

Minimum-Time Flow Control of Timed Continuous Choice-Free Nets

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Abstract—This paper addresses the problem of reaching (and then maintaining) an (optimal) flow of timed continuous Choice-Free Petri nets in minimum-time. First, we compute the optimal flow by solving a linear programming problem. Then we focus on driving the system to a steady-state (marking) corresponding to the optimal flow. The main challenge of solving this problem is the fact that, in general, the steady-state marking corresponding to a given optimal flow is not unique. We propose a heuristic algorithm, in which at each time step we estimate a “best” firing count vector that drives the system to the convex region where the optimal flow is obtained; then an ON/OFF strategy is applied. Later, we show that some additional firings can further decrease the time spent to obtain the optimal flow.

I. INTRODUCTION

Petri Nets (PN) is a well known paradigm used for modeling, analysis, and synthesis of *discrete event systems* (DES). With a strong ability to depict sequences, concurrency, conflicts and synchronizations, it is widely applied in industry for the analysis of manufacturing, traffic, software systems, or communication networks. Similar to other modeling formalisms for DES, it also suffers from the *state explosion* problem. To overcome this, a classical relaxation technique called *fluidization* can be used.

Continuous PN (CPN) ([1], [2]) are fluid approximations of classical *discrete PN* obtained by removing the integrality constraints, which means that firing count vectors and consequently the markings are no longer restricted to be in the natural but relaxed to be the non-negative real numbers. An important advantage of this relaxation is that more efficient algorithms are available for their analysis.

Some (heuristic) minimum-time controllers for a *target marking control* problem of Timed CPN (TCPN) can be found in, for example, [3], [4]. In these methods the goal is to drive the system to a given target (final) marking that is determined in preliminarily *planning* stage, according to some optimization objectives. In this paper we handle a different problem, focusing on reaching (and then maintaining) the optimal flow from a given initial state, while minimizing the time spent on the trajectory. In particular, we are interested in reaching the *maximal flow* in minimum-time. The main difference from the target marking control problem, also the main challenge, is that we may not be able to uniquely

determine a desired final state. And actually, these steady states with the maximal flow belong to a convex region in the reachability space. Then, two issues arise: which steady state with maximal flow can be reached fastest and which control method should be applied. Assuming that all the transitions are *controllable*, we already know that the ON/OFF controller is a minimum-time controller assuming a given firing count vector, and here we first focus on choosing the “best” one. We propose a heuristic algorithm for CF nets, in which we compute and update at each time step the “best” firing count vector according to an estimation of the number of time steps for firing. We also show that by means of some additional firings, the time to reach the maximal flow can be further reduced.

This paper is organized as follows: Section II briefly recalls some basic concepts of CPN. The problem addressed in this paper, named Minimum-time Flow Control, is stated in Section III. A heuristic algorithm for CF nets is proposed in Section IV and some examples are given in Section V. Some final comments are in Section VI.

II. BASIC CONCEPTS AND NOTATIONS

The reader is assumed to be familiar with the basic concepts of continuous Petri nets (see [1], [2] for a gentle introduction).

Definition 2.1: A continuous PN (CPN) system is a pair $\langle \mathcal{N}, \mathbf{m}_0 \rangle$ where $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$ is a net structure where:

- P and T are the sets of places and transitions respectively.
- $\mathbf{Pre}, \mathbf{Post} \in \mathbb{N}_{\geq 0}^{|P| \times |T|}$ are the pre and post incidence matrices.
- $\mathbf{m}_0 \in \mathbb{R}_{\geq 0}^{|P|}$ is the initial marking (state).

For $v \in P \cup T$, the sets of its input and output nodes are denoted as $\bullet v$ and $v \bullet$, respectively. Let $p_i, i = 1, \dots, |P|$ and $t_j, j = 1, \dots, |T|$ denote the places and transitions. Each place can contain a non-negative real number of tokens, its marking. The distribution of tokens in places is denoted by \mathbf{m} . The enabling degree of a transition $t_j \in T$ is given by:

$$\text{enab}(t_j, \mathbf{m}) = \min_{p_i \in \bullet t_j} \left\{ \frac{\mathbf{m}[p_i]}{\mathbf{Pre}[p_i, t_j]} \right\}$$

which represents the maximal amount in which t_j can fire. Transition t_j is called *k-enabled* at marking \mathbf{m} , if $\text{enab}(t_j, \mathbf{m}) = k$, being enabled if $k > 0$. An enabled transition t_j can fire in any real amount α , with $0 < \alpha \leq \text{enab}(t_j, \mathbf{m})$ leading to a new state $\mathbf{m}' = \mathbf{m} + \alpha \cdot \mathbf{C}[\cdot, t_j]$

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where $C = \text{Post} - \text{Pre}$ is the *token flow matrix* and $C[\cdot, j]$ is its j^{th} column.

