2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Convergence Bounds for Discrete-Time Second-Order Multi-Agent-Systems

A. Eichler and H. Werner, *Member, IEEE*

Abstract— **This paper presents convergence bounds for discrete-time second-order multi-agent systems with undirected or directed communication graphs. As has been shown before, the convergence depends on the eigenvalues of the Laplace matrix of the communication graph. For each eigenvalue (or eigenvalue pair) analytic bounds for the parameter set are given to render the protocol for that eigenvalue pair stable. In addition it is shown examplarily, that for the case of normalized Laplacian, the stabilizing solution set for the whole topology is non-empty.**

I. INTRODUCTION

The consensus problem for multi-agent systems has received considerable attention over the past ten years, because of its broad variety of applications in cooperative control, formation control, flocking, coverage control, task assignment and so on. Extensive surveys about the fields of applications are given e.g. in [1], [2].

When multi-vehicle systems are required to reach consensus, the model of a single agent may be rather complex. Using a separation principle proposed in [3], which is based on an information flow filter approach, it is possible to design stabilizing local controllers for individual agents and a stabilizing information flow filter for the whole multi-vehicle system separately, see [4]. In practice, the most important filter types are single- and double-integrators, where typically a system is required to agree on the position and/or the velocity.

Consensus theory for single-integrators has been intensively studied in the literature, see e.g. [5], [6], [7], where both continuous- and discrete-time consensus protocols are considered for networks with switching communication graphs and time-delays. Similar attention has been paid to double-integrators in [8] for the continuous-time case and in [9], [10], [11], [12], [13], [14] for the discrete-time case. Since in practice communication between the agents involves sampled rather than continuous-time data, in this paper we focus on discrete-time systems.

The convergence of consensus protocols for discrete-time second-order systems has been explored, e.g. in [9] for undirected and in [10] for undirected as well as directed communication graphs. Whereas for the undirected case explicit bounds are given in [10], for the directed case it is proved that there exists a choice of variables to achieve convergence and that those variables fulfill a certain condition without giving explicit bounds for them. In [11] a different protocol is discussed, where aspects like switching communication graphs are discussed. Concerning convergence it is stated that it has not been possible to find convergence bounds for the parameters there for a given non-switching graph and sampling time. To the best of the authors knowledge such bounds have not been reported yet. The protocol of [11] has been further explored concerning convergence speed in [15]. In [14] the n-th order consensus protocol is discussed with the double-integrator as a special case.

In this paper the protocol of [11] with an additional degree of freedom is analyzed. Analytic bounds for the free parameters are given for each eigenvalue of the Laplacian, whose intersection leads to convergence of the whole protocol.

This paper is organized as follows. In Section II some concepts from graph theory are reviewed and the discrete-time consensus protocol considered here is presented and analyzed. In Section III convergence bounds for undirected communication are proposed and in Section IV the same is done for the directed case. Numerical analysis and example cases in Section V illustrate the results of the paper. Finally conclusions are drawn in Section VI.

II. PRELIMINARIES

A. Graph Theory

Consider a multi-agent system with n agents and m communication links, described by the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, W)$ with the node set $\mathcal{N} = \{1, \ldots, n\}$, which represents the agents, and the edge set $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ describing the communication topology. If there is an edge $\{ij, i \neq j\} \in \mathcal{E}$ then agent i receives information from agent j; for undirected graphs agent j then also receives information from agent i, for directed graphs this is not necessarily the case.

The adjacency matrix $A \in \mathbb{R}^{n\times n}$ is associated with the graph $\mathcal{G}$. Its elements are defined as $a_{ij} = 1$ if $\{ij\} \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. By this definition every edge is weighted equally by 1. For the weighted adjacency A_w with differently weighted edges, the elements of the adjacency matrix are chosen as $a_{w_{ij}} = w_{ij}$, where w_{ij} is the weight of the edge $\{i, j\} \in \mathcal{E}$.

With the adjacency matrix the Laplacian $L \in \mathbb{R}^{n\times n}$ can be constructed by

$$l_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j \\ \sum_{j=1}^{n} a_{ij} & \text{if } i = j \end{cases} \tag{1}$$

for the unweighted case. For the weighted case a_{ij} in (1) has to be replaced by $a_{w_{ij}}$ to get L_w. In the following we will only consider equal weighting by a constant weighting factor $c > 0$, thus $L_w = cL$. Due to construction 0 is an eigenvalue of L (and L_w respectively) corresponding to the eigenvector $\mathbf{1}$ (a vector with all elements equal to 1).

A. Eichler and H. Werner are with the Institute of Control Systems, Hamburg University of Technology,Eissendorfer Str. 40, 21073 Hamburg, Germany e-mail: {annika.eichler, h.werner}@tu-harburg.de.

978-3-952-41734-8/©2013 EUCA

Let λ_i be the ith eigenvalue of L (and $c\lambda_i$ the ith eigenvalue of L_w), then if and only if the graph is connected, we have $\lambda_1 = 0 < \mathrm{Re}(\lambda_2) \leq \ldots \leq \mathrm{Re}(\lambda_n)$. In the undirected case we have $\lambda_i \in \mathbb{R}$ for $i = 1, \ldots, n$, because of symmetry.

B. *Consensus Protocol*

The discrete-time double-integrator dynamics for the ith agents information considered here is given by

$$\xi_i(k+1) = \xi_i(k) + \tau\zeta_i(k) \tag{2}$$

$$\zeta_i(k+1) = \zeta_i(k) + \tau u_i(k) \tag{3}$$

where the parameter $\tau > 0$ can be seen as sampling interval.

