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Stability analysis of networked systems with similar dynamics

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Abstract— **The paper concerns systems that are composed of subsystems with similar dynamics. A method for analyzing the stability of such systems is developed, where the subsystems are decomposed into nominal models with identical dynamics and individual models describing the deviation of the subsystems from the common nominal models. The analysis is divided into three steps. First, the stability of the network controlled and physically interconnected nominal subsystems is checked by a state transformation based on the structure of the interconnection. The second step is to find an upper bound for the dynamics of the deviation model. The third step is to prove the stability of the overall system by considering the stabilized nominal system and the error bound. A sufficient condition for the stability of the overall system is derived and the method is demonstrated by proving the stability of platooning vehicles.**

I. Introduction

The use of communication networks enables systems to have an information exchange among their subsystems which are spatially distributed but have a physical interconnection (Fig. 1a). The control performance of the local controllers can be improved by using the information channels to exchange appropriate data between the subsystems. If the controllers of the subsystems alone lead to a sufficient performance, no communication is necessary and the controller of the subsystems only uses local information to generate the control input. If a subsystem state exceeds a threshold, the controllers have to cooperate by exchanging information in order to guarantee the required performance of the overall system.

There exist various design and analysis methods for interconnected subsystems with identical dynamics (cf. [2], [3], [4]). However, in many technological systems, the subsystems just have similar dynamics rather than being identical, e.g. platooning vehicles with different loads and different engines or multi-area power systems that have different power generation capacities. For these systems the results in literature have to be extended to systems with similar subsystems. Such an extension is presented in this paper.

A. Idea of the stability analysis

The idea of the stability analysis developed in this paper is to decompose all subsystem models into a nominal model $\Sigma_{\mathrm{n}i}$, $(i \in \mathcal{N} := \{1, \ldots, N\})$, which is identical for all subsystems, and into an error model $\Sigma_{\mathrm{e}i}$ that describes the deviation of the subsystem dynamics from this common nominal model $\Sigma_{\mathrm{n}i}$. The subsystems $\Sigma_{\mathrm{d}i}$, $(i \in \mathcal{N})$ are physically interconnected (denoted by $\boldsymbol{L}_{\mathrm{p}}$) and interconnected over the local controllers C_i, which use the communication network (denoted by $\boldsymbol{L}_{\mathrm{c}}$) as shown in Fig. 1b. When coupling these controlled nominal subsystems $\bar{\Sigma}_{\mathrm{n}i}$, $(i \in \mathcal{N})$ through $\boldsymbol{L}_{\mathrm{p}}$ and $\boldsymbol{L}_{\mathrm{c}}$, a nominal overall system Σ_{n} results (Fig. 1c). Thus, every controlled nominal subsystem $\bar{\Sigma}_{\mathrm{n}i}$, $(i \in \mathcal{N})$ has a corresponding error subsystem $\Sigma_{\mathrm{e}i}$, which together build the controlled subsystem $\bar{\Sigma}_{\mathrm{d}i}$.

The benefit of this decomposition is twofold. First, the controllers C_i, $(i \in \mathcal{N})$ can be designed for the identical subsystems $\Sigma_{\mathrm{n}i}$ by using the methods that have been developed in literature for systems composed of identical subsystems (cf. [2], [5]). Second, the behavior of the overall system can be analyzed by separately considering the system Σ_{n}, which is composed of identical controlled subsystems $\bar{\Sigma}_{\mathrm{n}i}$, $(i \in \mathcal{N})$, and the models $\Sigma_{\mathrm{e}i}$ describing the error bounds. If the error bounds satisfy the stability test elaborated in this paper, this analysis shows that the subsystems are similar enough to be controlled by the identical designed controllers.

B. Interconnection of the subsystems

There exist three interconnection cases that can be described by the structure in Fig. 1:

1) **Control of multi-agent systems**: No physical interconnection and an arbitrary information exchange among the controllers ($\boldsymbol{L}_{\mathrm{p}} = \boldsymbol{0}$)
2) **Decentralized control of physically interconnected systems**: An arbitrary physical interconnection and no information exchange among the controllers ($\boldsymbol{L}_{\mathrm{c}} = \boldsymbol{0}$)
3) **Networked control of physically interconnected systems**: An arbitrary physical interconnection and an arbitrary information exchange among the controllers

The presented stability analysis holds for all three cases. If a control structure which is divided into an *autonomous control mode* and a *cooperative control mode* as described in [3] is considered, the stability analysis has to be performed for both communication modes separately.

C. Literature

There are several papers that deal with analysis and design problems for interconnected non-identical subsystems and which perform a decomposition, e.g. overlapping systems [13], [15] or the decomposition of symmetric interconnected systems [11], [12]. Furthermore, consensus and synchronization problems of heterogeneous subsystems are investigated in [9], [14].

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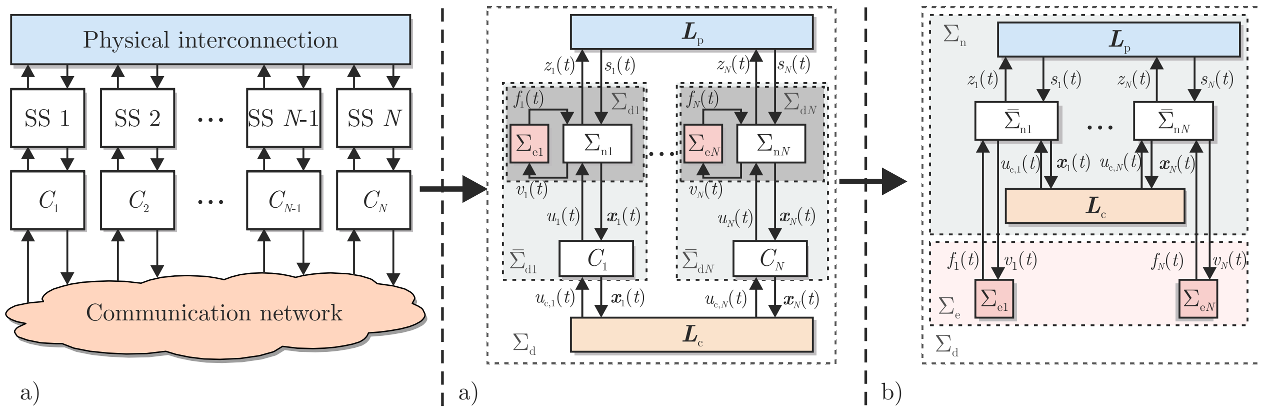