Non negative left and right natural annullers of the token flow matrix C are called P-*semiflows* (denoted by y) and T-*semiflows* (denoted by x), respectively. If $\exists y > 0$, $y \cdot C = 0$, then the net is said to be conservative. If $\exists x > 0$, $C \cdot x = 0$, it is said to be consistent.

A PN system is bounded when every place is bounded, i.e., its token content is less than some bounds at every reachable marking. It is live when every transition is live, i.e., it can ultimately occur from every reachable marking.

If m is reachable from m_0 through a finite sequence σ , the state (or fundamental) equation is satisfied: $m = m_0 + C \cdot \sigma$, where $\sigma \in \mathbb{R}_{\geq 0}^{[T]}$ is the *firing count vector*, i.e., $\sigma[t_j]$ is the cumulative amount of firings of t_j in sequence σ . A firing count vector σ is said to be *minimal* if for any T-*semiflow* x , $\|x\| \not\subseteq \|\sigma\|$, where $\|\cdot\|$ stands for the support of a vector, i.e., the index of the elements different than zero. One way to compute a minimal firing count vector σ that drives the system from m_0 to m_f is by solving the LPP (if σ contains a T-*semiflow*, it cannot be a solution minimizing the objective function):

$$\begin{aligned} \min \quad & \mathbf{1}^T \cdot \sigma \\ \text{s.t.} \quad & m_f = m_0 + C \cdot \sigma \\ & \sigma \geq 0 \end{aligned} \quad (1)$$

If for all $p \in P$, $|p^\bullet| \leq 1$ then \mathcal{N} is called *Choice-Free* (CF) net. A net is said to be a Marked Graph (MG) when the weight of every arc is equal to 1, and each place has exactly one input and exactly one output.

In timed continuous PN (TCPN) the state equation has an explicit dependence on time: $m(\tau) = m_0 + C \cdot \sigma(\tau)$ which through time differentiation becomes $\dot{m}(\tau) = C \cdot \dot{\sigma}(\tau)$. The derivative of the firing count $f(\tau) = \dot{\sigma}(\tau)$ is called the *firing flow*. Hence, the time evolution of the marking can be expressed as: $\dot{m}(\tau) = C \cdot f(\tau)$. Depending on how the flow is defined, many firing server semantics appear, being the most used ones *infinite* (or variable speed) and *finite* (or constant speed) server semantics [1], [2], for which a firing rate $\lambda[t_j] \in \mathbb{R}_{>0}$, or denoted by λ_j is associated to transition t_j . This paper deals with *infinite server semantics* for which the flow of a transition t_j at time τ is the product of its firing rate, λ_j , and its enabling degree at $m(\tau)$:

$$f(t_j, \tau) = \lambda_j \cdot \text{enab}(t_j, m(\tau)) = \lambda_j \cdot \min_{p_i \in \bullet t_j} \left\{ \frac{m(p_i, \tau)}{\text{Pre}[p_i, t_j]} \right\} \quad (2)$$

In this paper, the net system is subject to external control actions. We assume that the only admissible control law, denoted by u , consists in *slowing down* the firing flow of transitions defined for *uncontrolled* systems in (2) [2]. This means that transitions modelling machines, for example, cannot work faster than their nominal speeds. Under this assumption, the controlled flow of a TCPN system is denoted as:

$$w(\tau) = f(\tau) - u(\tau), \text{ with } 0 \leq u(\tau) \leq f(\tau)$$

The overall behavior of the system is ruled by:

$$\dot{m}(\tau) = C \cdot (f(\tau) - u(\tau))$$

For the sake of clarity, τ will be omitted in the rest of the paper when there is no confusion.

By sampling the continuous-time CPN system with a *sampling period* Θ , we obtain the discrete-time CPN ([5]) given by:

$$\begin{aligned} m_{k+1} &= m_k + C \cdot w_k \cdot \Theta \\ 0 &\leq w_k \leq f_k \end{aligned} \quad (3)$$

Here m_k and w_k are the marking and controlled flow at sampling instant k , i.e., at $\tau = k \cdot \Theta$.

It is proved in [5] that if the sampling period satisfies (4), the reachability spaces of discrete-time and continuous-time CPN systems are the same, excepting at borders.

$$\forall p \in P : \sum_{t_j \in p^\bullet} \lambda_j \cdot \Theta < 1 \quad (4)$$

In this paper, we assume that (4) holds and every transition is *controllable* (t_j is uncontrollable if the only control that can be applied is $u[t_j] = 0$).

