

Consider, as a benchmark, the solution of an Infinite Horizon Optimal Control Problem (IHOCP) with stage cost (4). Then, an inexact terminal cost or terminal constraint will certainly cause an FOCOCP to be "sub-optimal" with respect to this benchmark.[3] Regulating the terminal cost/constraint tradeoff could be, in principle, a way to reduce sub-optimality. Moreover, numerical difficulties could occur if α is very large or $\mathbb{X}_f$ very small.[4] In order to approximate the volume of a polytope $\mathcal{S} = \{x \,|Lx \leq \underline{1}\}$ it is possible to consider the ellipsoid $\varepsilon_{\mathcal{S}} = \{x \mid x^T\Sigma_L x \leq 1\}$, where $\Sigma_L = \frac{1}{q}L^T L$, where q is the number of rows of L. Define $\tilde{L} = \Sigma_L^{1/2}$. Then, the volume of $\varepsilon_{\mathcal{S}}$ is proportional to $\mathbf{d}_{\mathcal{S}} = \det(\tilde{L}^{-1})$. Hence, it is possible to select the γ providing the best $\mathbf{d}_{\mathbb{X}_N} \mid \alpha$ tradeoff. This requires the computation of $\mathbb{X}_N$, that can be done using polytope projections. However, if one wishes not to compute $\mathbb{X}_N$, the value of γ itself can be used, to approximate $\mathbf{d}_{\mathcal{O}_\infty}$.[5]

The following heuristic procedure is proposed.

[2] If $\lambda = 1$, then the set $\mathcal{S}$ is control-invariant.

[3] The correct cost-to-go cannot be computed [5].

[4] For example, with an interior-point method [9], if $\mathbb{X}_f$ has empty interior.

[5] For LTI systems, if $\mathbb{X}$ and $\mathbb{U}$ are polytopes, so is $\mathcal{O}_\infty$.

Algorithm 2 $\alpha|\gamma$-procedure

Require: $(t > 0,\ r > 0,\ t >> r,\ \text{step} > 0,\ \gamma_{\min} > 0)$
  $\mathbf{L}_\gamma \leftarrow \{1 : -\text{step} : \gamma_{\min}\}$, $\mathbf{L}_K = \emptyset$,
  for $i = 1$ to length$\{\mathbf{L}_\gamma\}$ **do**
    Solve (19) s.t. $\gamma \leq \mathbf{L}_\gamma\{i\}$
    Append $K^\star$ to $\mathbf{L}_K$
    if $\beta(K^\star) > 0$ **then**
      Compute $\alpha(K^\star)$ from Algorithm 1
    else
      $\alpha(K^\star) \leftarrow \infty$
    end if
    $J(i) \leftarrow t\ (1-\gamma^\star)^2 + r\ \alpha(K^\star)$
  end for
  $i^\star \leftarrow \arg\min_i J(i)$
  return $K \leftarrow \mathbf{L}_K\{i^\star\}$.

V. EXAMPLE

The following example is considered

$$A = \begin{bmatrix} 0.15 & 0.1 \\ 0 & 1.1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$Q = \begin{bmatrix} 20 & 0 \\ 0 & 60 \end{bmatrix} \quad R = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \tag{21}$$

with state and input magnitude constraints, $|x_j| \leq 30$, $|u_j| \leq 5$, $N = 3$, and $S = \eta I$, $\eta > 0$.

The following cases are investigated:

1) `con-sets`: Completion-based contractive-set approach [6], with 4 sets. K places all eigenvalues at 0.001.
2) `1-norm`: ℓ_1-terminal cost.
3) `1-norm` (γ): ℓ_1-terminal cost. Maximum γ.
4) `1-norm` ($\alpha|\gamma$): ℓ_1-terminal cost, using Algorithm 2, with $t = 1000$, $r = 1/100$.
5) `mixed`: mixed terminal cost. K places all eigenvalues at 0.001.
6) `mixed` (γ): mixed terminal cost. Maximum γ.
7) `mixed` ($\alpha|\gamma$): mixed terminal cost, using Algorithm 2, with $t = 100$, $r = 7/50$.
8) `inf-hor`: Open loop optimal control with $N = 200$.