Fig. 1. a) Network controlled subsystems (SS) with similar dynamics with local controllers (C_i); b) The overall system Σ_d consists of N subsystems $\Sigma_{\mathrm{d}i}$ ("$\Sigma_{\mathrm{d}i} = \Sigma_{\mathrm{n}i} + \Sigma_{\mathrm{e}i}$"), that are physically interconnected ($\boldsymbol{L}_\mathrm{P}$) and interconnected over the local controllers C_i which use the communication network ($\boldsymbol{L}_\mathrm{C}$); c) Structure for the stability analysis with Σ_d consisting of the controlled nominal system Σ_n and the error system Σ_e

In this paper, methods for networked systems with the structure shown in Fig. 1a are developed, which use approaches based on the decomposition of symmetric composite systems. The main idea for the stability analysis of interconnected subsystems with similar dynamics has been introduced in [11], [12] for symmetrical interconnections. Here, this approach is extended to arbitrary interconnections among subsystems with similar dynamics and an additional interconnection over a communication network is considered.

Furthermore, results for the stability analysis of subsystems with identical dynamics are used, in particular from vehicle formations (multi-agent systems) introduced in [5] and a decomposition approach for physically interconnected subsystems [2]. A part of the stability analysis presented here uses robust control approaches based on the small-gain theorem presented in [8], [16].

D. Overview

This paper is organized as follows. The model and structure of the overall system is introduced in Sec. II. Section III deals with the stability of the overall system and is divided into the analysis of the controlled nominal overall system Σ_n, the error overall system Σ_e and the overall system Σ_d. The main result of this paper is a method for analyzing the stability of the overall system which consists of similar subsystems presented in Theorem 1. Section IV discusses possible extensions of the stability analysis, e.g. special interconnections structures or nonlinear system models. In Sec. V the stability analysis is applied to the example of vehicle platooning.

II. System model

A. Subsystem models

According to Fig. 1b, the overall system Σ_d consists of the subsystems $\Sigma_{\mathrm{d}i}$, $(i\in\mathcal{N})$ each of which is controlled by an associated controller C_i. The subsystems are physically coupled ($\boldsymbol{L}_\mathrm{p}$), whereas the controllers communicate over the network ($\boldsymbol{L}_\mathrm{c}$).

In the following, the models of the subsystems $\Sigma_{\mathrm{d}i}$, $(i\in\mathcal{N})$ are represented by a nominal model $\Sigma_{\mathrm{n}i}$ together with an error model $\Sigma_{\mathrm{e}i}$:

$$\Sigma_{\mathrm{d}i}\!:\begin{cases}\Sigma_{\mathrm{n}i}\!:\begin{cases}\dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}\boldsymbol{x}_i(t) + \boldsymbol{b}u_i(t) + \boldsymbol{e}s_i(t) + \boldsymbol{p}f_i(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad \boldsymbol{x}_i(0) = \boldsymbol{x}_{0i} \\ y_i(t) = \boldsymbol{c}^\mathrm{T}\boldsymbol{x}_i(t) \\ z_i(t) = \boldsymbol{c}_\mathrm{z}^\mathrm{T}\boldsymbol{x}_i(t) \\ v_i(t) = \boldsymbol{c}_\mathrm{v}^\mathrm{T}\boldsymbol{x}_i(t) + d_\mathrm{v}u_i(t) + e_\mathrm{v}s_i(t) \end{cases} \\ \Sigma_{\mathrm{e}i}\!:\begin{cases}\dot{\boldsymbol{x}}_{\mathrm{e}i}(t) = \boldsymbol{A}_{\mathrm{e}i}\boldsymbol{x}_{\mathrm{e}i}(t) + \boldsymbol{b}_{\mathrm{e}i}v_i(t),\ \boldsymbol{x}_{\mathrm{e}i}(0) = \boldsymbol{x}_{\mathrm{e}0i} \\ f_i(t) = \boldsymbol{c}_{\mathrm{e}i}^\mathrm{T}\boldsymbol{x}_{\mathrm{e}i}(t) + d_{\mathrm{e}i}v_i(t). \end{cases}\end{cases} \tag{1}$$

$\boldsymbol{x}_i(t)$ is the nominal subsystem state and $\boldsymbol{x}_{\mathrm{e}i}(t)$ the error subsystem state, which are vectors of the dimension n and $n_{\mathrm{e}i}$, respectively. The control input $u_i(t)$, the control output $y_i(t)$, the error input $f_i(t)$, the error output $v_i(t)$, the interconnection input $s_i(t)$ and the interconnection output $z_i(t)$ are scalars.

The physical interconnections among the nominal subsystems $\Sigma_{\mathrm{n}i}$, $(i\in\mathcal{N})$ are described by

$$s_i(t) = \sum_{j=1}^{N} l_{\mathrm{P}ij} z_j(t), \quad i\in\mathcal{N}, \tag{2}$$

where $l_{\mathrm{P}ij}$ are the elements of the physical interconnection matrix $\boldsymbol{L}_\mathrm{p}$.

The inputs $u_i(t)$, $(i\in\mathcal{N})$ to $\Sigma_{\mathrm{d}i}$ in (1) are given by

$$u_i(t) = \begin{cases} u_{\mathrm{a},i}(t) & \text{autonomous} \\ u_{\mathrm{a},i}(t) + u_{\mathrm{c},i}(t) & \text{cooperative,} \end{cases} \tag{3}$$

where $u_{\mathrm{a},i}(t)$ is the autonomous part (local controller) with the state feedback

$$C_i : u_{\mathrm{a},i} = -\boldsymbol{k}_\mathrm{a}^\mathrm{T}\boldsymbol{x}_i(t) \tag{4}$$

and $u_{\mathrm{c},i}(t)$ is the additional cooperative input, see Fig. 1b.

The matrix $\boldsymbol{L}_\mathrm{c}$, describing the communicational interconnection, leads to the cooperative control input

$$u_{\mathrm{c},i}(t) = \sum_{j=1}^{N} l_{\mathrm{C}ij} \boldsymbol{k}_\mathrm{c}^\mathrm{T} \boldsymbol{x}_j(t), \quad i \in \mathcal{N}, \tag{5}$$

where $\boldsymbol{k}_\mathrm{c}^\mathrm{T}$ is the common gain matrix and $l_{\mathrm{C}ij}$ are the elements of $\boldsymbol{L}_\mathrm{c}$ (Fig. 1b).