III. PROBLEM STATEMENT: MINIMUM-TIME FLOW CONTROL AND THE MAIN DIFFICULTIES

The *Optimal flow control* problem is widely studied using different system models such as Petri nets, queueing networks, etc., see, for example, [6], [7], [8]. In [6] a control design for CPN was proposed, trying to obtain the flow minimising the production cost defined in *Wilson model*; contribution [7] studied the optimal flow control policies for a stochastic fluid-flow network, it aimed to minimize the total expected discounted cost defined by the reward for admission of fluid into the buffer and the cost incurred for holding fluid in the buffer; [8] proposed two flow control algorithms for networks with multiple paths between each source/destination pair, both are distributed algorithms over the network to maximize aggregate source utility, which can be described as a function of transmission rates. In this work, we focus on the optimal flow control of CF net systems, addressing the problem of reaching an optimal flow from a given initial state, while *minimizing the time* spent on the trajectory.

Recall that the optimal steady-state control problem of CPNs has been addressed in [9], trying to *maximize a profit function* depending on the marking in the steady-state (m_{ss}), the (controlled) flow in the steady state (w_{ss} , $C \cdot w_{ss} = 0$), and the initial marking (m_0). Here, we assume that the profit function is aiming to maximize the flow (under control inputs) of steady state for a given m_0 . Since only one minimal T-*semiflow* exists in (strongly connected and consistent) CF nets [10], it is equivalent to maximize the flow of any transition t_j , by means of the following LPP

[9]:

$$\begin{aligned}
\psi_j &= \max w_{ss}[t_j] \\
\text{s.t. } & \mathbf{m}_{ss} = \mathbf{m}_0 + \mathbf{C} \cdot \boldsymbol{\sigma} \\
& \mathbf{C} \cdot \mathbf{w}_{ss} = \mathbf{0} \\
& \mathbf{w}_{ss}[t] = \boldsymbol{\lambda}[t] \cdot \frac{\mathbf{m}_{ss}[p_i]}{\text{Pre}[p_i, t]} - \mathbf{v}[p_i, t], \\
& \quad \forall p_i \in \bullet t, \mathbf{v}[p_i, t] \geq 0 \\
& \mathbf{w}_{ss}, \boldsymbol{\sigma}, \mathbf{m}_{ss} \geq 0
\end{aligned} \tag{5}$$

where $\mathbf{v}[p_i, t]$ are *slack* variables.

Usually the solution of LPP (5) is not unique (different \mathbf{m}_{ss} may exist for a given \mathbf{w}_{ss}). Let us denote by $\psi_j = \mathbf{w}_{ss}[t_j]$ the optimal flow of transition t_j obtained by solving (5), and \mathcal{M} the set of markings with the maximal flow, i.e.,

$$\begin{aligned}
\mathcal{M} = \{ & \mathbf{m} | \mathbf{m} = \mathbf{m}_0 + \mathbf{C} \cdot \boldsymbol{\sigma}, \boldsymbol{\sigma} \geq 0 \text{ and } \exists \mathbf{0} \leq \mathbf{u} \leq \mathbf{f}, \\
& \mathbf{w} = \mathbf{f} - \mathbf{u}, \mathbf{C} \cdot \mathbf{w} = \mathbf{0}, \mathbf{w}[t_j] = \psi_j \}
\end{aligned} \tag{6}$$

where \mathbf{f} is the (uncontrolled) flow vector at marking \mathbf{m} . Any state in \mathcal{M} is an equilibrium point corresponding to the maximal flow that can be maintained by applying an appropriate control \mathbf{u} . Because all the constrains of (6) are linear, \mathcal{M} is a convex subset included in the reachability space [11]. It arises an interesting problem: which state $\mathbf{m} \in \mathcal{M}$ can be reached in minimum-time (by applying appropriate control methods)? or equivalently, how the maximal flow can be obtained in minimum-time? Here, we call this problem *Minimum-time Flow Control* problem.

We already know that for CF nets, given a final state and the corresponding minimal firing count vector, the minimum-time control strategy is ON/OFF [12]. However, in the Minimum-time Flow Control problem, we do not know which firing count vector (thus the marking) is the one that minimizes the time spent on the trajectory. In particular, the time spent is not monotonic with respect to the corresponding firing count vectors.