The resulting DOA and state trajectories are shown in Figures 1–4 (`con-sets`, `1-norm`, `1-norm` (γ) and `mixed` ($\alpha|\gamma$)), for $\eta = 300$, where the x (y) axis represents the first (second) state. In Figure 3–4, it can be noticed that the proposed strategies achieve the maximum DOA.

Table I shows the results for $\eta = 300$, where MSE is the mean squared error on x, and $\mathrm{E}[V_N^o] = \frac{1}{200}\sum_{i=0}^{200} \ell(x(i), u(i))$, is the expected closed-loop cost, both averaged over 50 simulations of the duration of 200 samples, with different initial conditions. For all of the considered cases, the Mean Absolute Input (MAI) value is approximately 0.047, and 100% of control comes from input 2.

Figures 5–7 compare the I/O behaviour for three of the approaches, with $x(0) = [15,\ 15]^T$, with $\eta = 300$ and $\eta = 5$. The new approaches show very similar input/output behaviour. For all approaches, when $\eta = 5$, both states are rapidly stabilised to the origin, by means of both actuators. On the other hand, for $\eta = 300$ only one actuator is used. This is exactly the sort of control allocation for which ℓ_{asso}-MPC is recommended.

TABLE I

CONTROL RESULTS PER ℓ_{asso}-MPC STRATEGY ($\lambda = 300$)

Strategy	MSE	$\rho(Z)$	α	γ	$\mathbf{d}_{\mathbb{X}_N}$	$\mathrm{E}[V_N^o]$
`con-sets`	1.767	62	-	-	1460	184.68
`1-norm`	1.772	-	2710	-	951	184.64
`1-norm` (γ)	1.772	-	10^5	1	1800	184.64
`1-norm` ($\alpha|\gamma$)	1.772	-	19260	0.5	1800	184.64
`mixed`	1.772	60	630	-	951	184.64
`mixed` (γ)	1.772	915	1500	1	1800	184.64
`mixed` ($\alpha|\gamma$)	1.772	277	857	0.5	1800	184.64
`inf-hor`	1.769	-	-	-	1800	184.62

From Table I, it can be noticed that the new approaches are "more optimal" than `con-sets`, with respect to $\mathrm{E}[V_N^o]$. On the other hand, the proposed strategies have all similar $\mathrm{E}[V_N^o]$, slightly higher than `inf-hor`. It can be noticed that `con-sets` is slightly more aggressive, having a smaller MSE than `inf-hor`, having smoother input 2 behaviour of the others.

Algorithm 2 gives no guarantees for the DOA. With the selected t and r, it provides the smallest γ giving maximal $\mathbb{X}_N$. It is recommended to take $t > r$, and to accept values of $\gamma > 0.3$, for Algorithm 2 to be used. In general, it is recommended to use the mixed ℓ_1/ℓ_2^2 cost and, if α is acceptable for the solver, to take the largest possible γ.

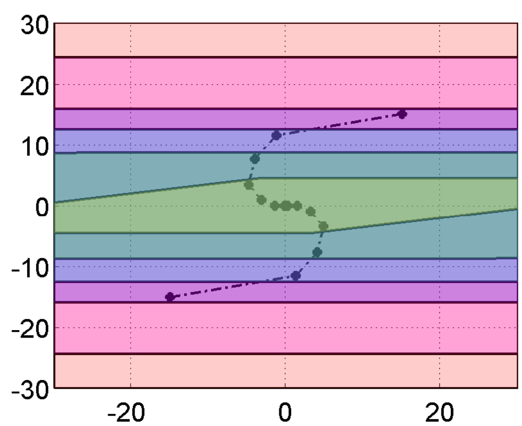

Fig. 1. `con-sets`: 4 invariant sets, $\mathbb{X}_N$, $\mathbb{X}$, and trajectories ($\eta = 300$).