The models for the decentralized controlled subsystems $\bar{\Sigma}_{\mathrm{d}i}$, $(i \in \mathcal{N})$ of the overall system Σ_d follow from (1), (2), (3), (4):

$$\bar{\Sigma}_{\mathrm{d}i}\!:\begin{cases} \bar{\Sigma}_{\mathrm{n}i}\!:\begin{cases} \dot{\boldsymbol{x}}_i(t) = \bar{\boldsymbol{A}}\boldsymbol{x}_i(t) + \boldsymbol{b}u_{\mathrm{c},i}(t) + \boldsymbol{e}s_i(t) + \boldsymbol{p}f_i(t), \\ \hfill \boldsymbol{x}_i(0) = \boldsymbol{x}_{0i} \\ y_i(t) = \boldsymbol{c}^\mathrm{T}\boldsymbol{x}_i(t) \\ z_i(t) = \boldsymbol{c}_\mathrm{z}^\mathrm{T}\boldsymbol{x}_i(t) \\ v_i(t) = \bar{\boldsymbol{c}}_\mathrm{v}^\mathrm{T}\boldsymbol{x}_i(t) + d_\mathrm{v}u_{\mathrm{c},i}(t) + e_\mathrm{v}s_i(t) \end{cases} \\ \Sigma_{\mathrm{e}i}:\begin{cases} \dot{\boldsymbol{x}}_{\mathrm{e}i}(t) = \boldsymbol{A}_{\mathrm{e}i}\boldsymbol{x}_{\mathrm{e}i}(t) + \boldsymbol{b}_{\mathrm{e}i}v_i(t) \\ f_i(t) = \boldsymbol{c}_{\mathrm{e}i}^\mathrm{T}\boldsymbol{x}_{\mathrm{e}i}(t) + d_{\mathrm{e}i}v_i(t) \end{cases} \end{cases} \tag{6}$$

with the matrices $\bar{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_\mathrm{a}^\mathrm{T}$ and $\bar{\boldsymbol{c}}_\mathrm{v}^\mathrm{T} = \boldsymbol{c}_\mathrm{v}^\mathrm{T} - d_\mathrm{v}\boldsymbol{k}_\mathrm{a}^\mathrm{T}$, see Fig 1c.

B. Overall system model

The overall physical interconnection follows from (2) and is described by the algebraic relation

$$\boldsymbol{s}(t) = \boldsymbol{L}_\mathrm{p}\boldsymbol{z}(t)\,, \tag{7}$$

where $\boldsymbol{s}(t) = (s_1(t), \ldots, s_N(t))^\mathrm{T}$ and $\boldsymbol{z}(t) = (z_1(t), \ldots, z_N(t))^\mathrm{T}$ are the vectors of interconnection inputs and interconnection outputs, respectively (Fig. 1c).

With (5), the overall cooperative control input $\boldsymbol{u}_\mathrm{c}(t)$ is

$$\boldsymbol{u}_\mathrm{c}(t) = (\boldsymbol{L}_\mathrm{c} \otimes \boldsymbol{k}_\mathrm{c}^\mathrm{T})\boldsymbol{x}(t), \tag{8}$$

where $\boldsymbol{x}(t) = (\boldsymbol{x}_1^\mathrm{T}(t), \ldots, \boldsymbol{x}_N^\mathrm{T}(t))^\mathrm{T}$, $\boldsymbol{u}_\mathrm{c}(t) = (u_{\mathrm{c},1}(t), \ldots, u_{\mathrm{c},N}(t))^\mathrm{T}$ and the operator "$\otimes$" denotes the *Kronecker product* (cf. [10]). Note that $\boldsymbol{L}_\mathrm{p}$ and $\boldsymbol{L}_\mathrm{c}$ do not have to be Laplacian matrices.

From (6), (7), (8), the nominal overall system

$$\Sigma_\mathrm{n}:\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_\mathrm{L}\boldsymbol{x}(t) + \boldsymbol{P}_\mathrm{L}\boldsymbol{f}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) = \boldsymbol{C}_\mathrm{L}\boldsymbol{x}(t) \\ \boldsymbol{v}(t) = \boldsymbol{C}_\mathrm{vL}\boldsymbol{x}(t) \end{cases} \tag{9}$$

follows, with the matrices

$$\begin{aligned} \boldsymbol{A}_\mathrm{L} &= (\boldsymbol{I}_N \otimes \bar{\boldsymbol{A}}) + (\boldsymbol{L}_\mathrm{p} \otimes \boldsymbol{e}\boldsymbol{c}_\mathrm{z}^\mathrm{T}) + (\boldsymbol{L}_\mathrm{p} \otimes \boldsymbol{b}\boldsymbol{k}_\mathrm{c}^\mathrm{T}) \\ \boldsymbol{P}_\mathrm{L} &= (\boldsymbol{I}_N \otimes \boldsymbol{p}) + (\boldsymbol{L}_\mathrm{p} \otimes \boldsymbol{e}) \\ \boldsymbol{C}_\mathrm{L} &= (\boldsymbol{I}_N \otimes \boldsymbol{c}^\mathrm{T}) \\ \boldsymbol{C}_\mathrm{vL} &= (\boldsymbol{I}_N \otimes \bar{\boldsymbol{c}}_\mathrm{v}^\mathrm{T}) + (\boldsymbol{L}_\mathrm{p} \otimes e_\mathrm{v}\boldsymbol{c}_\mathrm{z}^\mathrm{T}) + (\boldsymbol{L}_\mathrm{c} \otimes d_\mathrm{v}\boldsymbol{k}_\mathrm{c}^\mathrm{T})\,, \end{aligned}$$

where $\boldsymbol{I}_N$ is the identity matrix of size N. The signals of Σ_n in (9) are $\boldsymbol{f}(t) = (f_1(t), \ldots, f_N(t))^\mathrm{T}$, $\boldsymbol{y}(t) = (y_1(t), \ldots, y_N(t))^\mathrm{T}$ and $\boldsymbol{v}(t) = (v_1(t), \ldots, v_N(t))^\mathrm{T}$.

