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Robustness of Synchronization in Heterogeneous Multi-Agent Systems

Jaeyong Kim, Jongwook Yang, Hyungbo Shim, and Jung-Su Kim

***Abstract*— This paper studies robustness of synchronization in heterogeneous multi-agent systems, which is gained by interactions with other agents through the network. In order to effectively deal with the heterogeneous cases, we introduce the concept of the averaged dynamics which is the average of all agents' dynamics, and then claim that two sources enhance the robustness of the group behavior against differences among agents. First, we show that strong coupling among heterogeneous agents makes the trajectories of all agents remain in an arbitrarily small neighborhood of the trajectory of the averaged dynamics. Second, we observe that the amount of individual variation of each agent, that contributes to the averaged dynamics, gets smaller as the number of agents increases, and thus, the averaged dynamics becomes more robust to the differences among agents. Simulation results confirm our claim that a large number of agents with strong couplings have robust synchronization.**

I. INTRODUCTION

We study the behavior of a group of N dynamic agents represented by

$$\dot{x}_i = f_i(t, x_i) + u_i, \quad i \in \mathcal{N} = \{1, 2, \cdots, N\} \tag{1}$$

where $x_i \in \mathbb{R}$ is the state and $u_i \in \mathbb{R}$ indicates interactions with other agents through the network. Here the function $f_i(t, x_i)$ also includes possible time-varying disturbances entering each agent as well as parametric variations or uncertainties of each agent. And it is claimed that, under the interaction given by the diffusive-type coupling [1]

$$u_i = k \sum_{j=1}^{N} \alpha_{ij}(x_j - x_i) \tag{2}$$

where k represents the coupling strength and α_{ij} is the (i, j)-th entry of the adjacency matrix of the given network, the robustness of the group behavior can be enhanced. By the group behavior, we mean the set of solution $x_i(t)$'s. Robustness of group behavior is understood in the following two perspectives. First, it will be shown that each solution $x_i(t)$ satisfies that

$$\limsup_{t \to \infty} |x_i(t) - s(t)| \leq \sigma(1/k), \qquad \forall k > \bar{K} \tag{3}$$

where $\bar{K}$ is a minimal required strength and σ is a class-$\mathcal{K}$ function. Here, $s(t)$ is the solution of the *averaged dynamics*:

$$\dot{s}(t) = \frac{1}{N} \sum_{i=1}^{N} f_i(t, s(t)) \tag{4}$$

with the averaged initial condition $s(0) = \frac{1}{N} \sum_{i=1}^{N} x_i(0)$. It is seen from (3) that, even though the dynamics of each agent are all different, or uncertain, their behaviors approach close to that of $s(t)$ and remain close to each other when the coupling strength k is large. It is therefore the robustness of $x_i(t)$'s against the differences of f_i's in (1). Second, we claim that $s(t)$ becomes more robust against individual variation of f_i as the number of agents, N, increases. For example, if $f_i(t, x_i) = a_i x_i + \Delta_i$, where a_i and Δ_i are independent and identically distributed random variables with the average $\bar{a}$ and 0, respectively, then the averaged dynamics becomes

$$\dot{s} = \left(\frac{1}{N} \sum_{i=1}^{N} a_i \right) s + \frac{1}{N} \sum_{i=1}^{N} \Delta_i.$$

When N gets large, the effects of individual variations in a_i and Δ_i get weakened in the sense that the averaged dynamics can be regarded as $\dot{s} = \bar{a}s$ (which we may regard as a 'nominal' averaged dynamics) with the standard deviations are in the order of $1/\sqrt{N}$.

In the next subsection, some motivation of the studied problem is presented. Then, in Section II, we present a theorem on the robustness by strong coupling. Section III discusses the robustness by a large number of agents, and also illustrates that strong coupling and a large number of agents yield that every agent's dynamics synchronizes with the nominal averaged dynamics. Finally, Section IV concludes the paper.

A. *Motivation of the Study*

During the last decade, synchronization in engineering and nature has received considerable interests. This is because it turns out that there are several dynamical properties of synchronized dynamics that are not found in dynamics of an individual system [2], [3]. For instance, the circadian oscillator is the representative of this. Many life phenomena in biological systems are heavily dependent on the time of day. In order to provide these biological systems with *robust clocks*, circadian oscillators are required to show collective and synchronized behaviors. If the oscillators work separately, it might be easy to be destroyed by stochastic nature of biological systems. This can lead to a hazardous situation. To avoid such a situation, a robust clock is indispensable. The heart's pacemaker cells and spike-bursting neurons are good examples of these kinds of synchronization behaviors [2], [4].