VI. CONCLUSIONS

This paper presented two stabilising ℓ_{asso}-MPC controllers for redundantly-actuated LTI systems, by means of terminal constraint set and terminal cost. Two stabilising terminal costs have been presented, featuring a scaled 1-norm. A strategy has been proposed for the computation of the required candidate terminal controller, that minimises the magnitude of this scaling, while achieving the maximal domain of attraction. A priori computation of the terminal set facilitates design, as well as online tuning. The performed

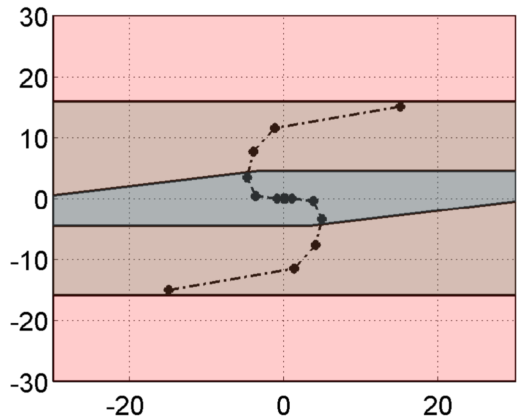

Fig. 2. `1-norm`: Sets, $\mathcal{O}_\infty$, $\mathbb{X}_N$, $\mathbb{X}$, and trajectories ($\eta = 300$).

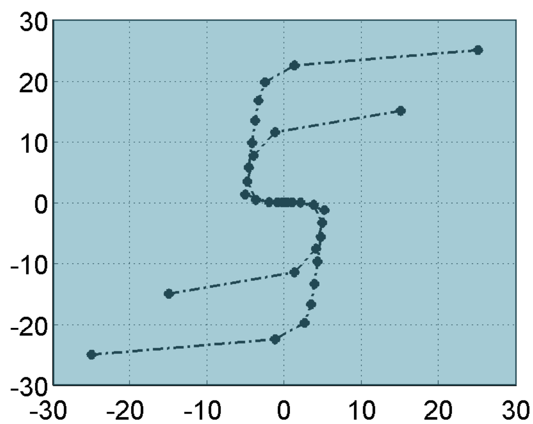

Fig. 3. `1-norm` (γ): Sets, $\mathcal{O}_\infty \equiv \mathbb{X}_N \equiv \mathbb{X}$, and trajectories ($\eta = 300$).

simulations suggest that the proposed strategy is better than the completion-based contractive-sets approach of [6], for the control of redundantly-actuated systems.

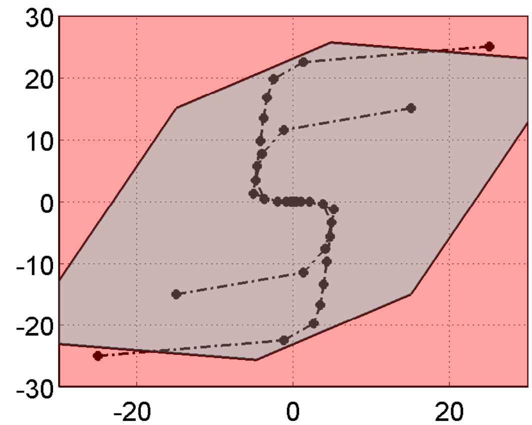

Fig. 4. `mixed` ($\alpha|\gamma$): Sets, $\mathcal{O}_\infty$, $\mathbb{X}_N \equiv \mathbb{X}$, and trajectories ($\eta = 300$).

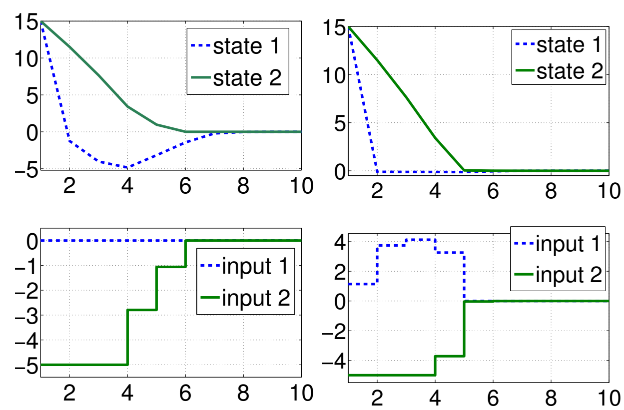

Fig. 5. `con-sets`: I/O trajectories for $\eta = 300$ (left), $\eta = 5$ (right).