III. Stability analysis

A. Nominal overall system Σ_n

At the first step, the stability of Σ_n in (9) is analyzed by extending ideas of [2] on physically symmetrically interconnected subsystems with identical dynamics and of [5] on the cooperative control of vehicle formations. Therefore, the matrices $\boldsymbol{L}_\mathrm{p}$ in (7) and $\boldsymbol{L}_\mathrm{c}$ in (8) are assumed to be simultaneously diagonalizable.

Lemma 1: The controller (3), (4), (5) stabilizes the nominal overall system Σ_n in (9), if and only if the matrices

$$\tilde{\boldsymbol{A}}_{\mathrm{L}ii} = \bar{\boldsymbol{A}} + \lambda_i(\boldsymbol{L}_\mathrm{p})\boldsymbol{e}\boldsymbol{c}_\mathrm{z}^\mathrm{T} + \lambda_i(\boldsymbol{L}_\mathrm{c})\boldsymbol{b}\boldsymbol{k}_\mathrm{c}^\mathrm{T}, \quad (i \in \mathcal{N}) \tag{10}$$

are Hurwitz, where $\lambda_i(\boldsymbol{L}_\mathrm{p})$, $(i \in \mathcal{N})$ and $\lambda_i(\boldsymbol{L}_\mathrm{c})$ are the eigenvalues of the interconnection matrices $\boldsymbol{L}_\mathrm{p}$ in (7) and $\boldsymbol{L}_\mathrm{c}$ in (8).

Proof: The proof is based on [2], but has to take into account another structure of the system model (cf. Σ_n in (9)) for the identical subsystems. Σ_n is transformed by the following relation

$$\tilde{\boldsymbol{x}}(t) = (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1}\boldsymbol{x}(t) = \boldsymbol{T}_n^{-1}\boldsymbol{x}(t), \tag{11}$$

where $\boldsymbol{T}$ diagonalizes the interconnection matrices $\boldsymbol{L}_\mathrm{p}$ and $\boldsymbol{L}_\mathrm{c}$ into $\tilde{\boldsymbol{L}}_\mathrm{p} = \mathrm{diag}(\lambda_i(\boldsymbol{L}_\mathrm{p})) = \boldsymbol{T}^{-1}\boldsymbol{L}_\mathrm{p}\boldsymbol{T}$, $(i \in \mathcal{N})$ and $\tilde{\boldsymbol{L}}_\mathrm{c} = \mathrm{diag}(\lambda_i(\boldsymbol{L}_\mathrm{c})) = \boldsymbol{T}^{-1}\boldsymbol{L}_\mathrm{c}\boldsymbol{T}$, respectively. The transformation (11) leads to the transformed nominal overall system

$$\tilde{\Sigma}_\mathrm{n}:\begin{cases} \dot{\tilde{\boldsymbol{x}}}(t) = \tilde{\boldsymbol{A}}_\mathrm{L}\tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{P}}_\mathrm{L}\boldsymbol{f}(t) \\ \boldsymbol{y}(t) = \tilde{\boldsymbol{C}}_\mathrm{L}\tilde{\boldsymbol{x}}(t) \\ \boldsymbol{v}(t) = \tilde{\boldsymbol{C}}_\mathrm{vL}\tilde{\boldsymbol{x}}(t)\,, \end{cases} \tag{12}$$

where the matrix $\tilde{\boldsymbol{A}}_\mathrm{L}$ is block diagonal and its block diagonal elements are $\tilde{\boldsymbol{A}}_{\mathrm{L}ii}$, $(i \in \mathcal{N})$, described by (11). The other matrices appearing in (12) are $\tilde{\boldsymbol{P}}_\mathrm{L} = \boldsymbol{T}^{-1} \otimes \boldsymbol{p}$, $\tilde{\boldsymbol{C}}_\mathrm{L} = \boldsymbol{T} \otimes \boldsymbol{c}^\mathrm{T}$ and $\tilde{\boldsymbol{C}}_\mathrm{vL} = \boldsymbol{T} \otimes \bar{\boldsymbol{c}}_\mathrm{v}^\mathrm{T} + \boldsymbol{L}\boldsymbol{T} \otimes e_\mathrm{v}\boldsymbol{c}_\mathrm{z}^\mathrm{T} + \boldsymbol{L}\boldsymbol{T} \otimes d_\mathrm{v}\boldsymbol{k}_\mathrm{c}^\mathrm{T}$. Hence, the system matrix $\tilde{\boldsymbol{A}}_\mathrm{L}$ in (12) is block diagonal and the models $\tilde{\Sigma}_{\mathrm{n}i}$, $(i \in \mathcal{N})$ of $\tilde{\Sigma}_\mathrm{n}$ (12) are defined by

$$\tilde{\Sigma}_{\mathrm{n}i}\!:\begin{cases} \dot{\tilde{\boldsymbol{x}}}_i(t) = \tilde{\boldsymbol{A}}_{\mathrm{L}ii}\tilde{\boldsymbol{x}}_i(t) + \displaystyle\sum_{j=1}^{N} \tilde{\boldsymbol{p}}_{\mathrm{L}ij} f_j(t) \\ y_i(t) = \displaystyle\sum_{j=1}^{N} \tilde{\boldsymbol{c}}_{\mathrm{L}ij}^\mathrm{T}\tilde{\boldsymbol{x}}_j(t) \\ v_i(t) = \displaystyle\sum_{j=1}^{N} \tilde{\boldsymbol{c}}_{\mathrm{vL}ij}^\mathrm{T}\tilde{\boldsymbol{x}}_j(t)\,. \end{cases} \tag{13}$$

As a consequence, the stability of $\tilde{\Sigma}_\mathrm{n}$ is equivalent to the stability of the N separate models $\tilde{\Sigma}_{\mathrm{n}i}$, which completes the proof. ■

Lemma 1 reveals that the stability of Σ_n in (9), which is of the order nN, can be analyzed by considering the stability of the N single subsystems $\tilde{\Sigma}_{\mathrm{n}i}$, $(i \in \mathcal{N})$ in (13) of order n separately. The system matrices $\tilde{\boldsymbol{A}}_{\mathrm{L}ii}$, $(i \in \mathcal{N})$ only differ from each other by the eigenvalues $\lambda_i(\boldsymbol{L}_\mathrm{p})$ and

$\lambda_i(\boldsymbol{L}_\mathrm{c})$. The case that a complex value for $\lambda_i(\boldsymbol{L}_\mathrm{p})$ or $\lambda_i(\boldsymbol{L}_\mathrm{c})$ leads to complex-valued models $\tilde{\Sigma}_{\mathrm{n}i}$ is discussed in [5] for vehicle formations. If $\lambda_i(\boldsymbol{L}_\mathrm{p})$ and $\lambda_i(\boldsymbol{L}_\mathrm{c})$, $(i\in\mathcal{N})$ are equal to zero, the stability of the corresponding subsystem $\tilde{\Sigma}_{\mathrm{n}i}$ is independent of the interconnection with the other models $\tilde{\Sigma}_{\mathrm{n}j}$, $(j\in\mathcal{N})$ in (13).