Let us consider the MG (a subclass of CF nets) in Fig.1 and assume that sampling period is $\Theta = 0.01$. By solving LPP (5), we obtain that the maximal flow of any transition (because in MGs 1 is the unique T-semiflow) is $\psi = 1$. Two of the possible steady-states (in the same region) corresponding to the maximal flow are: $\mathbf{m}_1 = [100 \ 170 \ 20 \ 94 \ 6 \ 190 \ 10 \ 10 \ 4]^T$ and $\mathbf{m}_{11} = [100 \ 2 \ 188 \ 94 \ 6 \ 190 \ 10 \ 10 \ 4]^T$, $\mathbf{m}_1, \mathbf{m}_{11} \in \mathcal{M}$, with corresponding minimal firing count vectors $\boldsymbol{\sigma}_1 = [0 \ 30 \ 6 \ 0 \ 10 \ 0 \ 0]^T$ and $\boldsymbol{\sigma}_{11} = [0 \ 198 \ 6 \ 0 \ 10 \ 0 \ 0]^T$, respectively. Obviously, $\boldsymbol{\sigma}_1 \leq \boldsymbol{\sigma}_{11}$. Let us consider 9 intermediate points on the straight line from \mathbf{m}_1 to \mathbf{m}_{11} , obtained by $\mathbf{m}_i = (1 - \alpha) \cdot \mathbf{m}_1 + \alpha \cdot \mathbf{m}_{11}$, $i = 2, 3, \dots, 10, \alpha = 0.1, 0.2, \dots, 0.9$. Intermediate markings \mathbf{m}_2 to \mathbf{m}_{10} belong to \mathcal{M} , hence they are also steady-states with the optimal flow, and the corresponding minimal firing count vectors satisfy $\boldsymbol{\sigma}_1 \leq \boldsymbol{\sigma}_2 \leq \dots \leq \boldsymbol{\sigma}_{10} \leq \boldsymbol{\sigma}_{11}$. By applying the ON/OFF controller, the numbers of time steps for reaching \mathbf{m}_1 to \mathbf{m}_{11} starting from $\mathbf{m}_0 = [100 \ 200 \ 0 \ 100 \ 0 \ 200 \ 0 \ 0 \ 10]^T$ are shown in Fig.2.

For reaching \mathbf{m}_1 by applying the ON/OFF controller, the required number of time steps is 827. For \mathbf{m}_2 , it is decreased to 507 (remember that $\boldsymbol{\sigma}_1 \leq \boldsymbol{\sigma}_2$). The required number of

Fig. 1. A simple MG with the maximal flow $\psi = 1$, firing rate vector $\boldsymbol{\lambda} = [1 \ 0.5 \ 0.25 \ 1/6 \ 0.05 \ 0.1 \ 0.1]^T$

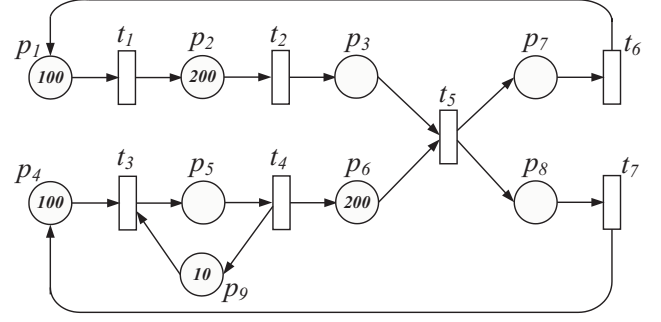
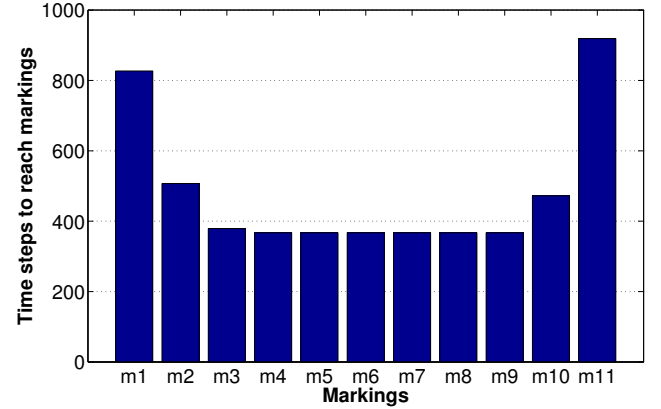


Fig. 2. The time steps required to reach different steady-states with the maximal flow by applying the ON/OFF controller to the MG shown in Fig.1: non-monotonicity appears with respect to the corresponding firing count vectors



time steps is further decreased to 379 for \mathbf{m}_3 to \mathbf{m}_9 . But it starts to increase from \mathbf{m}_{10} . For reaching \mathbf{m}_{11} , 919 time steps are required. We can easily observe that a smaller $\boldsymbol{\sigma}$ does not provide less time to obtain the maximal flow. Furthermore, the non-monotonicity respect to the firing count vector has been exhibited in this example.

The difference between $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, for example, is that in $\boldsymbol{\sigma}_2$ transition t_2 fires more than in $\boldsymbol{\sigma}_1$. Therefore, p_3 receives more tokens and t_5 may fire faster (its flow is increased). In the cases of $\boldsymbol{\sigma}_1$ to $\boldsymbol{\sigma}_9$, transition t_5 is the one that fires “slowest”, i.e., the one that requires more time steps to fire the given firing amount. Therefore by increasing the flow of t_5 , the overall number of time steps is decreased. On the other hand, if t_2 fires too much, as in $\boldsymbol{\sigma}_{10}$ and $\boldsymbol{\sigma}_{11}$, t_2 becomes the one that requires more time steps, so the overall time steps starts to increase.