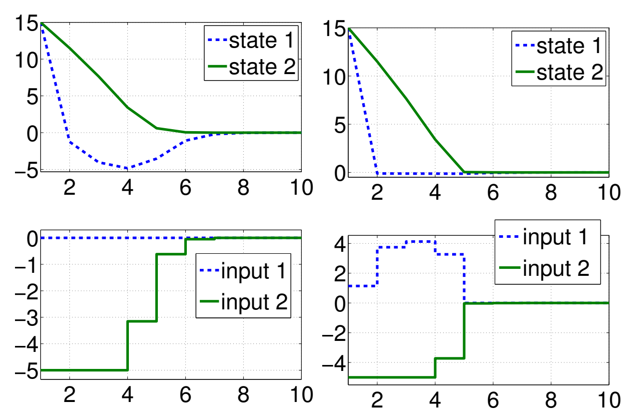

Fig. 6. `1-norm`: I/O trajectories for $\eta = 300$ (left), $\eta = 5$ (right).

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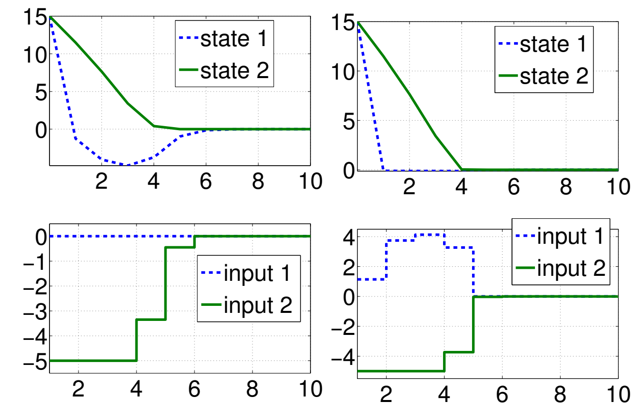

Fig. 7. `mixed` $(\alpha|\gamma)$: I/O trajectories for $\eta = 300$ (left), $\eta = 5$ (right).

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APPENDIX

Proof. (Lemma III.3)

Assume $x \in \mathbb{X}_f$. Substituting (11) in (12), we have

$$\begin{aligned} &1 \geq \|\hat{P}(A+BK)\hat{P}^{-1}\|_1 + \tfrac{1}{\alpha}\|SK\hat{P}^{-1}\|_1 + \\ &+\tfrac{1}{\alpha}\max_{(x\in\mathbb{X}_f)} \|\hat{P}x\|_1 \left(\|\tilde{Q}\hat{P}^{-1}\|_2^2 + \|\tilde{R}K\hat{P}^{-1}\|_2^2\right) \\ &\geq \|\hat{P}(A+BK)\hat{P}^{-1}\|_1 + \tfrac{1}{\alpha}\|SK\hat{P}^{-1}\|_1 + \\ &+\tfrac{1}{\alpha}\|\hat{P}x\|_1 \left(\|\tilde{Q}\hat{P}^{-1}\|_2^2 + \|\tilde{R}K\hat{P}^{-1}\|_2^2\right). \end{aligned} \tag{22}$$

Recalling that $P = \alpha\hat{P}$, the last part of the inequality (22) is equal to

$$\begin{aligned} &\|P(A+BK)P^{-1}\|_1 + \|SKP^{-1}\|_1 + \\ &+\|Px\|_1 \left(\|\tilde{Q}P^{-1}\|_2^2 + \|\tilde{R}KP^{-1}\|_2^2\right) \\ &= \|P(A+BK)P^{-1}\|_1 + \|SKP^{-1}\|_1 + \\ &+\|Px\|_1^2/\|Px\|_1 \left(\|\tilde{Q}P^{-1}\|_2^2 + \|\tilde{R}KP^{-1}\|_2^2\right), \end{aligned} \tag{23}$$

that is an upper bound of (since $\|x\|_2 \leq \|x\|_1$)

$$\begin{aligned} &\|P(A+BK)P^{-1}\|_1 + \|SKP^{-1}\|_1 + \\ &+\|Px\|_2^2/\|Px\|_1 \left(\|\tilde{Q}P^{-1}\|_2^2 + \|\tilde{R}KP^{-1}\|_2^2\right). \end{aligned} \tag{24}$$