As mentioned above, an example of remarkable properties of synchronized dynamics is robustness. It is reported

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J. Kim, J. Yang, and H. Shim are with ASRI, Department of Electrical Engineering and Computer Science, Seoul National University, Korea.

J.-S. Kim is with Department of Control and Instrumentation Engineering, Seoul National University of Science and Technology, Korea.

that synchronized dynamics via coupling between oscillators enhances robustness of oscillating dynamics against noise or parameter variations; see [5]–[7] and references therein. This robustness implies that the oscillating behavior under noise or parameter variations is as close as possible to the nominal oscillating behavior (protected from them). This means that although an individual oscillating behavior is not robust under the uncertainties, collective oscillators can be indeed so by achieving synchronization via coupling between them.

This paper focuses on the question; what is the most general principle behind this finding? To be more specific, this finding might hint at (or motivates to make) the hypothesis that a large-scale synchronization-like behavior shows robustness against noise. As already mentioned, the noise can be attenuated in the synchronization-like behavior if the number of agents N is sufficiently large. Thus, in this paper, we will consider the interconnected systems to be robust against noise if their behaviors are close to their averaged dynamics without noise.

In the case of the identical agents with noise, both strong coupling and a large number of agents are required to achieve the noise-free synchronization [5], [7]. Under these preconditions (strong coupling and a large number of agents), the behavior of the identical individual dynamics is arbitrary close to that of a noise-free system [5]. However, the behavior of complex networks with nonidentical systems is much more complicated than that of the identical case [15]. In this paper, we introduce the averaged dynamics (4) which is the average not of the states, but of the dynamics, in order to effectively describe the group behavior of nonidentical subsystems. Then, we will show that strong coupling guarantees the synchronization with the averaged dynamics, and this will show that strong coupling contributes to the robustness. Furthermore, as seen in the above example (i.e., $f_i(t, x_i) = a_i x_i + \Delta_i$), the noise attenuation effect of a large number of agents is better understood in the averaged dynamics.

On the other hand, the considered problem can be viewed as achieving the *practical consensus* from the viewpoint of the consensus problems discussed in, e.g., [8]–[12]. The practical consensus problem can be defined as follows: for any given $\epsilon > 0$, design the consensus controller (2) for the multi-agent system (1) such that

$$\limsup_{t\to\infty} |x_i(t) - x_j(t)| \le \epsilon, \quad \forall i, j \in \mathcal{N}.$$

This approximate consensus is the best we can achieve because (1) is the heterogeneous multi-agent system for which exact (or asymptotic) consensus is not possible if there is no common internal model among them. See [13]–[15] for details. In particular, Zhao *et al.* [15] have considered a similar problem, but they assume that the averaged solution $s(t)$ of (4) is also a solution of each subsystems, i.e., $\dot{s}(t) = f_i(t, s(t))$, $i \in \mathcal{N}$. Unlike this, the averaged solution $s(t)$ may not be the solution to any subsystem (1) in this paper.

B. Notation

An undirected graph is denoted by $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$, where $\mathcal{N} = \{1, 2, \ldots, N\}$ is a finite nonempty set of nodes, $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is an edge set of ordered pairs of nodes, and the *adjacency matrix* $\mathcal{A} = [\alpha_{ij}] \in \mathbb{R}^{N\times N}$ is defined such that $\alpha_{ji} = 1$ if $(i, j) \in \mathcal{E}$, while $\alpha_{ji} = 0$ if $(i, j) \notin \mathcal{E}$. The (symmetric) *Laplacian* matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ of $\mathcal{G}$ is defined as $l_{ii} := \sum_{j\neq i} \alpha_{ij}$ and $l_{ij} := -\alpha_{ij}$ for all $i \neq j$. By its construction, it contains at least one eigenvalue of zero, whose corresponding eigenvector is 1_N ($N \times 1$ vector comprising all ones), and all the other eigenvalues are nonnegative. Thus, we sort them as $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$, where λ_i's are eigenvalues of $\mathcal{L}$. For matrices $A_1, \cdots, A_k$, the block diagonal matrix whose i-th diagonal entry is A_i is denoted by $\mathrm{diag}(A_1, \cdots, A_k)$. For a vector x and a matrix A, $|x|$ and $\|A\|$ denote the Euclidean norm and the induced matrix 2-norm, respectively.

II. Robustness by Strong Coupling

We study the problem under the following assumptions.