Using the distributed discrete-time consensus protocol

$$u_i(k) = -\sum_{j=1}^{n} a_{w_{ij}} \Big[\big(\xi_i(k) - \xi_j(k)\big) + \alpha\big(\zeta_i(k) - \zeta_j(k)\big)\Big] \tag{4}$$

proposed in [11] with $\alpha > 0$, leads to

$$\begin{bmatrix}\xi(k+1)\\ \zeta(k+1)\end{bmatrix} = \begin{bmatrix} I & \tau I\\ -\tau c L & I - \tau\alpha c L\end{bmatrix}\begin{bmatrix}\xi(k)\\ \zeta(k)\end{bmatrix} = \Psi\begin{bmatrix}\xi(k)\\ \zeta(k)\end{bmatrix}. \tag{5}$$

Here $\xi(k) = \begin{bmatrix}\xi_1(k) & \ldots & \xi_n(k)\end{bmatrix}^T$ and $\zeta(k)$ is defined respectively.

It is assumed here for ease of notation that ξ_i and ζ_i are one-dimensional signals. The results can easily be extended to vector-valued signals by using the Kronecker product.

C. *Convergence Analysis of Consensus Protocol*

The protocol (5) is said to converge if

$$\lim_{k\to\infty} |\xi_i(k) - \xi_j(k)| = 0 \quad \text{and} \quad \lim_{k\to\infty} |\zeta_i(k) - \zeta_j(k)| = 0\,.$$

To explore under which conditions (5) converges, we start by calculating the characteristic polynomial of Ψ in (5) as

$$\begin{aligned} p(s) &= \det(sI - \Psi)\\ &= \det\left(s^2 I + s(-2I + \tau\alpha c L) + I + \tau^2 c L - \tau\alpha c L\right)\\ &= \prod_{i=1}^{n}\left(s^2 + s(-2 + \tau\alpha c\lambda_i) + 1 + \tau^2 c\lambda_i - \tau\alpha c\lambda_i\right) \end{aligned} \tag{6}$$

where λ_i denotes the ith eigenvalue of L. It is obvious that for every λ_i there are two corresponding eigenvalues of Ψ, denoted by ψ_{2i-1} and ψ_{2i}, which can be calculated as

$$\psi_{2i-1,2i} = 1 - \frac{\tau}{2}\alpha\lambda_i \pm \frac{\tau}{2}\sqrt{\alpha^2\lambda_i^2 - 4\lambda_i}\,. \tag{7}$$

Thus $\lambda_1 = 0$ leads to $\psi_1 = \psi_2 = 1$. The consensus protocol (5) achieves consensus if and only if the remaining eigenvalues of Ψ are inside the unit disc. In [11] it is shown that if and only if $|\psi_j| < 1$ for $j = 3, \ldots, 2n$, (5) converges to

$$\lim_{k\to\infty} \xi(k) = \mathbf{1}v_l^T\xi(0) + k\tau\mathbf{1}v_l^T\zeta(0) \tag{8}$$

$$\lim_{k\to\infty} \zeta(k) = \mathbf{1}v_l^T\zeta(0) \tag{9}$$

and thus consensus is achieved. Here v_l is the left eigenvector of L corresponding to the eigenvalue $\lambda_1 = 0$.

Note that for undirected graphs due to symmetry right and left eigenvectors are equal up to scaling, thus $v_l = \frac{1}{n}\mathbf{1}$ such that $v_l\mathbf{1} = 1$. In that case average consensus is reached. For directed graphs the consensus value is weighted by v_l.

Lemma 1: The consensus protocol (5) achieves consensus, i.e. $|\psi_j| < 1$ for $j = 3, \ldots, 2n$ if and only if all roots of the polynomial

$$\begin{aligned} &t^2(\tau^2 c\lambda_i) + t(2\tau\alpha c\lambda_i - 2\tau^2 c\lambda_i) + \ldots\\ &\ldots(4 + \tau^2 c\lambda_i - 2\tau\alpha c\lambda_i) = 0 \end{aligned} \tag{10}$$

are in the open left half plane for $i = 2, \ldots, n$.

Proof: The bilinear transformation

$$s = \frac{t+1}{t-1} \tag{11}$$

maps the roots inside the unit disc of a polynomial

$$s^2 + as + b = 0, \tag{12}$$

into roots in the open left half plane of

$$t^2(1 + a + b) + t(2 - 2b) + b - a + 1 = 0. \tag{13}$$

Applying (11) to the characteristic polynomial (6) leads to (10) for $j = 3, \ldots, 2n$. This completes the proof. ■

D. *Problem Statement*

In the following the sampling interval τ as well as the communication topology, represented by the Laplacian L, are assumed to be given. The problem considered here is to determine α and to choose the edge weight c such that (5) converges, i.e. the condition of Lemma 1 is fulfilled. This problem is solved in the following section for the undirected case and in Section IV for the directed case.