B. Error overall system Σ_e

In the second step, the I/O behavior of the stable error overall system Σ_e is investigated, which is described by the I/O model

$$\Sigma_\mathrm{e}: \quad \boldsymbol{f} = \boldsymbol{G}_\mathrm{e} * \boldsymbol{v}, \tag{14}$$

where "$*$" symbolizes the convolution operation

$$\boldsymbol{G} * \boldsymbol{u} = \int_0^t \boldsymbol{G}(t-\tau)\boldsymbol{u}(\tau)\mathrm{d}\tau.$$

With the impulse response matrix $\boldsymbol{G}_\mathrm{e}(t) = \mathrm{diag}(g_{\mathrm{e}i}(t))$, $(i\in\mathcal{N})$ of Σ_e (14), the impulse response matrices of the error subsystems $\Sigma_{\mathrm{e}i}$ are

$$g_{\mathrm{e}i}(t) = \boldsymbol{c}_{\mathrm{e}i}^\mathrm{T}\mathrm{e}^{\boldsymbol{A}_{\mathrm{e}i}t}\boldsymbol{b}_{\mathrm{e}i} + d_{\mathrm{e}i}, \quad (i\in\mathcal{N}).$$

$\boldsymbol{G}_\mathrm{e}(t)$ is a block diagonal matrix, since in the error overall system Σ_e each signal v_i, $(i\in\mathcal{N})$ only has an effect on the signal $f_i(t)$ of the same error subsystems $\Sigma_{\mathrm{e}i}(t)$.

For the stability analysis of Σ_d, the error subsystems $\Sigma_{\mathrm{e}i}$, $(i\in\mathcal{N})$ are described by upper bounds $\bar{g}_{\mathrm{e}i}(t)$

$$|g_{\mathrm{e}i}(t)| \le \bar{g}_{\mathrm{e}i}(t) \quad \forall t, \tag{15}$$

which show that a bounded error output $v_i(t)$ leads to a bounded error input $f_i(t)$

$$|f_i(t)| \le \bar{g}_{\mathrm{e}i}(t) * |v_i(t)|.$$

Let $\boldsymbol{M}$ be

$$\boldsymbol{M} = \int_0^\infty |\boldsymbol{G}(t)|\,\mathrm{d}t, \tag{16}$$

then the matrix $\bar{\boldsymbol{M}}_\mathrm{e}$ is finite, since the error overall subsystems Σ_e is I/O stable.

C. Decomposed overall system Σ_d

The I/O behavior of the overall system Σ_d is described by the model

$$\Sigma_\mathrm{d}: \begin{cases} \Sigma_\mathrm{n}: \begin{cases} \boldsymbol{y} = \boldsymbol{G}_\mathrm{yf} * \boldsymbol{f} \\ \boldsymbol{v} = \boldsymbol{G}_\mathrm{vf} * \boldsymbol{f} \end{cases} \\ \Sigma_\mathrm{e}\ : \quad \boldsymbol{f} = \boldsymbol{G}_\mathrm{e} * \boldsymbol{v}\,. \end{cases} \tag{17}$$

The impulse response matrices of Σ_n in (9) are defined by:

$$\boldsymbol{G}_\mathrm{yf}(t) = \boldsymbol{C}_\mathrm{L}\mathrm{e}^{\boldsymbol{A}_\mathrm{L}t}\boldsymbol{P}_\mathrm{L}$$
$$\boldsymbol{G}_\mathrm{vf}(t) = \boldsymbol{C}_\mathrm{vL}\mathrm{e}^{\boldsymbol{A}_\mathrm{L}t}\boldsymbol{P}_\mathrm{L}\,.$$

The third step of the stability analysis is given by the following I/O stability criterion for Σ_d.

Lemma 2: Consider the overall system Σ_d (17), where the impulse response matrix $\boldsymbol{G}_\mathrm{e}(t)$ of the error overall system Σ_e (14) satisfies the inequality

$$|\boldsymbol{G}_\mathrm{e}(t)| \le \bar{\boldsymbol{G}}_\mathrm{e}(t) \quad \forall t \tag{18}$$

for a given matrix $\bar{\boldsymbol{G}}_\mathrm{e}(t)$. Suppose that the nominal overall system Σ_n (9) and the error overall system Σ_e with $\boldsymbol{G}_\mathrm{e}(t) = \bar{\boldsymbol{G}}_\mathrm{e}(t)$ are I/O stable, i.e. the matrices $\boldsymbol{M}_\mathrm{yf}$, $\boldsymbol{M}_\mathrm{vf}$ and $\bar{\boldsymbol{M}}_\mathrm{e}$ from (16) have finite elements. Then the overall system Σ_d (17) is I/O stable, if

$$\lambda_\mathrm{p}(\bar{\boldsymbol{M}}_\mathrm{e}\boldsymbol{M}_\mathrm{vf}) < 1, \tag{19}$$

where λ_p denotes the maximum eigenvalue (Perron root) of the non-negative matrix $\bar{\boldsymbol{M}}_\mathrm{e}\boldsymbol{M}_\mathrm{vf}$.

Lemma 2 is based on the results in [11], where also a proof is given. However, the analysis is extended here to an arbitrary physical interconnection matrix $\boldsymbol{L}_\mathrm{p}$ in (7) and an additional interconnection matrix $\boldsymbol{L}_\mathrm{c}$ in (8). To use the stability criterion (19), the scalars $m_{\mathrm{vf}ij}$, which are the elements of the matrix $\boldsymbol{M}_\mathrm{vf}$, have to be determined. For this, the I/O behavior of the transformed nominal models $\tilde{\Sigma}_{\mathrm{n}i}$, $(i\in\mathcal{N})$ in (13) is investigated.