Assumption 1: (Individual system) The function $f_i(t, x_i)$ of the individual system (1) is uniformly bounded in t, continuously differentiable, and globally Lipschitz in x_i uniformly in t; i.e., there exist a non-decreasing continuous function $M : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ and a constant L such that

$$|f_i(t, a)| \le M(|a|), \quad \left|\frac{\partial f_i}{\partial x_i}(t, x_i)\right| \le L, \qquad \forall x_i \in \mathbb{R},\ \forall t \ge 0,\ \forall i \in \mathcal{N}. \quad (5)$$

Assumption 1 guarantees the uniqueness of solution for the individual system. Moreover, if the domain of f_i is restricted, a constant L can always be found. This assumption makes the analysis simple.

By letting $x := [x_1, \ldots, x_N]^T$ and $f(t, x) := [f_1(t, x_1), \ldots, f_N(t, x_N)]^T$, the dynamics of the overall system, composed of (1) and (2), is written as

$$\dot{x} = -k\mathcal{L}x + f(t, x), \quad (6)$$

where $\mathcal{L}$ is the Laplacian matrix describing the network connection.

Assumption 2: (Network property) The coupling network topology under consideration is undirected and connected.

A direct consequence of the assumption is that the Laplacian matrix $\mathcal{L}$ is symmetric and has zero eigenvalue which is simple [10]. Therefore, by Schur's lemma, there exists a normal (real unitary) matrix $U \in \mathbb{R}^{N\times N}$ such that $U\mathcal{L}U^{-1} = \mathrm{diag}(0, \Lambda)$ with a real diagonal matrix $\Lambda \in \mathbb{R}^{(N-1)\times(N-1)}$. From the property of the Laplacian matrix of a connected symmetric graph, it follows that $\Lambda = \mathrm{diag}(\lambda_2, \lambda_3, \cdots, \lambda_N)$ with all positive λ_i's, and without loss of generality, we suppose that the first row of U is $(1/\sqrt{N})[1, 1, \cdots, 1]$. Define the matrix $W := (1/\sqrt{N})U$. Then,

$$W = \begin{bmatrix} \frac{1}{N} 1_N^T \\ R^T \end{bmatrix}, \qquad W^{-1} = [1_N,\ Q],$$

where R and Q are real matrices of size $N \times (N-1)$ such that $R^T R = (1/N)I$, $Q^T Q = NI$, $R^T 1_N = 0$, $Q^T 1_N = 0$, and $R^T Q = I$. Hence, $\|Q\| = \sqrt{N}$ and $\|R\| = 1/\sqrt{N}$.

Now, for the averaged dynamics (4) which is simply written as

$$\dot{s} = \frac{1}{N} 1_N^T f(t, 1_N s), \tag{7}$$

the following properties are required in the analysis.

Assumption 3: (Averaged dynamics) (i) The solution $s(t)$ from any initial condition is ultimately bounded in the sense that there exists B such that

$$\limsup_{t\to\infty} |s(t)| \le B. \tag{8}$$

(ii) The average of Jacobians of individual systems is strictly negative. More specifically, there exists a constant $p > 0$ such that

$$\frac{1}{N}\frac{\partial(1_N^T f(t,x))}{\partial x} 1_N = \frac{1}{N}\sum_{i=1}^N \frac{\partial f_i}{\partial x_i}(t, x_i) \le -p, \qquad \forall x_i \in \mathbb{R},\ \forall t \ge 0. \tag{9}$$

Remark 1: The condition (9) is stronger than asking that the averaged dynamics (7) is a contracting system [16] since, from (9) with $x = 1_N s$, it holds that $(1/N)(\partial 1_N^T f(t, 1_N s))/(\partial s) \le -p$ for all $s \in \mathbb{R}$. $\diamondsuit$

Under the assumptions so far, we obtain the following theorem.

Theorem 1: Under Assumptions 1, 2, and 3, there exists a class-$\mathcal{K}$ function σ^* such that the solutions of the overall system, composed of (1) and (2), with arbitrary initial conditions and the solution $s(t)$ to the averaged dynamics (4) with $s(0) = \frac{1}{N}\sum_{i=1}^N x_i(0)$ satisfy

$$\limsup_{t\to\infty} |x_i(t) - s(t)| \le \sigma^*\left(\frac{1}{k\lambda_2 - L}\right), \quad \forall k > \bar{K}, \tag{10}$$

for all $i = 1, \dots, N$, where

$$\bar{K} = \frac{L^2}{p\lambda_2} + \frac{L}{\lambda_2}. \tag{11}$$

In particular, the function σ^* is defined on $[0, p/L^2)$ and given by

$$\sigma^*(\chi) = M(B)\sqrt{N}\sqrt{r(\chi)} \tag{12}$$

in which,

$$r(\chi) = \begin{cases} 0, & \chi = 0, \\ \frac{2\chi}{p - L^2\chi}, & 0 < \chi \le \frac{2p}{p^2 + 4L^2}, \\ \frac{(p^2 + 2L^2)\chi^2}{(p - L^2\chi)^2}, & \frac{2p}{p^2+4L^2} < \chi < \frac{p}{L^2}. \end{cases} \tag{13}$$