III. UNDIRECTED COMMUNICATION

Theorem 1: Consider a multi-agent system with undirected connected communication graph represented by L and sampling time τ. The consensus protocol (5) converges if and only if

$$\begin{aligned} 0 &< c < \frac{4}{\tau^2\lambda_n}\\ \tau &< \alpha < \frac{2}{\tau c\lambda_n} + \frac{\tau}{2}\,. \end{aligned} \tag{14}$$

Proof: According to Lemma 1, (5) converges if and only if (10) has all its roots in the open left plane for $j = 3, \ldots, 2n$. With the Routh-Hurwitz stability criterion this is fulfilled if and only if

$$\begin{aligned} \tau^2 c\lambda_i &> 0\,,\\ 2\tau\alpha c\lambda_i - 2\tau^2 c\lambda_i &> 0\,,\\ 4 + \tau^2 c\lambda_i - 2\tau\alpha c\lambda_i &> 0\,. \end{aligned} \tag{15}$$

Since λ_i for $i = 2, \ldots, n$ is positive, the first condition of (15) is fulfilled. The second condition leads to $\alpha > \tau$ and the third to $\alpha < \frac{2}{\tau c\lambda_i} + \frac{\tau}{2}$. This condition has to hold for all $i = 2, \ldots, n$; using λ_n leads to the least upper bound as given in (14).

To find a value for α, the weighting c has to fulfill $\tau < \frac{2}{\tau c\lambda_n} + \frac{\tau}{2}$, which proves the condition for c in (14). Since only equivalent transformations have been used, the 'only if'-part follows. ■

IV. DIRECTED COMMUNICATION

The Laplacian of a directed communication graph can have real and complex eigenvalues, $\Lambda_r(L)$ and $\Lambda_c(L)$ respectively, with $\Lambda_r(L) = \{\lambda_i \in \mathrm{eig}(L) \mid \mathrm{Im}(\lambda_i) = 0,\ \lambda_i \neq 0\}$ and $\Lambda_c(L) = \{\lambda_i \in \mathrm{eig}(L) \mid \mathrm{Im}(\lambda_i) \neq 0,\ \lambda_i \neq 0\}$.

Lemma 2: Consider a multi-agent system with directed connected communication graph represented by L and sampling time τ. Condition (10) is satisfied for all $\lambda_i \in \Lambda_r(L)$ if and only if $(c, \alpha) \in \Omega_r$, where

$$\Omega_r = \left\{ (c,\alpha) \middle| \begin{array}{l} 0 < c < \min_{\lambda_i \in \Lambda_r} \frac{4}{\tau^2 \lambda_i} \\ \tau < \alpha < \min_{\lambda_i \in \Lambda_r} \frac{2}{\tau c \lambda_i} + \frac{\tau}{2} \end{array} \right\} \tag{16}$$

Proof: The proof is analogous to Theorem 1. ■

Lemma 3: Consider a multi-agent system with directed connected communication graph represented by L and sampling time τ. Condition (10) is satisfied for a $\lambda_i \in \Lambda_c(L)$ if and only if $(c_i, \alpha_i) \in \Omega_{c_i}$, where

$$\Omega_{c_i} = \left\{ (c_i,\alpha_i) \middle| \begin{array}{l} 0 < c_i < c_u(\lambda_i) \\ \alpha_l(\lambda_i, c_i) < \alpha_i < \alpha_u(\lambda_i, c_i) \end{array} \right\} \tag{17}$$

with $c_u(\lambda_i)$, $\alpha_l(\lambda_i, c_i)$ and $\alpha_u(\lambda_i, c_i)$ given in the appendix in (35), (32) and (33).

Proof: The task is to find bounds for α_i and c_i such that (10) has its roots in the open left half plane for $\lambda_i \in \Lambda_c$.

Assume as in [10] that t_1 and t_2 are solutions of (10), then

$$t_1 + t_2 = \frac{-2\tau\alpha_i c_i \lambda_i + 2\tau^2 c_i \lambda_i}{\tau^2 c_i \lambda_i} = \frac{2}{\tau}(\tau - \alpha_i)\ , \tag{18}$$

$$t_1 t_2 = \frac{4 + \tau^2 c_i \lambda_i - 2\tau\alpha_i c_i \lambda_i}{\tau^2 c_i \lambda_i} = \frac{4}{\tau^2 c_i \lambda_i} + 1 - \frac{2\alpha_i}{\tau}\ . \tag{19}$$

Because the right hand side of (18) is real, it is clear that $\mathrm{Im}(t_1) + \mathrm{Im}(t_2) = 0$, thus $t_1 = a_1 + jb$ and $t_2 = a_2 - jb$.

The consensus protocol converges if $\mathrm{Re}(t_1) = a_1 < 0$ and $\mathrm{Re}(t_2) = a_2 < 0$, which is equivalent to $a_1 + a_2 < 0$ and $a_1 a_2 > 0$. With (18) the first condition is fulfilled for $\alpha_i > \tau$, leading to a first lower bound for alpha, which is equal to that of the undirected case. But as will be shown in the following, the condition $a_1 a_2 > 0$ may lead to tighter bounds.

To fulfill condition $a_1 a_2 > 0$, $t_1 = a_1 + jb$ and $t_2 = a_2 - jb$ are replaced in (19), leading to

$$a_1 a_2 + b^2 + j(a_2 - a_1)b = \frac{4}{\tau^2 c_i \lambda_i} + 1 - \frac{2\alpha_i}{\tau}\ . \tag{20}$$

Comparing the imaginary parts of both sides in (20) leads to

$$(a_2 - a_1)b = -\frac{4\,\mathrm{Im}(\lambda_i)}{\tau^2 c_i |\lambda_i|^2} \tag{21}$$

and the comparison of the real parts to

$$a_1 a_2 + b^2 = \frac{4\,\mathrm{Re}(\lambda_i)}{\tau^2 c_i |\lambda_i|^2} + 1 - \frac{2\alpha_i}{\tau}\ . \tag{22}$$