Lemma 3: Transform the system Σ_n in (17) by the relation in (11). Then the elements of the impulse response matrix $\boldsymbol{G}_\mathrm{vf}$ are given by

$$g_{\mathrm{vf}ij}(t) = \sum_{k=1}^{N} \tilde{\boldsymbol{c}}_{\mathrm{vL}ik}^\mathrm{T}\mathrm{e}^{\tilde{\boldsymbol{A}}_{\mathrm{L}kk}t}\tilde{\boldsymbol{p}}_{\mathrm{L}kj}, \quad (i,j\in\mathcal{N})\,, \tag{20}$$

where $\tilde{\boldsymbol{c}}_{\mathrm{vL}ik}^\mathrm{T}$, $\tilde{\boldsymbol{A}}_{\mathrm{L}kk}$ and $\tilde{\boldsymbol{p}}_{\mathrm{L}kj}$, $(i,j,k\in\mathcal{N})$ are matrices of the transformed models $\tilde{\Sigma}_{\mathrm{n}i}$ in (13).

Lemma 3 shows that the non-negative matrix $\boldsymbol{M}_\mathrm{vf}$ in (19) can be determined just by analyzing the n-th order system $\tilde{\Sigma}_{\mathrm{n}i}$ in (13). The stability analysis described in Lemma 2 can be further simplified, if multiple eigenvalues of the interconnection matrices $\boldsymbol{L}_\mathrm{p}$ and $\boldsymbol{L}_\mathrm{c}$ lead to multiple subsystems $\tilde{\Sigma}_{\mathrm{n}i}$, $(i\in\mathcal{N})$ in (13), hence, the I/O behavior $g_{\mathrm{vf}ij}(t)$ from the error input $f_i(t)$ to the error output $v_i(t)$ in (20) can be merged.

D. Summary

With the results presented above, all three steps for the stability analysis can be performed as follows:

1) Check the stability of Σ_n (9) with Lemma 1.
2) Define an upper bound for the I/O behavior of the error overall system Σ_e (17), with the result that the matrix $\bar{\boldsymbol{M}}_\mathrm{e}$ given by (16) and (18) is finite.
3) Check the I/O stability of Σ_d in (9) with Lemma 2 while taking into account the simplification made in Lemma 3.

Hence, the stability analysis for the overall system can be summarized in the following theorem.

Theorem 1: Consider systems that consist of subsystems with similar dynamics. Decompose the subsystem models into identical nominal models $\Sigma_{\mathrm{n}i}$, $(i \in \mathcal{N})$ and individual error models $\Sigma_{\mathrm{e}i}$ as shown in (6). The subsystems are controlled by identical controllers (4). Assume that the interconnection matrix $\boldsymbol{L}_{\mathrm{p}}$ from (7) and $\boldsymbol{L}_{\mathrm{c}}$ from (8) can be diagonalized by the transformation (11). Then the overall system Σ_{d} is I/O stable, if the following conditions hold:

1) The matrices $\tilde{\boldsymbol{A}}_{\mathrm{L}ii}$, $(i \in \mathcal{N})$ defined in (10) are Hurwitz.
2) The elements of $\bar{m}_{\mathrm{e}i}$, $(i \in \mathcal{N})$, given by (15) and (16) are finite.
3) The condition (19) holds, where $\boldsymbol{M}_{\mathrm{vf}} = (m_{\mathrm{vf}ij})$, $(i, j \in \mathcal{N})$ is given by (16) and its entries are defined by (20).

Theorem 1 extends the stability analysis of systems composed of identical subsystems to systems composed of similar subsystems. For this, the simplifications on the stability analysis of identical subsystems based on the state transformation (11) as shown in [2] and [5] can also be applied to the nominal overall system Σ_{n} in (9). Furthermore, the determination of the matrix $\boldsymbol{M}_{\mathrm{vf}}$, which is needed for the stability criterion in (19), can be simplified by the same state transformation.

The elements $\bar{m}_{\mathrm{e}i}$ can be interpreted as a "degree of similarity". In general, greater elements $\bar{m}_{\mathrm{e}i}$ lead to a greater value of $\lambda_{\mathrm{p}}(\bar{\boldsymbol{M}}_{\mathrm{e}}\boldsymbol{M}_{\mathrm{vf}})$ due to the monotony property of the Perron root [1]. Hence, if the condition in (19) holds, the subsystem $\Sigma_{\mathrm{d}i}$, $(i \in \mathcal{N})$ in (1) can be called similar.

IV. Extensions

In this section four possible extensions for the stability analysis in Sec. III are given.

At first, Lemma 1 also holds for interconnection matrices $\boldsymbol{L}_{\mathrm{p}}$ and $\boldsymbol{L}_{\mathrm{c}}$ that are not simultaneously diagonalizable, but that can be transformed into upper triangular form. Following (12) and with the transformation matrix $\boldsymbol{T}$ that leads to upper triangular matrices $\tilde{\boldsymbol{L}}_{\mathrm{p}} = \boldsymbol{T}^{-1}\boldsymbol{L}_{\mathrm{p}}\boldsymbol{T}$ and $\tilde{\boldsymbol{L}}_{\mathrm{c}} = \boldsymbol{T}^{-1}\boldsymbol{L}_{\mathrm{c}}\boldsymbol{T}$, the matrix $\tilde{\boldsymbol{A}}_{\mathrm{L}}$ is also upper triangular. Therefore, the stability of the nominal overall system Σ_{n} in (12) also only depends on the block diagonal elements of the upper triangular matrix $\tilde{\boldsymbol{A}}_{\mathrm{L}}$ from (12). Thus, Theorem 1 holds for transformed interconnection matrices $\tilde{\boldsymbol{L}}_{\mathrm{p}} = \boldsymbol{T}^{-1}\boldsymbol{L}_{\mathrm{p}}\boldsymbol{T}$ and $\tilde{\boldsymbol{L}}_{\mathrm{c}} = \boldsymbol{T}^{-1}\boldsymbol{L}_{\mathrm{c}}\boldsymbol{T}$, that are upper triangular, without the simplification for the determination of $\boldsymbol{M}_{\mathrm{vf}}$ in (20).