Remark 2: The ultimate bound expressed by the function σ^* and the value of $\bar{K}$ may be conservative. However, the current expressions (11) and (12) yield a reasonable interpretation. For example, (11) indicates that the minimal coupling strength k increases as the Lipschitz constant L increases while it decreases as the degree of stability p and the smallest non-zero eigenvalue λ_2 of the network get larger.

Remark 3: Theorem 1 may be considered as a solution to the practical consensus problem (for the multi-agent system (1)), as discussed in the Introduction. In fact, for any given ϵ, there is a sufficiently large k such that $\limsup_{t\to\infty} |x_i(t) - x_j(t)| \le \epsilon$ for $i, j \in \mathcal{N}$. Since the terminology 'practical consensus' is used differently in [17] where just boundedness of the difference $|x_i(t) - x_j(t)|$ is of interest, we emphasize that the error could be made arbitrarily small in Theorem 1.

A. Proof of Theorem 1

By the coordinate transformation

$$\xi = \begin{bmatrix} \xi_1 \\ \tilde{\xi} \end{bmatrix} = Wx = \begin{bmatrix} \frac{1}{N}1_N^T \\ R^T \end{bmatrix} x \tag{14}$$

where $\tilde{\xi} = [\xi_2, \dots, \xi_N]^T$, the overall system (6) is transformed into

$$\begin{aligned} \dot{\xi}_1 &= \frac{1}{N}1_N^T f(t, 1_N\xi_1 + Q\tilde{\xi}) \\ \dot{\tilde{\xi}} &= -k\Lambda\tilde{\xi} + R^T f(t, 1_N\xi_1 + Q\tilde{\xi}) \end{aligned} \tag{15}$$

because $W^{-1}\xi = 1_N\xi_1 + Q\tilde{\xi}$. With $e := \xi_1 - s$, equation (15) can be rewritten as

$$\dot{e} = \frac{1}{N}1_N^T f(t, 1_N e + 1_N s + Q\tilde{\xi}) - \frac{1}{N}1_N^T f(t, 1_N s) \tag{16a}$$

$$\dot{\tilde{\xi}} = -k\Lambda\tilde{\xi} + R^T f(t, 1_N e + 1_N s + Q\tilde{\xi}). \tag{16b}$$

Let a Lyapunov function be

$$V(e, \tilde{\xi}) = \frac{1}{2}e^2 + \frac{1}{2}|\tilde{\xi}|^2.$$

Then, the time derivative of V along (16) becomes

$$\begin{aligned} \dot{V} = \frac{e}{N}&\left[1_N^T f(t, 1_N e + 1_N s + Q\tilde{\xi}) - 1_N^T f(t, 1_N s)\right] \\ &- k\tilde{\xi}^T\Lambda\tilde{\xi} + \left[\tilde{\xi}^T R^T f(t, 1_N e + 1_N s + Q\tilde{\xi})\right. \\ &\left. - \tilde{\xi}^T R^T f(t, 1_N s)\right] + \tilde{\xi}^T R^T f(t, 1_N s). \end{aligned}$$

By the mean-value theorem, we obtain

$$\begin{aligned} \dot{V} = \frac{e}{N}&\frac{\partial(1_N^T f)}{\partial x}\Big|_z \cdot (1_N e + Q\tilde{\xi}) \\ &- k\tilde{\xi}^T\Lambda\tilde{\xi} + \frac{\partial(\tilde{\xi}^T R^T f)}{\partial x}\Big|_w \cdot (1_N e + Q\tilde{\xi}) \\ &+ \tilde{\xi}^T R^T f(t, 1_N s) \end{aligned}$$

in which, $z \in \mathbb{R}^N$ and $w \in \mathbb{R}^N$ are some points on the line segment connecting $1_N e + Q\tilde{\xi} + 1_N s$ and $1_N s$. Since

$$\frac{\partial(\tilde{\xi}^T R^T f)}{\partial x}\Big|_w = \tilde{\xi}^T R^T \mathrm{diag}\Big(\frac{\partial f_1}{\partial x_1}(t, w_1), \cdots, \frac{\partial f_N}{\partial x_N}(t, w_N)\Big),$$

it is seen by (5) that

$$\left|\frac{\partial(\tilde{\xi}^T R^T f)}{\partial x}\Big|_w\right| \le L\|R\||\tilde{\xi}|, \quad \forall t \ge 0 \tag{17}$$

and similarly that

$$\left|\frac{\partial(1_N^T f)}{\partial x}\Big|_z\right| \le L\sqrt{N}, \quad \forall t \ge 0. \tag{18}$$