Solving (21) for b^2 and replacing it in (22) leads to

$$a_1 a_2 + \frac{16\,\mathrm{Im}(\lambda_i)^2}{\tau^4 c_i^2 |\lambda_i|^4 (a_2 - a_1)^2} = \frac{4\,\mathrm{Re}(\lambda_i)}{\tau^2 c_i |\lambda_i|^2} + 1 - \frac{2\alpha_i}{\tau} \tag{23}$$

where $(a_2 - a_1)$ has to be replaced to get an expression only in terms of the product $a_1 a_2$ as desired. From (18) it is known that

$$(a_2 - a_1)^2 = (a_2 + a_1)^2 - 4a_1 a_2 = \frac{4}{\tau^2}(\tau - \alpha_i)^2 - 4a_1 a_2\ .$$

Substituting this in (23) leads to an equation that only depends on the product $a_1 a_2$ and that can be reformulated as a polynomial

$$4(a_1 a_2)^2 + A a_1 a_2 + B = 0 \tag{24}$$

with

$$A = -\frac{4}{\tau^2}(\alpha_i - \tau)^2 - 4\left[\frac{4\,\mathrm{Re}(\lambda_i)}{\tau^2 c_i |\lambda_i|^2} - \frac{2\alpha_i}{\tau} + 1\right], \tag{25}$$

$$B = -\frac{16\,\mathrm{Im}(\lambda_i)^2}{\tau^4 c_i^2 |\lambda_i|^4} + \frac{4}{\tau^2}(\alpha_i - \tau)^2 \left[\frac{4\,\mathrm{Re}(\lambda_i)}{\tau^2 c_i |\lambda_i|^2} - \frac{2\alpha_i}{\tau} + 1\right]. \tag{26}$$

According to the solutions of a quadratic equation all solutions of $a_1 a_2 > 0$, if and only if $A < 0$ and $B > 0$. It is easy to verify that if $B > 0$, therefore $A < 0$ since

$$\frac{4\,\mathrm{Re}(\lambda_i)}{\tau^2 c_i |\lambda_i|^2} - \frac{2\alpha_i}{\tau} + 1 > 0\ .$$

Thus only B has to be considered. As function of α, $B(\alpha_i)$ is a third order polynomial in α with

$$\lim_{\alpha_i \to -\infty} B(\alpha_i) = \infty\ , \qquad \lim_{\alpha_i \to \infty} B(\alpha_i) = -\infty$$

and has one minimum at $\alpha_{i_{\min}} = \tau$ and one maximum at $\alpha_{i_{\max}} = \frac{2\tau^2|\lambda_i| + 4\,\mathrm{Re}(\lambda_i)}{3\tau|\lambda_i|}$. Since $B(\alpha_{i_{\min}}) = -\frac{16\,\mathrm{Im}(\lambda_i)^2}{\tau^4|\lambda_i|^4} < 0$, there only exists a range of $\alpha_i > \tau$ such that $B(\alpha_i) > 0$ if and only if $B(\alpha_{i_{\max}}) > 0$.

Let $B_{\max} = B(\alpha_{i_{\max}})$. We have

$$B_{\max} = -\frac{4}{27} + 16\left[\frac{\mathrm{Re}(\lambda_i)}{9c_i\tau^2|\lambda_i|^4} + \frac{16\,\mathrm{Re}(\lambda_i)^3}{27c_i^3\tau^6|\lambda_i|^6} - \frac{9\,\mathrm{Im}(\lambda_i)^4 + 13\,\mathrm{Re}(\lambda_i)^2\,\mathrm{Im}(\lambda_i)^2 + 4\,\mathrm{Re}(\lambda_i)^4}{9c_i^2\tau^4|\lambda_i|^6}\right]. \tag{27}$$

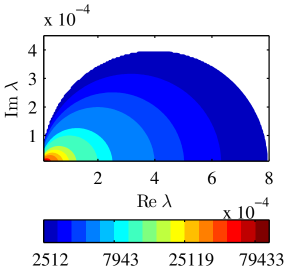
which is a function of c_i. The task is now to find the range of $c_i > 0$, for which $B_{\max}(c_i) > 0$. By definition $c_i > 0$, thus $B_{\max}(c_i)$ is analyzed for positive c_i. Since the limits are

$$\lim_{c_i \to +0} B_{\max} = \infty\ , \quad \lim_{c_i \to \infty} B_{\max} = -\frac{4}{27}$$

for each complex eigenvalue of $\lambda_i \in \Lambda_c(L)$ there exists a region $0 < c_i < c_u(\lambda_i)$, given in the appendix in (35), such that $B_{\max}(c_i) > 0$. Thus there exists a non-empty range $\alpha_l(\lambda_i, c_i) < \alpha_i < \alpha_u(\lambda_i, c_i)$ such that (10) is fulfilled. Those bounds of α_i are the respective roots of $B(\alpha_i)$; they are given in the appendix in (32) and (33). Since only equivalent transformations have been used, the 'only if'-part follows. ■

Definition 1: The intersection of Ω_{c_i} for all $\lambda_i \in \Lambda_c(L)$ is defined as

$$\Omega_c = \bigcap_{\lambda_i \in \Lambda_c} \Omega_{c_i}\ . \tag{28}$$



(a) Upper bound α_u (b) Lower bound α_l

Fig. 1. Bounds on α for λ in the complex plane, $\tau = 1$

Theorem 2: Given a directed communication graph with Laplacian L and a sampling time τ, the condition of Lemma 1 is fulfilled, if $(c, \alpha) \in \Omega_r \cap \Omega_c$.