Second, the error bounds $\bar{g}_{\mathrm{e}i}(t)$ in (15) of the error overall system Σ_{e} can also include model uncertainties of the subsystems, even an extension to nonlinear systems can be made, if all or a part of the error subsystems $\Sigma_{\mathrm{e}i}$, $(i \in \mathcal{N})$ in (6) are replaced by nonlinear error models.

Third, a direct throughput between the input $u_i(t)$ and the output $y_i(t)$ of the nominal subsystems $\Sigma_{\mathrm{n}i}$, $(i \in \mathcal{N})$ in (6) is not investigated, but can be easily extended. Then, in principal, the same result will be received, because the state transformation does not affect the throughput.

Fourth, systems with multi-variable subsystems can also be investigated with the presented method, if just the control of multi-agent systems or the decentralized control of physically interconnected systems is considered.

V. Application Example

A. Platooning vehicles

Consider a string of N vehicles, one behind the other, moving in a straight line. Each vehicle represents a subsystem $\bar{\Sigma}_{\mathrm{d}i}$, $(i \in \mathcal{N})$ of the overall system Σ_{d}, which are not physically interconnected but can communicate over a common network. The velocity $\dot{\alpha}_i(t)$ and the position $\alpha_i(t)$ of the vehicles are used as state variables $x_{i1}(t)$ and $x_{i2}(t)$, respectively. The leading vehicle is not considered as a subsystem of the overall system, but its position is the reference input for the first following vehicle that is $\bar{\Sigma}_{\mathrm{d}1}$.

The vehicles are controlled by a distance controller $\alpha_{\mathrm{ref}i}(t) = k_{\mathrm{P}}(\alpha_{i-1}(t) - \alpha_i(t)) + k_{\mathrm{I}}\alpha_{\mathrm{I}i}(t)$ with an integral term $k_{\mathrm{I}}\alpha_{\mathrm{I}i}(t)$ that leads to the state variable $x_{i3}(t) = \alpha_{\mathrm{I}i}(t)$ and a velocity controller with the gain k_{V}. The vehicles have different masses m_i, $(i \in \mathcal{N})$, where an unloaded truck has a mass of 20 000 kg and a fully loaded truck weighs 40 000 kg.

B. System model

The identical nominal subsystems $\bar{\Sigma}_{\mathrm{n}i}$, $(i \in \mathcal{N})$ in (6) are defined by the unloaded truck with $m_{\mathrm{n}} = 20\,000$ kg and

$$\bar{\boldsymbol{A}} = \begin{pmatrix} -\frac{k_{\mathrm{d}}+k_{\mathrm{V}}k_{\mathrm{u}}}{m_{\mathrm{n}}} & 0 & \frac{k_{\mathrm{I}}k_{\mathrm{u}}}{m_{\mathrm{n}}} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\boldsymbol{b} = (\frac{k_{\mathrm{u}}k_{\mathrm{P}}}{m_{\mathrm{n}}}\ 0\ 1)^{\mathrm{T}}$, $\boldsymbol{p} = (1\ 0\ 0)^{\mathrm{T}}$, $\boldsymbol{c}^{\mathrm{T}} = (1\ 0\ 0)$, $\bar{\boldsymbol{c}}_{\mathrm{v}}^{\mathrm{T}} = (k_{\mathrm{d}}\ 0\ k_{\mathrm{I}}k_{\mathrm{u}})$ and $d_{\mathrm{v}} = k_{\mathrm{u}}k_{\mathrm{P}}$. The vectors $\boldsymbol{e}$, $\boldsymbol{c}_{\mathrm{z}}^{\mathrm{T}}$ and $\boldsymbol{e}_{\mathrm{v}}$ are zero, because there does not exist any physical interconnection ($\boldsymbol{L}_{\mathrm{p}} = \boldsymbol{0}$).

The error subsystems $\Sigma_{\mathrm{e}i}$, $(i \in \mathcal{N})$ are given by

$$\Sigma_{\mathrm{e}i} : f_i(t) = m_{\mathrm{e}i}v_i(t) = \frac{m_{\mathrm{n}} - m_i}{m_{\mathrm{n}}m_i}v_i(t). \tag{21}$$

The numeric values and the physical interpretation of the parameters of the subsystems $\bar{\Sigma}_{\mathrm{d}i}$, $(i \in \mathcal{N})$ are specified in Table I.

The communication via the network is given by the matrix

$$\boldsymbol{L}_{\mathrm{c}} = \begin{pmatrix} 1 & -\frac{0.06}{N} & \cdots & -\frac{0.06}{N} \\ -\frac{0.06}{N} & 1 & \cdots & -\frac{0.06}{N} \\ \vdots & \cdots & \ddots & -\frac{0.06}{N} \\ -\frac{0.06}{N} & \cdots & -\frac{0.06}{N} & 1 \end{pmatrix}. \tag{22}$$

TABLE I

Parameters of the subsystem $\bar{\Sigma}_{\mathrm{d}i}$

Parameter	Physical interpretation	Value
k_{d}	Aerodynamic drag coefficient	$2.83 \cdot 10^3\,\frac{\mathrm{kg}}{\mathrm{s}}$
k_{u}	Driving force coefficient	$1.42 \cdot 10^4\,\frac{\mathrm{kg}}{\mathrm{s}}$
k_{P}	Proportional gain dist. controller	2.4
k_{I}	Integral gain dist. controller	0.1
k_{V}	Gain velocity controller	-2
m_i	Mass truck $(i \in \mathcal{N})$	$[20, \ldots, 40] \cdot 10^3\mathrm{kg}$

C. Stability analysis

The matrix $\boldsymbol{L}_\mathrm{c}$ in (22) has two different eigenvalues $\lambda_1(\boldsymbol{L}_\mathrm{c}) = 1 - 0.06\frac{1}{N}$ and $\lambda_2(\boldsymbol{L}_\mathrm{c}) = 1 - 0.06\frac{N-1}{N}$, where the first eigenvalue $\lambda_1(\boldsymbol{L}_\mathrm{c})$ occurs $N-1$ times. For the stability of the overall system Σ_d the three conditions of Theorem 1 have to be fulfilled:

1) The matrices $\tilde{\boldsymbol{A}}_{\mathrm{L}ii}$, $(i \in \mathcal{N})$ from (10) have only two different eigenvalue triples. The characteristic polynomial of the matrix $\tilde{\boldsymbol{A}}_{\mathrm{L}ii}$ $(i\in\mathcal{N})$, with $\lambda_i(\boldsymbol{L}_\mathrm{c})$ as a variable, is
$$p_{\tilde{\boldsymbol{A}}_{\mathrm{L}ii}}(\lambda) = \lambda^3 + 1.56\lambda^2 + 1.7\lambda_i(\boldsymbol{L}_\mathrm{c})\lambda + 0.07\lambda_i(\boldsymbol{L}_\mathrm{c}).$$
Following the Routh-Hurwitz stability criterion, the matrix $\tilde{\boldsymbol{A}}_{\mathrm{L}ii}$, $(i \in \mathcal{N})$ is stable for $\lambda_i(\boldsymbol{L}_\mathrm{c}) > 0$. With this, the nominal overall system Σ_n is stable for an arbitrary number of subsystems.
2) Equation (21) yields finite values $\bar{m}_{\mathrm{e}i} = |m_{\mathrm{e}i}|$, $(i\in\mathcal{N})$.
3) Considering the *Gershgorin circle theorem* [6], [7] and the fact that $\bar{\boldsymbol{M}}_\mathrm{e}\boldsymbol{M}_\mathrm{vf}$ is a nonnegative matrix (cf. [1]) the maximum eigenvalue of the matrix $\bar{\boldsymbol{M}}_\mathrm{e}\boldsymbol{M}_\mathrm{vf}$ is less or equal to the maximum value of its column sum. Hence, the overall system Σ_d is stable, if the following condition holds
$$m_{\mathrm{vf}ii}|m_{\mathrm{e}i}| + m_{\mathrm{vf}ij}\sum_{j=1, j\neq i}^{N}|m_{\mathrm{e}j}| < 1,$$
where $|m_{\mathrm{e}i}|$ and $m_{\mathrm{vf}ij}$, $(i,j\in\mathcal{N})$ are the scalar entries of the matrices $\bar{\boldsymbol{M}}_\mathrm{e}$ and $\boldsymbol{M}_\mathrm{vf}$, respectively. The scalars $m_{\mathrm{vf}ii}$ and $m_{\mathrm{vf}ij}$ are given by the definition in (16) and (20)
$$\begin{aligned} g_{\mathrm{nvf}ii} &= \frac{N-1}{N}(\bar{\boldsymbol{c}}_\mathrm{v}^\mathrm{T} + \lambda_1(\boldsymbol{L}_\mathrm{c})\boldsymbol{b}\boldsymbol{k}_\mathrm{c}^\mathrm{T})\mathrm{e}^{\tilde{\boldsymbol{A}}_{\mathrm{L}11}t}\boldsymbol{p} \\ &\quad + \frac{1}{N}(\bar{\boldsymbol{c}}_\mathrm{v}^\mathrm{T} + \lambda_2(\boldsymbol{L}_\mathrm{c})\boldsymbol{b}\boldsymbol{k}_\mathrm{c}^\mathrm{T})\mathrm{e}^{\tilde{\boldsymbol{A}}_{\mathrm{L}22}t}\boldsymbol{p} \\ g_{\mathrm{nvf}ij} &= -\frac{1}{N}(\bar{\boldsymbol{c}}_\mathrm{v}^\mathrm{T} + \lambda_1(\boldsymbol{L}_\mathrm{c})\boldsymbol{b}\boldsymbol{k}_\mathrm{c}^\mathrm{T})\mathrm{e}^{\tilde{\boldsymbol{A}}_{\mathrm{L}11}t}\boldsymbol{p} \\ &\quad + \frac{1}{N}(\bar{\boldsymbol{c}}_\mathrm{v}^\mathrm{T} + \lambda_2(\boldsymbol{L}_\mathrm{c})\boldsymbol{b}\boldsymbol{k}_\mathrm{c}^\mathrm{T})\mathrm{e}^{\tilde{\boldsymbol{A}}_{\mathrm{L}22}t}\boldsymbol{p}\,. \end{aligned}$$

If the number of trucks is set to six ($N = 6$), the scalars $m_{\mathrm{vf}ii} = 2.625 \cdot 10^4$ and $m_{\mathrm{vf}ij} = 0.0328 \cdot 10^4$ $(i,j = 1,\ldots,6; j \neq i)$ are given. Hence, Σ_d is I/O stable for
$$2.625 \cdot 10^4|m_{\mathrm{e}i}| + 0.033 \cdot 10^4 \sum_{j=1, j\neq i}^{6}|m_{\mathrm{e}j}| < 1$$
This inequality shows that the error bound $|m_{\mathrm{e}i}|$, $(i\in\mathcal{N})$ of the investigated vehicle $\bar{\Sigma}_{\mathrm{d}i}$ has the greatest influence on the stability of the overall system Σ_d, since the factor before the sum is 100 times smaller.

If only one truck is loaded and the other trucks have the nominal mass $m_\mathrm{n} = 20\,000$ kg, then the platoon is stable, if this vehicle has a mass that is smaller than $84\,045$ kg. If all 6 trucks (vehicles) are assumed to have the same mass, then the maximum value for a stable platoon is $m_i < m_\mathrm{max} = 70\,739$ kg. So Σ_d is always stable, since a fully loaded truck ($40\,000$ kg) has a mass that is smaller than m_max. The analysis can of course be extended to a larger number of trucks.

VI. Conclusion

In the paper a method for analyzing the stability of interconnected subsystems with similar dynamics has been derived, that takes into account the structural properties of the interconnection between the subsystems to reduce the complexity of the stability proof. The stability analysis of systems composed of identical subsystems is extended to systems composed of similar subsystems. It shows that the simplifications on the stability analysis of identical subsystems based on a state transformation can also be applied to the identical part of the decomposed overall system. The stability condition in Theorem 1 gives a sufficient condition for the stability of the original overall system. It is divided into a necessary and a sufficient condition for the stability of the interconnected nominal subsystems of the decomposed overall system. Based on this, Theorem 1 also gives a sufficient condition for the nature of the similarity of the subsystems. Therefore, an upper bound for the deviation of the subsystems from the nominal subsystems with identical dynamics had to be determined. An application example with symmetrically interconnected platooning vehicles has demonstrated that these stability criteria can be used successfully to prove the stability of subsystems with similar dynamics.

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