Therefore, using (9) and the fact that $\|Q\| = \sqrt{N}$ and $\|R\| = 1/\sqrt{N}$, it follows that

$$\begin{aligned}\dot V &\le -p|e|^2 + \frac{L\|Q\|}{\sqrt{N}}|e||\tilde\xi| - k\lambda_2|\tilde\xi|^2 \\ &\quad + L\|R\||\tilde\xi|(\sqrt{N}|e| + \|Q\||\tilde\xi|) + |R^T f(t, 1_N s)||\tilde\xi| \\ &\le -p|e|^2 + 2L|e||\tilde\xi| - (k\lambda_2 - L)|\tilde\xi|^2 + |R^T f(t, 1_N s)||\tilde\xi|.\end{aligned}$$

With $k_1 := k\lambda_2 - L$ and $a = -L$, the following lemma can be employed to find the region for $\dot V < 0$.

Lemma 1: Let

$$\rho_{k_1}(x, y) = -\begin{bmatrix}|x| \\ |y|\end{bmatrix}^T \begin{bmatrix}p & a \\ a & k_1\end{bmatrix}\begin{bmatrix}|x| \\ |y|\end{bmatrix} + M|y|$$

with $x \in \mathbb{R}^l$, $y \in \mathbb{R}^m$, $p > 0$, $M > 0$ and a is a constant. Then, there is a class-$\mathcal{K}$ function $r(\cdot)$ such that

$$\rho_{k_1}(x, y) < 0 \quad \text{on } \{(x, y) : |x|^2 + |y|^2 > M^2 r(1/k_1)\},$$

for all $k_1 > a^2/p$. $\Diamond$

Proof: Note that

$$\begin{aligned}\rho_{k_1}(x, y) &= -p|x|^2 - 2a|x||y| - k_1|y|^2 + M|y| \\ &= -p\left(|x| + \frac{a}{p}|y|\right)^2 - \left(k_1 - \frac{a^2}{p}\right)|y|^2 + M|y|.\end{aligned}$$

Let $\delta(k_1) := k_1 - a^2/p$ and $k_1 > a^2/p$ so that $\delta(k_1) > 0$, and let

$$Y_{k_1}(x, y) := \left[\frac{\sqrt{p}}{\sqrt{\delta(k_1)}}(|x| + \frac{a}{p}|y|),\ |y|\right]^T.$$

Then, since $|y| \le |Y_{k_1}(x, z)|$,

$$\begin{aligned}\rho_{k_1}(x, y) &= -\delta(k_1)\left[\frac{p}{\delta(k_1)}\left(|x| + \frac{a}{p}|y|\right)^2 + |y|^2\right] + M|y| \\ &\le -\delta(k_1)|Y_{k_1}(x, y)|^2 + M|Y_{k_1}(x, y)|.\end{aligned}$$

Therefore, $\rho_{k_1}(x, y) < 0$ if $|Y_{k_1}(x, y)| > M/\delta(k_1)$.

For convenience, let

$$Z := \frac{\sqrt{p}}{\sqrt{\delta(k_1)}}(|x| + \frac{a}{p}|y|).$$

Then, we have

$$\begin{aligned}|x|^2 + |y|^2 &= \left(\frac{\sqrt{\delta(k_1)}}{\sqrt{p}}Z - \frac{a}{p}|y|\right)^2 + |y|^2 \\ &\le \frac{2\delta(k_1)}{p}Z^2 + \frac{2a^2}{p^2}|y|^2 + |y|^2 \\ &\le \eta(k_1)|Y_{k_1}(x, y)|^2,\end{aligned}$$

where $\eta(k_1) = \max\left\{2\delta(k_1)/p, 1 + 2a^2/p^2\right\}$. Define $r : [0, p/a^2) \to [0, \infty)$ as follows:

$$r(\chi) := \begin{cases}0 & \text{if } \chi = 0, \\ \sup_{\mu \ge \frac{1}{\chi}} \frac{\eta(\mu)}{\delta^2(\mu)} & \text{if } 0 < \chi < \frac{p}{a^2}.\end{cases}$$

(If $a = 0$, the number p/a^2 is replaced by ∞.) Then, it can be verified that the function $r(\chi)$ belongs to class-$\mathcal{K}$ because it is equivalently written as (13).