IV. A HEURISTIC ALGORITHM FOR CF NETS

In a (strongly connected and consistent) CF net there exists a unique minimal T-semiflow \mathbf{x} and its *support* contains all the transitions ($\mathbf{C} \cdot \mathbf{x} = 0, \mathbf{x} > 0$) [10]. Therefore, if ψ_j is the maximal flow of transition t_j (the optimal solution of LPP (5)), then the maximal flow of every transition can be deduced (because $\boldsymbol{\psi}$ is a steady-state flow, $\mathbf{C} \cdot \boldsymbol{\psi} = 0$ and $\boldsymbol{\psi} = \alpha \cdot \mathbf{x}, \alpha > 0$). Moreover, the minimal required marking of a place p_i to ensure the maximal flow can be easily determined

by the firing rate of its unique output transition and weight on the arc:

Definition 4.1: Let $\langle \mathcal{N}, \lambda, \mathbf{m}_0 \rangle$ be a CF system, \mathbf{x} be the minimal T-semiflow and ψ_j be the optimal flow of transition t_j . Then, $\boldsymbol{\mu}$ is said to be the minimal required marking for the optimal flow ¹, if:

$$\boldsymbol{\mu}[p_i] = (\psi_j / \lambda[t]) \cdot (\mathbf{x}[t] / \mathbf{x}[t_j]) \cdot \mathit{Pre}[p_i, t], \quad \forall p_i \in P, \{t\} = p_i^\bullet \quad (7)$$

An immediate consequence of Definition 4.1 is the following: given $\mathbf{m} \geq \boldsymbol{\mu}$, the uncontrolled flow of any transition t_j corresponding to \mathbf{m} satisfies $\mathbf{f}[t_j] \geq \psi_j$. Therefore, if all the transitions are controllable, there exists $0 \leq \mathbf{u} \leq \mathbf{f}$, such that the controlled flow $\mathbf{w}[t_j] = \psi_j$. In other words, for any reachable marking \mathbf{m} , $\mathbf{m} \in \mathcal{M}$ iff $\mathbf{m} \geq \boldsymbol{\mu}$. Thus, the Minimum-time Flow Control problem of CF nets is equivalent to *reach a marking $\mathbf{m} \geq \boldsymbol{\mu}$ in minimum-time*. Moreover, a firing count vector $\boldsymbol{\sigma}$ that leads to a steady state \mathbf{m}_{ss} with the maximal flow is a solution of the following equations:

$$\begin{aligned} \mathbf{m}_{ss} &= \mathbf{m}_0 + \mathbf{C} \cdot \boldsymbol{\sigma} \\ \mathbf{m}_{ss} &\geq \boldsymbol{\mu} \\ \boldsymbol{\sigma} &\geq 0 \end{aligned} \quad (8)$$

The ON/OFF controller [12] is a minimum-time controller of CF nets assuming a given firing count vector. However, the firing count vector satisfying (8) is not unique in general. Therefore, we need to compute the best one, i.e., the one that leads to the maximal flow in minimum-time by applying the ON/OFF controller.

According to the ON/OFF strategy, every transition fires as fast as possible, until each one completes its required amount given by the corresponding firing count vector. Therefore, the overall time is determined by the “slowest” transition, i.e., the one that costs most time steps to fire its given amount. Since the firing speed is variable in TCPN under infinite server semantics, depending on the state evolution, we will consider an *estimation of the number of time steps*.

Let us assume that the current marking at time step k is \mathbf{m}_k and let $\boldsymbol{\sigma}_k$ be a firing count vector that should be fired to reach a state in \mathcal{M} . Then $\mathbf{S}_k[t_j] = \lceil \frac{\boldsymbol{\sigma}_k[t_j]}{\lambda_j \cdot \mathit{enab}(\mathbf{m}_k, t_j) \cdot \Theta} \rceil$ can be viewed as an estimation of the number of time steps that transition t_j needs to fire (it is an estimation because it is assumed a constant speed for t_j). Given transitions t_a and t_b , if $\mathbf{S}_k[t_a] > \mathbf{S}_k[t_b]$, then it would be said that t_a is “slower” than t_b . Notice that \mathbf{S}_k does not give neither a lower nor an upper bound because \mathbf{m}_k changes dynamically.

In the heuristics we propose, at each time step k we minimize the number of time steps of the slowest transition, i.e., to minimize the infinity norm of \mathbf{S}_k , $\|\mathbf{S}_k\|_\infty = \max\{|\mathbf{S}_k[t_j]|\}, t_j \in T$. The minimization of $\|\mathbf{S}_k\|_\infty$ can be done by solving the following LPP, in which a new variable d is introduced:

$$\begin{aligned} \min \quad & d \\ \text{s.t.} \quad & \mathbf{m}_{ss} = \mathbf{m}_k + \mathbf{C} \cdot \boldsymbol{\sigma}_k \\ & \mathbf{m}_{ss} \geq \boldsymbol{\mu} \\ & d \geq \boldsymbol{\sigma}_k[t] / (\lambda[t] \cdot \mathit{enab}(\mathbf{m}_k, t) \cdot \Theta), \forall t \in T \\ & \boldsymbol{\sigma}_k \geq \mathbf{0} \end{aligned} \quad (9)$$

where \mathbf{m}_k is a the current marking at time step k .