From the above statements

$$\begin{aligned} &1 \geq \|P(A+BK)P^{-1}\|_1 + \|SKP^{-1}\|_1 \\ &+1/\|Px\|_1 \left(\|\tilde{Q}P^{-1}\|_2^2\|Px\|_2^2 + \|\tilde{R}KP^{-1}\|_2^2\|Px\|_2^2\right). \end{aligned} \tag{25}$$

Multiply both sides by (25) to $\|Px\|_1$ to get

$$\begin{aligned} \|Px\|_1 \geq\ & \|P(A+BK)P^{-1}\|_1\|Px\|_1 + \\ & \|SKP^{-1}\|_1\|Px\|_1 + \|\tilde{Q}P^{-1}\|_2^2\|Px\|_2^2 \\ & +\|\tilde{R}KP^{-1}\|_2^2\|Px\|_2^2 \\ & \geq \|P(A+BK)x\|_1 + \|SKx\|_1 + \|\tilde{Q}x\|_2^2 \\ & +\|\tilde{R}Kx\|_2^2, \end{aligned} \tag{26}$$

from which condition (10) follows. ■

Proof. (Lemma III.4)

From Z being the solution of (13) it follows that

$$\begin{aligned} x^T(A+BK)^T Z(A+BK)x - x^T Zx = \\ -x^T(Q+K^T RK)x, \end{aligned} \tag{27}$$

and, consequently,

$$\begin{aligned} F((A+BK)x) - F(x) + \ell(x,Kx) = \\ \|P(A+BK)x\|_1 - \|Px\|_1 + \|SKx\|_1. \end{aligned} \tag{28}$$

In order to verify $F((A+BK)x) - F(x) \leq -\ell(x,Kx)$, it is necessary to have

$$\|P(A+BK)x\|_1 - \|Px\|_1 + \|SKx\|_1 \leq 0. \tag{29}$$

Notice that,

$$\begin{aligned} &\|P(A+BK)x\|_1 - \|Px\|_1 + \|SKx\|_1 \leq \\ &\|\hat{P}(A+BK)\hat{P}^{-1}\|_1\|Px\|_1 - \|Px\|_1 \\ &+\tfrac{1}{\alpha}\|SK\hat{P}^{-1}\|_1\|Px\|_1 \leq 0, \end{aligned} \tag{30}$$

if

$$\frac{1}{\alpha}\underbrace{\|SK\hat{P}^{-1}\|_1}_{\zeta} \leq \underbrace{1 - \|\hat{P}(A+BK)\hat{P}^{-1}\|_1}_{\beta}, \tag{31}$$

from which the claim follows. ■

Proof. (Theorem IV.3)

By Theorem IV.2, the constraints on the vertices in (19), imply that $\exists\lambda > 0$ such that $\gamma^\star\mathbb{X}$ is λ-contractive, under the admissible $u = K^\star x$. In particular, since $u = K^\star x$ is admissible on the vertices, it is admissible in $\gamma^\star\mathbb{X}$. As a consequence of contractivity, from Theorem IV.1, $\psi_{\gamma^\star\mathbb{X}}(x)$ is a local Lyapunov function for the system. Then, the system origin is LAS under $u = K^\star x$, implying $\rho(A+BK^\star) < 1$. ■

Proof. (Lemma IV.4)

From the constraints in (19), it follows that, for any feasible γ and K there exists a $\lambda \in [0,\ 1)$ such that

$$x \in \gamma\mathbb{X} \Rightarrow (A+BK)^k x \in \lambda\gamma\mathbb{X} \subset \gamma\mathbb{X},\ \forall k \geq 0. \tag{32}$$

Moreover, since $Kx \in \mathbb{U},\ \forall x \in \gamma\mathbb{X} \subseteq \mathbb{X}$, it follows that $\gamma\mathbb{X} \subseteq \bar{X}$, with $\bar{X}$ being the input admissible set in (15). Recalling the definition of $\mathcal{O}_\infty$, in (15), from (32) if follows that $\gamma\mathbb{X} \subseteq \mathcal{O}_\infty$ for any feasible γ and K. ■