Finally, if $|x|^2 + |y|^2 > M^2 r(1/k_1)$, then

$$\begin{aligned}|Y_{k_1}(x, y)|^2 &\ge \frac{1}{\eta(k_1)}\left(|x|^2 + |y|^2\right) > \frac{M^2}{\eta(k_1)}r\left(\frac{1}{k_1}\right) \\ &\ge \frac{M^2}{\eta(k_1)}\frac{\eta(k_1)}{\delta^2(k_1)} = \left(\frac{M}{\delta(k_1)}\right)^2,\end{aligned}$$

and thus $\rho_{k_1}(x, y) < 0$. This completes the proof. ■

By Lemma 1, it is seen that

$$\dot V < 0 \quad \text{if } 2V = e^2 + |\tilde\xi|^2 > |R^T f(t, 1_N s)|^2 r(1/k_1),$$

which implies that

$$\limsup_{t\to\infty} 2V(t) \le \limsup_{t\to\infty} |R^T f(t, 1_N s(t))|^2 r(1/k_1). \quad (19)$$

By (5) and (8), we have that

$$\begin{aligned}\limsup_{t\to\infty} |R^T f(t, 1_N s(t))| &\le \|R\|\sqrt{N}M(\limsup_{t\to\infty} |s(t)|) \\ &\le M(B). \end{aligned} \quad (20)$$

Finally, note that

$$x - 1_N s = W^{-1}\xi - 1_N s = 1_N\xi_1 - 1_N s + Q\tilde\xi = [1_N, Q]\begin{bmatrix}e \\ \tilde\xi\end{bmatrix}.$$

Then, the vector norm of the i-th row of $[1_N, Q]$ is $\sqrt{N}$ by the construction of W, and thus,

$$|x_i - s| \le \sqrt{N}\sqrt{|e|^2 + |\tilde\xi|^2} = \sqrt{N}\sqrt{2V}.$$

Therefore, for any $i \in \mathcal{N}$,

$$\limsup_{t\to\infty} |x_i(t) - s(t)| \le M(B)\sqrt{N}\sqrt{r(1/k_1)} \quad (21)$$

if $k_1 = k\lambda_2 - L > L^2/p$. From this, the class-$\mathcal{K}$ function σ^* in (12) and the constant $\bar K$ of (11) are found.

B. Discussions

The proof of Theorem 1 enlightens the following:

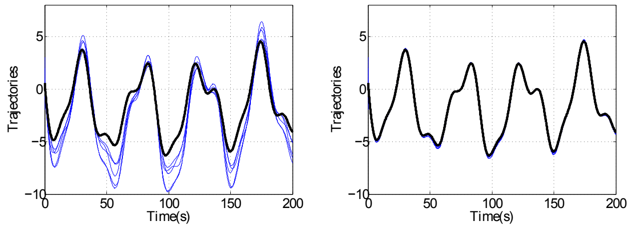
1) The quantity $|R^T f(t, 1_N s)|$ has the meaning of 'measure of heterogeneity' in the sense that, if all agents are identical; $f_i(t, s) = f_0(t, s)$ for all s and $i \in \mathcal{N}$, then $R^T f(t, 1_N s) = R^T 1_N f_0(t, s) = 0$. More specifically, if we denote the first column of R^T by r_1 so that $R^T = [r_1, \tilde R]$ with a matrix $\tilde R$, then it follows from $R^T 1_N = 0$ that $r_1 = -\tilde R 1_{N-1}$. Hence,

$$R^T f(t, s) = [r_1, \tilde R]\begin{bmatrix}f_1(t, s) \\ \vdots \\ f_N(t, s)\end{bmatrix} = \tilde R\begin{bmatrix}f_2(t, s) - f_1(t, s) \\ \vdots \\ f_N(t, s) - f_1(t, s)\end{bmatrix}.$$

2) If $|R^T f(t, 1_N s)| = 0$, the consensus is made for any positive k. This can be easily seen from (16b), where $\lim_{t\to\infty}\tilde\xi(t) = 0$ implies $\lim_{t\to\infty}|x_i(t) - x_1(t)| = 0$ for all $i = 2, \ldots, N$, because

$$\tilde\xi = R^T x = \tilde R\begin{bmatrix}x_2 - x_1 \\ \vdots \\ x_N - x_1\end{bmatrix}.$$

Even in this case, it is seen from (16a) that convergence of $e(t)$ to zero (or, $x_i(t)$ to $s(t)$) requires a certain stability, such as incremental stability, of the averaged dynamics (4).