The control progress is given in Algorithm 1, which is a close-loop control and at each time step we recompute the “best” firing count vector $\boldsymbol{\sigma}_k$ according to the current state. Then, $\boldsymbol{\sigma}_k$ is fired by applying the ON/OFF strategy, obtaining a heuristics for the Minimum-time Flow Control. Since the stability of applying the ON/OFF controller to CF nets has been proved, the convergence of Algorithm 1 can be easily obtained.

Algorithm 1 An algorithm of Minimum-time Flow Control problem for CF nets

Input: $\langle \mathcal{N}, \lambda, \mathbf{m}_0 \rangle, t_j, \Theta$

Output: sequence of controlled flows: $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_k$

- 1: compute ψ_j by solving LPP (5);
 - 2: compute $\boldsymbol{\mu}$ that satisfies (7);
 - 3: $k \leftarrow 0$;
 - 4: **while** not $(\mathbf{m}_k \geq \boldsymbol{\mu})$ **do**
 - 5: compute $\boldsymbol{\sigma}_k$ by solving LPP (9);
 - 6: compute the controlled flow \mathbf{w}_k corresponding to the ON/OFF strategy;
 - 7: update state: $\mathbf{m}_{k+1} \leftarrow \mathbf{m}_k + \Theta \cdot \mathbf{C} \cdot \mathbf{w}_k$;
 - 8: $k \leftarrow k + 1$;
 - 9: **end while**
 - 10: compute the steady state control \mathbf{w}_k , such that: $\mathbf{C} \cdot \mathbf{w}_k = \mathbf{0}, \mathbf{w}_k[t_j] = \psi_j$;
 - 11: **Return** $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_k$;
-

Algorithm 1 can be further improved, considering the *persistence* property of CF nets: the (additional) firing of one transition does not disable the firing of the others [10]. However, it may increase the flow of the other transitions (as in the net system in Fig.1, additional firings of t_2 increased the flow of t_5). Based on this observation and because our problem is to drive the system to a marking $\mathbf{m} \in \mathcal{M}$, i.e., $\mathbf{m} \geq \boldsymbol{\mu}$, for any transition t , if at time step k all of its input place $p_i \in \bullet t$ satisfy $\mathbf{m}_k[p_i] > \boldsymbol{\mu}[p_i]$, we can fire t without increasing the time to reach \mathcal{M} . So, in the improved algorithm we distinguish the following two cases:

- (1) for any transition t with $\boldsymbol{\sigma}_k[t] = 0$, we consider the markings of the input places of t : if for any $p_i \in \bullet t$, $\mathbf{m}_k[p_i] > \boldsymbol{\mu}[p_i]$, then t is fired as fast as possible; else, t is blocked;
- (2) for any transition t with $\boldsymbol{\sigma}_k[t] > 0$ the ON/OFF strategy is applied.

The strategy of case (1) can only decrease the time for reaching a marking in \mathcal{M} , but not increase. This is because we would fire t only if all of its input places already have *more-than-enough* markings to obtain the maximal flow; at

¹It is a marking vector that may not be reachable.

the same time this firing will not “slow down”, but may “speed up”, the firing of others. This improved process is given in Algorithm 2.

Algorithm 2 Improved algorithm of Minimum-time Flow Control problem for CF nets

Input: $\langle \mathcal{N}, \lambda, m_0 \rangle, t_j, \Theta$

Output: w_0, w_1, \dots, w_k

- 1: compute ψ_j by solving LPP (5);
 - 2: compute μ that satisfies (7);
 - 3: $k \leftarrow 0$;
 - 4: compute σ_k by solving LPP (9);
 - 5: **while** not $(m_k \geq \mu)$ **do**
 - 6: **for all** $t \in T$ **do**
 - 7: **if** $\sigma_k[t] > 0$ **then**
 - 8: compute the controlled flow $w_k[t]$ corresponding to the ON/OFF strategy;
 - 9: **else if** $m_k[p_i] > \mu[p_i]$ for any $p_i \in \bullet t$ **then**
 - 10: $w_k[t] \leftarrow \lambda[t] \cdot \text{enab}(m_k, t)$;
 - 11: **else**
 - 12: $w_k[t] \leftarrow 0$;
 - 13: **end if**
 - 14: **end for**
 - 15: update state: $m_{k+1} \leftarrow m_k + \Theta \cdot C \cdot w_k$;
 - 16: $k \leftarrow k + 1$;
 - 17: **end while**
 - 18: compute the steady state control w_k , such that: $C \cdot w_k = 0, w_k[t_j] = \psi_j$;
 - 19: **Return** w_0, w_1, \dots, w_k ;
-

Proposition 4.2: Let $\langle \mathcal{N}, \lambda, m_0 \rangle$ be a CF net system. By applying Algorithm 2, the system converges to a steady-state marking $m \in \mathcal{M}$ that maximizes the flow.