Proof: Since λ_i for $i = 2, \ldots, n$ is equal to $\Lambda_r(L) \cup \Lambda_c(L)$, Lemma 1 is satisfied, if Lemma 2 and 3, e.g. $(c, \alpha) \in \Omega_r \cap \Omega_c$. ■

Note that the presented results are all valid for other matrices than the Laplacian defined in (1) as long as the eigenvalues satisfy $\lambda_1 = 0 < \mathrm{Re}(\lambda_2) \leq \ldots \leq \mathrm{Re}(\lambda_n)$, like e.g. the normalized Laplacian [4] with all its eigenvalues inside a disc with radius 1 centered at 1.

Note that although with Lemma 3 it can be proven that for each $\lambda_i \in \Lambda_c(L)$ there exists α_i and c_i, such that (10) has its poles in the left half plane, this does not prove that Ω_c is non-empty. In Figure 1 it is illustrated, that for the normalized Laplacian for each possible eigenvalue selection a non-empty region can be found by choosing c small enough. Here α_u and α_l are plotted for all possible eigenvalues of the normalized Laplacian with $c = 0.004$. The legend shows that $\alpha_u > 1000$ and $\alpha_l < 500$. General analytic bounds for Ω_c and thus for $\Omega_r \cap \Omega_c$ have not been found yet.

V. NUMERICAL ANALYSIS AND EXAMPLES

A. Numerical Analysis

In the following the results of Section IV for directed communication are illustrated for a complex pole pair $\lambda = 2 \pm i1$ and a sampling interval $\tau = 1$. To check if for this eigenvalue pair and a weighting $c = 1$, a nonempty region of α can be found such that (10) has its roots in the open left half plane, $B(\alpha)$ has to be analyzed. If $B(\alpha)$ has its maximum at a value greater than zero, then (10) has its roots in the open left half plane. In Figure 2 the curve $B(\alpha)$ is shown in solid blue. The minimum at $\alpha = \tau = 1$ is highlighted by the red dashed line. It is obvious that there is no region of α, such that (10) has its roots in the open left half plane, since the maximum is not above zero.

The green dashed and red dashed dotted curves show $B(\alpha)$ for a scaled λ; In green dashed $\lambda = 2 \pm i1$ is scaled by 0.4 and in red dashed dotted by 0.3. Naturally the location of the minimum is unchanged at $\alpha = \tau$, but the location of the maximum is shifted to higher values of α and its value increases with smaller scaling factors. For a scaling factor of 0.4 the maximum is exactly zero and for a scaling of 0.3 it is larger. Since scaling is nothing else than the weighting factor c, for the complex eigenvalue pair $\lambda = 2 \pm i1$ there exists a $c < 0.4$ such that an α can be found that stabilizes (10).

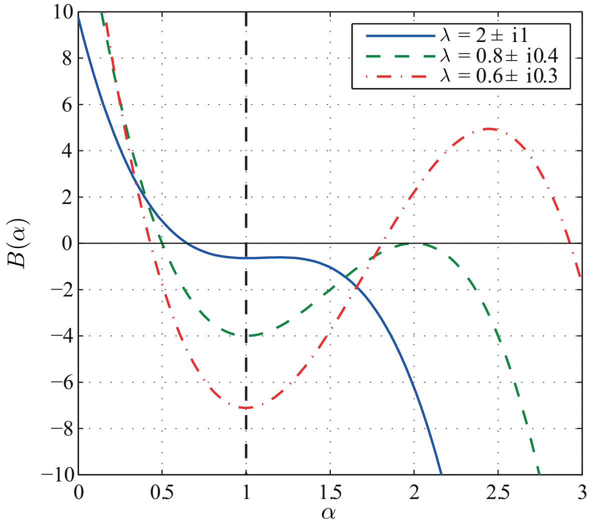


Fig. 2. $B(\alpha)$ analyzed for different λ and $\tau = 1$

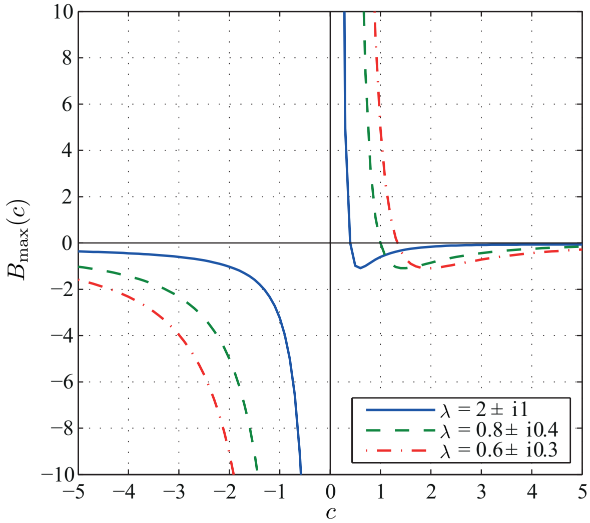
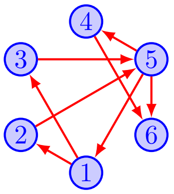


Fig. 3. $B_{\max}(c)$ analyzed for different λ and $\tau = 1$

This is illustrated further in Figure 3. Here $B_{\max}(c)$ is plotted over c for $\lambda = 2 \pm i1$ and the scaled versions. To get a non-empty stabilizing region for α, $B_{\max}$ has to be larger than zero. Therefore c has to be less than the zero of $B_{\max}$. As expected for smaller scaling the zero crossing is shifted to higher values of c. For $\lambda = 2 \pm i1$ the zero is with $c = 0.4$ less than 1. This confirms that for $\lambda = 2 \pm i1$ no stabilizing region of α can be found, unless it is scaled by $c < 0.4$. The green dashed curve with a scaling of 0.4 crosses zero at 1, since we are on the border of convergence. For the red dashed dotted curve and a scaling of 0.4, the zero of $B_{\max}$ is larger than 1, thus with this scaling a stabilizing region of α can be found.