(a) $N = 5,\ k = 5$ (b) $N = 5,\ k = 50$

Fig. 1. Simulation results with $k = 5$ and $k = 50$. The thin blue and the thick black lines represent the trajectories of 5-agent systems and the trajectory $s(t)$ of the averaged dynamics, respectively.

C. Simulation

Theorem 1 means that as the coupling strength k gets larger, the group behavior becomes closer to the trajectory of the averaged dynamics. To see this, let us consider the following group of heterogeneous subsystems: $f_1(t, x_1) = -x_1 + 2.7\sin(0.35t + 57.3)$, $f_2(t, x_2) = -0.75x_2 + 1.4\sin(0.22t + 40.1)$, $f_3(t, x_3) = -0.5x_3 + 4.7\sin(0.35t + 10.3)$, $f_4(t, x_4) = 0.5x_4 + 4.5\sin(0.13t + 17.2)$, and $f_5(t, x_5) = 0.75x_5 + 2\sin(0.01t + 23)$. In this case, the averaged dynamics of the subsystems is obtained as

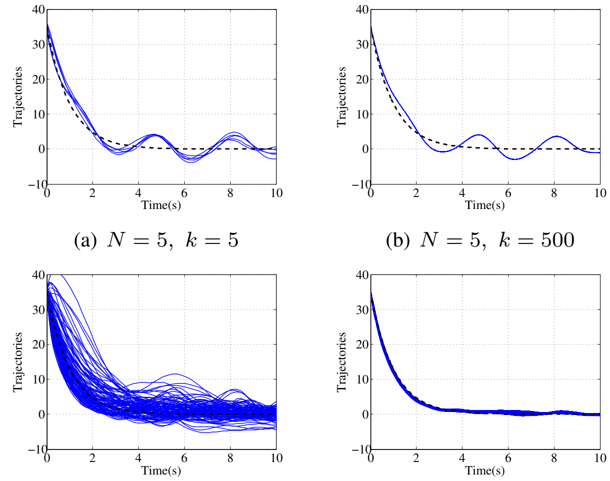
$$\begin{aligned}\dot{s} = &- s + \frac{1}{5}\big\{2.7\sin(0.35t + 57.3) + 1.4\sin(0.22t + 40.1)\\ &+ 4.7\sin(0.35t + 10.3) + 4.5\sin(0.13t + 17.2)\\ &+ 2\sin(0.01t + 23)\big\}.\end{aligned}$$

Although the group includes two unstable agents, the averaged dynamics is stable.

Here, we assume the network topology is the ring network with unit weights, in which each node connects to exactly two other nodes, forming a single continuous pathway for signals through each node. With $N = 5$, the second smallest eigenvalue of the Laplacian matrix is $\lambda_2 = 1.382$. The simulation results show that by increasing the coupling strength $k > \bar{K} = 1.447$ (with $L = 1$ and $p = 1$), the behaviors of all agents approach the trajectory of the averaged dynamics; see Fig. 1.

III. Robustness by a Large Number of Agents

In this section we illustrate, through simulation studies, that a large number of agents with strong couplings have robust synchronization. Here the robust synchronization comes from two sources; one is from the fact that the amount of individual variation of each agent, that contributes to the averaged dynamics, gets smaller as the number of agents increases, so that the averaged dynamics becomes more robust to those variations from heterogeneity. Depending on the cases, increasing number of agents may cause decrease of the second smallest eigenvalue λ_2 of the network Laplacian. As a result, the deviation of individual trajectory from the solution $s(t)$ of the averaged dynamics may get larger. To compensate this (or to maintain the same level of the deviation), the coupling gain k may need to be increased,



(a) $N = 5,\ k = 5$ (b) $N = 5,\ k = 500$

(c) $N = 100,\ k = 5$ (d) $N = 100,\ k = 500$

Fig. 2. Trajectories of N-agent systems with coupling strength k are depicted as blue solid curves, and the trajectory $s_0(t)$ of the nominal averaged system is given as the black dashed curve.

which is the second source of robustness as discussed in the previous section.

As an example, consider a group of N agents given by

$$f_i(t, x_i) = (-1 + \Delta_i)x_i + m_i \sin(2w_i t + \theta_i),$$

for $i = 1, 2, \ldots, N$, where Δ_i, m_i, and w_i are random variables of standard normal distribution $N(0, 1)$, and θ_i is a random variable of uniform distribution on $[0, 2\pi]$. It is assumed that these agents are interconnected by the ring network. The simulation results of sample runs with different N and k are given in Fig. 2, which can be interpreted as follows.