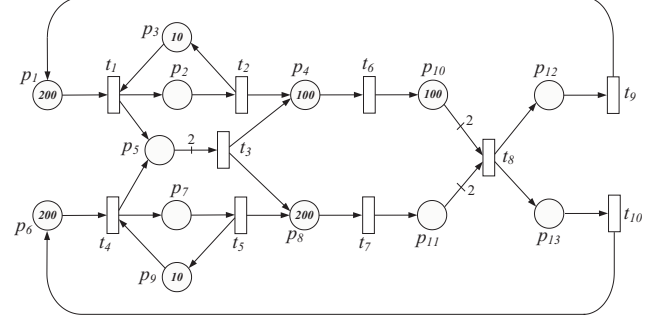
Proof: Since \mathcal{N} is a CF net, the additional firings (for a transition t with $\sigma_k[t] = 0$) do not disable the firing of σ_k that drives the system to a state $m \in \mathcal{M}$, i.e., $m \geq \mu$. On the other hand, for any place p_i with $m[p_i] \leq \mu[p_i]$, we do not decrease its marking, therefore the algorithm will converge to a marking $m' \geq \mu$ belonging to \mathcal{M} . ■

Algorithm 2 is still a heuristics for minimum-time. One clear reason is that we try to look for the “best” firing count vector (in terms of spending less time on firing it) based on an approximation of the time steps that is obtained from the current state and flow; nevertheless, the risk of choosing a very “bad” one is somehow reduced because after each time step we recompute it based on the actual state. Another possible reason is that only *local* information is considered. By means of some firings, the time spent for reaching a marking in \mathcal{M} may be decreased. But, in the case concerning a transition t with $\sigma_k[t] = 0$, it is allowed to fire t again only if its input places have tokens more than those in μ . However, this strategy may not be the optimal in some situations, even for MGs (a simple subclass of CF nets, see the net in Fig. 5 for a example).

V. EXAMPLES

Let us consider the CF net system in Fig.3, assuming $\Theta = 0.01$. The unique minimal T-semiflow of the net is $x = [1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 2 \ 1 \ 1 \ 1]^T$.

Fig. 3. A CF net system with the maximal flow $\psi_9 = 1$, firing rate vector $\lambda = [0.25 \ 1/6 \ 0.05 \ 0.25 \ 1/6 \ 0.1 \ 0.5 \ 0.05 \ 1/30 \ 1/30]^T$

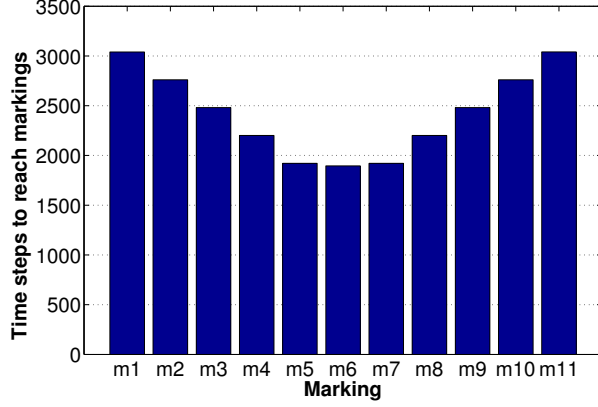


Assume that we want to maximize the flow of transition t_9 , by solving LPP (5) with $t_j = t_9$, it is obtained $\psi_j = 1$. From (7), the corresponding minimal required marking is $\mu = [4 \ 6 \ 4 \ 20 \ 40 \ 4 \ 6 \ 4 \ 4 \ 40 \ 40 \ 30 \ 30]^T$. However, the solution of LPP (8) is not unique. For instance, $\sigma_1 = [34 \ 28 \ 0 \ 6 \ 0 \ 0 \ 100 \ 30 \ 0 \ 0]^T$ and $\sigma_{11} = [6 \ 0 \ 0 \ 34 \ 28 \ 0 \ 100 \ 30 \ 0 \ 0]^T$ are both solutions of LPP (8), reaching optimal-flow steady states $m_1 = [166 \ 6 \ 4 \ 128 \ 40 \ 194 \ 6 \ 100 \ 4 \ 40 \ 40 \ 30 \ 30]^T$ and $m_{11} = [194 \ 6 \ 4 \ 100 \ 40 \ 166 \ 6 \ 128 \ 4 \ 40 \ 40 \ 30 \ 30]^T$. Similarly to the example of the MG in Fig.1, we also consider 9 more intermediate points on the straight line from m_1 to m_{11} and the maximal flow can be obtained from all of them. The time steps required for reaching m_i , $i = 1, 2, \dots, 11$, by using the ON/OFF controller, the results of applying Algorithm 1 and Algorithm 2, are illustrated in Fig.4.