B. Small Numerical Example

Instead of only analyzing one complex eigenvalue pair, a multi-agent system with $n = 6$ agents is considered in the following. The corresponding graph is shown in Figure 4. The Laplacian has the eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = 1$, $\lambda_4 = 2$ and $\lambda_{5,6} = 2 \pm i1$. Note that the latter eigenvalue pair has been analyzed in the previous section.



$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Fig. 4. Example graph and corresponding Laplacian L

For the real eigenvalues λ_r and $c = 1$ the stabilizing bounds are $1 < \alpha < 1.5$. For this region the eigenvalues of Ψ are calculated with (7) and the root locus w.r.t. α is shown in Figure 5. The circles and triangles mark the starting values for $\alpha = \tau = 1$. In blue dashed the root loci of

$$\psi_{2i-1} = 1 - \frac{\tau}{2}\alpha\lambda_i + \frac{\tau}{2}\sqrt{\alpha^2\lambda_i^2 - 4\lambda_i} \tag{29}$$

and in red solid those of

$$\psi_{2i} = 1 - \frac{\tau}{2}\alpha\lambda_i - \frac{\tau}{2}\sqrt{\alpha^2\lambda_i^2 - 4\lambda_i} \tag{30}$$

are shown for $i = 1, \dots, 6$. The blue circles and red triangles mark the starting values for $\alpha = \tau = 1$ of ψ_{2i-1} and ψ_{2i} respectively. It can be shown that for $\alpha = \tau$, all $\psi_{2i-1,2i}$ for λ_r are located on the unit circle. With increasing α they move on circles around 1 into the unit disc to the real axis, whereas ψ_{2i-1} converges along the positive and ψ_{2i} along the negative real axis.

To see how the complex pair behaves, $\lambda = a + ib$ is substituted in (29) leading to

$$\begin{aligned} \psi_{2i-1} =& 1 - \frac{\tau}{2}\alpha(a+ib) + \dots \\ & \dots \frac{\tau}{2}\underbrace{\sqrt{\alpha^2(a^2-b^2) - 4a + (2ab\alpha^2 - 4b)i}}_{\sqrt{z}}\,. \end{aligned}$$

For complex λ_c, ψ_{2i-1} is discontinuous if $\mathrm{Im}(z)$ changes sign. This can be seen in Figure 5, here for $\alpha = \tau = 1$, $\mathrm{Im}(z) = 0$ and thus $\arg(z) = \pi$; for $\alpha > \tau$, $\mathrm{Im}(z) < 0$ and thus $\arg(z) = -\pi$. If such a change of sign occurs, $\psi_{2i-1}(a + ib)$ and $\psi_{2i}(a - ib)$ change positions, and $\psi_{2i-1}(a - ib)$ and $\psi_{2i}(a + ib)$ respectively. The limits are $\lim_{\alpha\to\infty} \psi_{2i-1} = 1$ and $\lim_{\alpha\to\infty} \psi_{2i} = -\tau\alpha\lambda_i$, as can be expected from the figure.

As discussed in the previous section, there does not exist a stabilizing region of α, which is confirmed in Figure 5, because for every α some roots are outside the unit disc. The root locus for $c = 0.3$ is shown in Figure 6. From the previous section we know that this leads to a stabilizing region of α, which is confirmed here.