The effect of strong coupling k is seen rather clearly by comparing Fig. 2.(a) with (b), and (c) with (d), respectively. On the other hand, by comparing (a) with (c), and (b) with (d), it is seen that increasing N (under the same k) results in more deviation. This is because, in the case of the ring network with unit weights, increasing N leads to decrease of the second smallest eigenvalue λ_2 of the Laplacian [18]. (See also (10).) Therefore, in order to maintain the same level of deviations, the coupling strength k needs to be increased as well.

On the other hand, it is observed in the figure that, as N increases, the solutions of each agents (solid blue) are more concentrated around the dashed black curve, which is the solution of the *nominal* averaged system

$$\dot{s}_0 = -s_0, \qquad s_0(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0).$$

This system is introduced just for comparison, which is obtained from the averaged dynamics

$$\dot{s} = \left(-1 + \frac{1}{N}\sum_{i=1}^{N}\Delta_i\right)s + \frac{1}{N}\sum_{i=1}^{N} m_i \sin(2w_i t + \theta_i)$$

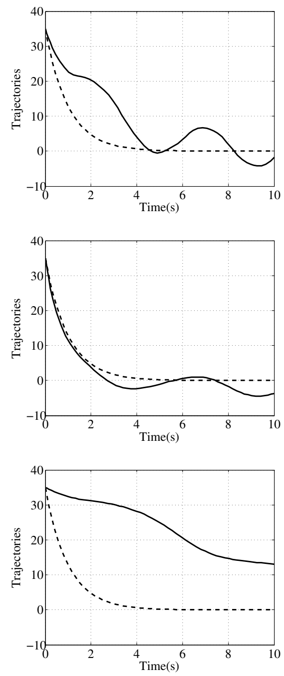


Fig. 3. Plots of $s(t)$ (solid) from 6 sample runs for $N = 5$. The solution $s_0(t)$ of the nominal averaged system is also drawn (dashed).

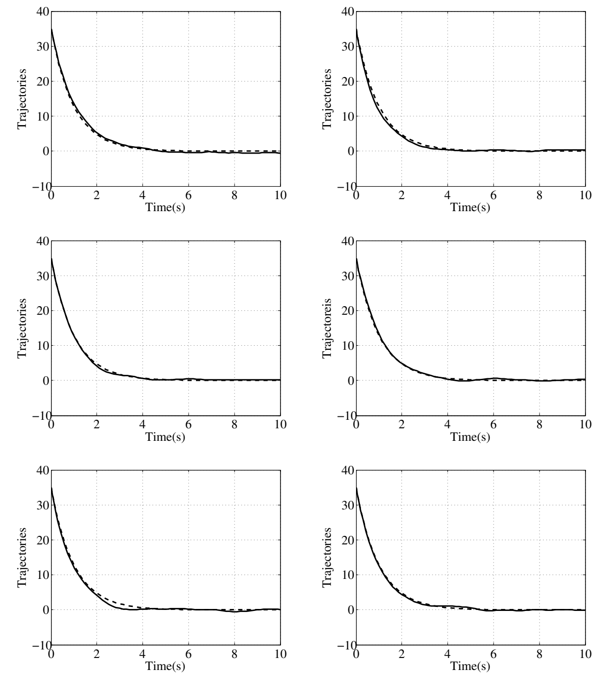


Fig. 4. Plots of $s(t)$ (solid) from 6 sample runs for $N = 100$. The solution $s_0(t)$ is also drawn (dashed).

by the fact that expectations of Δ_i and m_i are all zero. Indeed, it can be observed by Monte Carlo experiments that the solution $s(t)$ tends to $s_0(t)$ as N increases. For example, Fig. 3 is taken from six random samples of the group of 5 agents, and shows the trajectories of $s(t)$ (solid) and $s_0(t)$ (dashed), while Fig. 4 is for the case of 100 agents. It is seen that, as N increases, (the solution of) the averaged dynamics from the random samples becomes closer to (the solution of) the nominal averaged system, as expected. This observation confirms the results in Fig. 2.

IV. Concluding Remarks

We have claimed that strong coupling and a large number of agents both enhance robustness of the networked group behavior. While strong coupling enforces the behavior of each agent tend to that of averaged dynamics, a large number of agents attenuate the effects of differences among agents into the averaged dynamics. This phenomenon is illustrated by simulation study, which confirms that a large number of agents with strong coupling yields robust synchronization. Our future work will provide with a mathematical analysis of the robustness by a large number of agents.

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