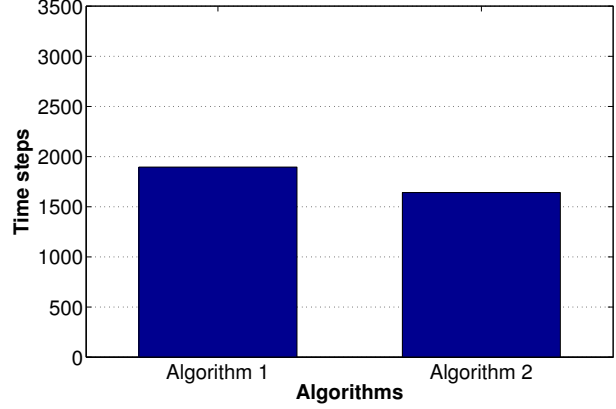
As shown in Fig. 4, by applying Algorithm 1 we can obtain the maximal flow in 1895 time steps, which is the same as using the ON/OFF controller to drive the system to m_6 . However, we should remember that we do not know *a priori* that driving the system to m_6 will cost less time than to other markings m_i , $i = 1, 2, \dots, 11$, $i \neq 6$. By applying Algorithm 2, the time to reach the maximal flow is further reduced to 1641 time steps.

Although Algorithm 2 can highly reduce the time spent for reaching a marking in \mathcal{M} , the minimum-time is not guaranteed in general, even for MGs. Let us consider a MG shown in Fig.5, assuming that the firing rate vector $\lambda = [1 \ 1 \ 1 \ 1 \ 1/3 \ 1 \ 1]^T$ and $\Theta = 0.01$. The maximal flow is $\psi = 1$ (obtained by solving LPP (5)) and the corresponding minimal required marking is $\mu = [1 \ 1 \ 1 \ 3 \ 3 \ 1 \ 1]^T$. By using Algorithm 1, we can reach the maximal flow in 138 time steps. By Algorithm 2, we can reduce the number of time steps to 102, reaching steady-state $m = [5.64 \ 2.941 \ 2.609 \ 3 \ 3 \ 1 \ 3.587 \ 7.323]^T$ and the corresponding firing count vector $\sigma = [9.09 \ 3.55 \ 1.609 \ 3 \ 0 \ 0.2 \ 6.413]^T$. Nevertheless, it is still not the minimum-time for reaching the maximal flow: by firing $\sigma' = [8.942 \ 3.437 \ 1.598 \ 3 \ 0 \ 0.2 \ 6.34]^T$ using the

Fig. 4. Comparison of time steps for reaching the maximal flow of the CF system in Fig.3



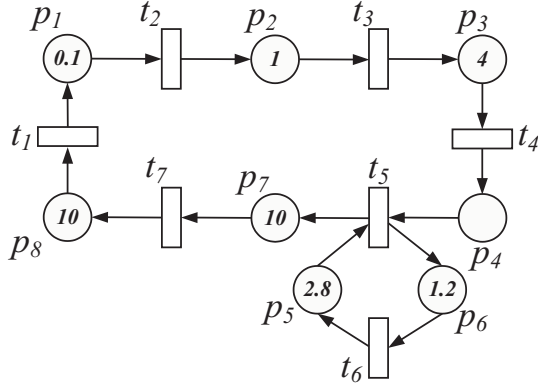
(a) time steps required to reach different steady-states with the maximal flow by applying the ON/OFF control



(b) time steps required to reach the maximal flow by applying the proposed algorithms

ON/OFF strategy, we reach another maximal flow steady-state $m' = [5.605 \ 2.84 \ 2.598 \ 3 \ 3 \ 1 \ 3.66 \ 7.398]^T$ in only 100 time steps.

Fig. 5. A MG example where Algorithm 2 does not give the minimum-time to the maximal flow



VI. CONCLUSIONS

In this paper we discuss the Minimum-time (optimal) Flow Control problem for CF net systems. The main difference from the target marking control problem (at the same time the main difficulty of solving this problem) is that, in general we cannot uniquely determine a steady state with the given optimal flow (in our case the maximal flow), and actually, they belong to a convex region. We propose a heuristic algorithm for CF nets. An ON/OFF strategy is applied to fire the estimated “best” firing count vector leading to the maximal flow and we show that the time spent on the trajectory can be further reduced by some additional firings. Concerning the computational complexity, in each time step we solve a LPP, therefore, in polynomial time. Nevertheless, the number of time steps depends on many variables as net structure, markings and firing rates. As a future work, we consider to extend the method to general net systems where the following two main issues should be addressed: (1) the

ON/OFF strategy may not be applicable; (2) the optimal flow may not be unique and thus the minimal required marking for the optimal flow may not be determined either. Another clear extension is to consider systems with uncontrollable transitions.

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