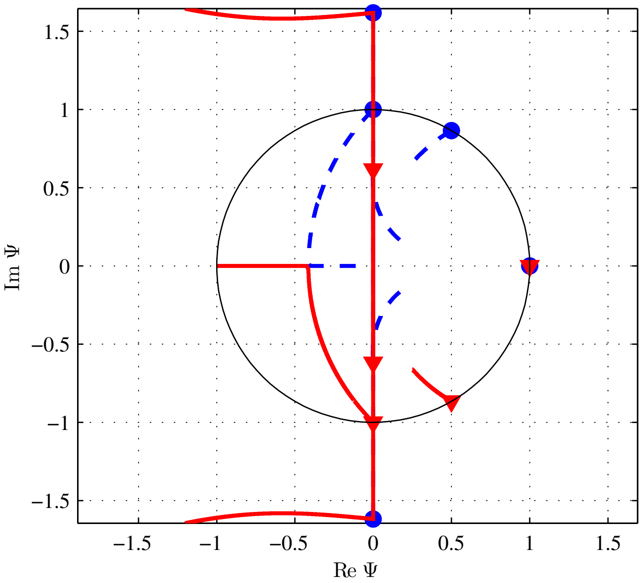


Fig. 5. Root locus of the eigenvalues of Ψ for $1 < \alpha < 1.5$ and $c = 1$

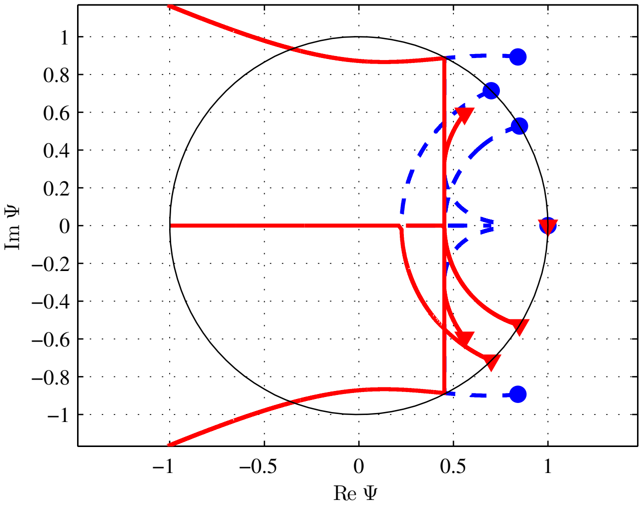


Fig. 6. Root locus of the eigenvalues of Ψ for $1 < \alpha < 3.83$ and $c = 0.3$

C. Large Numerical Example

In the following a multi-agent system with 12 agents and more than 1 complex eigenvalue pair is considered. The eigenvalues of L are $\lambda_1 = 0$, $\lambda_2 = 0.44$, $\lambda_{3,4} = 0.77 \pm i0.79$, $\lambda_{5,6} = 1$, $\lambda_{7,8} = 1.29 \pm i0.66$, $\lambda_{9,10} = 2.03 \pm i0.85$, $\lambda_{11} = 2.47$ and $\lambda_{12} = 2.93$. In Figure 7 the bounds on α depending on the choice of c are shown for the real as well as for the complex eigenvalues. The colored region is the intersection of all stabilizing regions for the single eigenvalues, and is thus the overall stabilizing solution.

VI. CONCLUSION

The convergence of a discrete-time second-order multi-agent system is discussed. For this purpose the characteristic polynomial of the considered protocol is analyzed, which

$$\Upsilon = \left[\frac{216\,\mathrm{Im}(\lambda_i)^2}{\tau^6|\lambda_i|^5} + \frac{96\,\mathrm{Re}(\lambda_i)^3}{\tau^6|\lambda_i|^6} - \frac{24\,\mathrm{Re}(\lambda_i)\left[9\,\mathrm{Im}(\lambda_i)^4 + 13\,\mathrm{Im}(\lambda_i)^2\,\mathrm{Re}(\lambda_i)^2 + 4\,\mathrm{Re}(\lambda_i)^4\right]}{\tau^6|\lambda_i|^8}\right]^{\frac{1}{3}} \tag{31}$$

$$\alpha_l(\lambda_i, c_i) = \frac{5c_i\tau^2|\lambda_i|^2 + 4\,\mathrm{Re}(\lambda_i)}{6c_i\tau|\lambda_i|^2} - \frac{\Gamma^{\frac{1}{3}}}{12c_i^2\tau|\lambda_i|^2} - \sqrt{3}\left(\frac{\Gamma^{\frac{1}{3}} i}{12c_i^2\tau|\lambda_i|^2} - \frac{\left(c_i\tau^2|\lambda_i|^2 - 4\,\mathrm{Re}(\lambda_i)\right)^2 i}{12\Gamma^{\frac{1}{3}}\tau|\lambda_i|^2}\right) - \frac{\left(c_i\tau^2|\lambda_i|^2 - 4\,\mathrm{Re}(\lambda_i)\right)^2}{12\Gamma^{\frac{1}{3}}\tau|\lambda_i|^2} \tag{32}$$

$$\alpha_u(\lambda_i, c_i) = \frac{\left(c_i\tau^2|\lambda_i|^2 - 4\,\mathrm{Re}(\lambda_i)\right)^2}{6\tau|\lambda_i|^2\Gamma^{\frac{1}{3}}} + \frac{5c_i\tau^2|\lambda_i|^2 + 4\,\mathrm{Re}(\lambda_i)}{6c_i\tau|\lambda_i|^2} + \frac{\Gamma^{\frac{1}{3}}}{6c_i^2\tau|\lambda_i|^2} \tag{33}$$

$$\begin{aligned}\Gamma =& 12c_i^5\tau^4\left[\mathrm{Im}(\lambda_i)^4\,\mathrm{Re}(\lambda_i) + 2\,\mathrm{Im}(\lambda_i)^2\,\mathrm{Re}(\lambda_i)^3 + \mathrm{Re}(\lambda_i)^5\right] + c_i^4\tau^2\left[-216\,\mathrm{Im}(\lambda_i)^4 - 264\,\mathrm{Im}(\lambda_i)^2\,\mathrm{Re}(\lambda_i)^2 - 48\,\mathrm{Re}(\lambda_i)^4\right]\\ & - |\lambda_i|^6 c_i^6\tau^6 + 12\sqrt{3}\,\tau\, c_i^{\frac{7}{2}}\,|\mathrm{Im}(\lambda_i)|\,|\lambda_i|\left(c_i^3\tau^6|\lambda_i|^6 - 12c_i^2\tau^4\left[\mathrm{Im}(\lambda_i)^4\,\mathrm{Re}(\lambda_i) + 2\,\mathrm{Im}(\lambda_i)^2\,\mathrm{Re}(\lambda_i)^3 + \mathrm{Re}(\lambda_i)^5\right]\right.\\ & \left. + c_i\tau^2\left[108\,\mathrm{Im}(\lambda_i)^4 + 156\,\mathrm{Im}(\lambda_i)^2\,\mathrm{Re}(\lambda_i)^2 + 48\,\mathrm{Re}(\lambda_i)^4\right] - 64\,\mathrm{Re}(\lambda_i)^3\right)^{\frac{1}{2}} + 64c_i^3\,\mathrm{Re}(\lambda_i)^3\end{aligned} \tag{34}$$

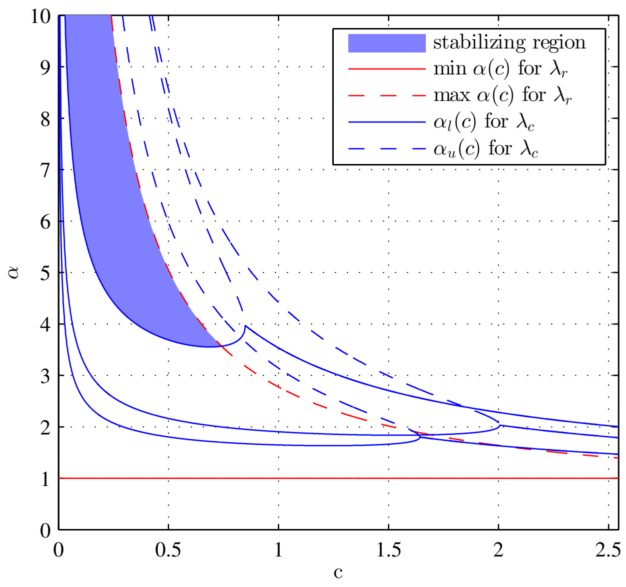


Fig. 7. Bounds on α depending on c

depends on the eigenvalues of the Laplace matrix. Analytic bounds on the parameters are given for arbitrary eigenvalues, such that the characteristic polynomial is stable. Here real as well as complex eigenvalues are considered. It is further shown exemplarily that for the case of a normalized Laplacian there always exists a non-empty parameter set, that leads to convergence.

APPENDIX

A. *Bounds for c and α*

For the complex eigenvalues $\lambda_i \in \lambda_c$ the upper bound of c_i is calculated as

$$c_u(\lambda_i) = \Upsilon + \frac{4\,\mathrm{Re}(\lambda_i)}{\tau^2|\lambda_i|^2} - \frac{36\,\mathrm{Im}(\lambda_i)^2}{\tau^4|\lambda_i|^4\Upsilon} \tag{35}$$

with Υ given in (31). For the complex eigenvalues $\lambda_i \in \lambda_c$ the bounds of α_i are calculated as in (32) and (33) with (34).

REFERENCES

[1] R. M. Murray, “Recent research in cooperative control of multivehicle systems,” *Journal of Dynamic Systems, Measurement, and Control*, vol. 129, no. 5, pp. 571–583, 2007.

[2] W. Ren and Y. Cao, *Distributed Coordination of Multi-agent Networks*. Springer-Verlag London Limited, 2011.

[3] J. A. Fax and R. M. Murray, “Information flow and cooperative control of vehicle formations,” *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1465–1476, 2004.

[4] U. Pilz, A. Popov, and H. Werner, “An information flow filter approach to cooperative vehicle control,” in *18th IFAC World Congress*, 2011, pp. 7432–7437.

[5] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Transactions on Automatic Control*, vol. 9, no. 49, pp. 1520–1533, 2004.

[6] R. Olfati-Saber, J. A. Fax, and R. M. Murray, “Consensus and cooperation in networked multi-agent systems,” *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[7] W. Ren, R. W. Beard, and E. M. Atkins, “Information consensus in multivehicle cooperative control,” *IEEE Control Systems Magazine*, vol. 27, no. 2, pp. 71–82, 2007.

[8] L. Peng, J. Yingmin, D. Junping, and Y. Shiying, “Distributed consensus control for second-order agents with fixed topology and time-delay,” in *26th Chinese Control Conference*, 2007, pp. 577–581.

[9] R. Carli, A. Chiuso, L. Schenato, and S. Zampieri, “Optimal synchronization for networks of noisy double integrators,” *IEEE Transaction on Automatic Control*, vol. 56, no. 5, pp. 1146–1152, 2011.

[10] W. Ren and Y. Cao, “Convergence of sampled-data consensus algorithms for double-integrator dynamics,” in *Proceedings of the IEEE Conference on Decision and Control*, 2008, pp. 3965–3970.

[11] D. W. Casbeer, R. Beard, and A. L. Swindlehurst, “Discrete double integrator consensus,” in *47th IEEE Conference on Decision and Control*, 2008, pp. 2264–2269.

[12] Y. Zhang and Y. Tian, “Consentability and protocol design of multi-agent systems with stochastic switching topology,” *Automatica*, vol. 45, no. 5, pp. 1195–1201, 2009.

[13] Y. Cao and W. Ren, “Sampled-data discrete-time coordination algorithms for double-integrator dynamics under dynamic directed interaction,” *International Journal of Control*, vol. 83, no. 3, pp. 506–515, 2010.

[14] M. Zhu and S. Martínez, “Discrete-time dynamic average consensus,” *Automatica*, vol. 46, no. 2, pp. 322–329, 2000.

[15] A. Eichler and H. Werner, “Closed-form solution for optimal convergence speed of multi-agent systems with discrete-time double-integrator dynamics,” *Systems & Control Letters*, 2013, submitted. [Online]. Available: http://www.tu-harburg.de/r̃tsae/rtsae_SCL.htm