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A max-plus based fundamental solution to a class of linear regulator problems with non-quadratic terminal payoff

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Abstract— **This paper studies a class of linear regulator problem where the terminal payoff function is not necessarily quadratic. The value function for this problem is generally not quadratic and thus it can not be reduced to solving the corresponding matrix Riccati equation as for the standard linear quadratic regulator (LQR) problem. The computational method of direct iteration using the dynamic programming equations is computationally expensive. In this paper, a new computational method based on max-plus techniques is developed for this problem which is demonstrated to be more efficient and more accurate. In particular, three max-plus fundamental solutions are obtained which can be used as the kernel of max-plus integration with respect to the max-plus dual of the terminal payoff to generate the value function of the linear regulator problem.**

I. INTRODUCTION

The Linear Quadratic Regulator (LQR) problem [2], [4] is one of the most well understood and most widely applied optimal control methods. In its standard formulation, the running cost is a quadratic function on both state and control variables and the terminal cost function is quadratic on state. The value functions (both finite horizon and infinite horizon) are quadratic on state. The Hessian matrices that define these value functions are the solutions of the associated Differential (Difference) Riccati Equations (DRE) in finite horizon or the stabilizing solution of the corresponding Algebraic Riccati Equation (ARE) in infinite horizon. These solutions can be computed very accurately and efficiently. However, the method of solving the LQR problem via solving DRE and ARE fails when the terminal cost function is not quadratic. In this case, the value functions are not necessarily quadratic so that they can not be characterized by their Hessian matrices. A method using direct iterations on the iterative equation derived from dynamic programming can be used to compute these value functions. However, this grid based method of direct iteration on dynamic programming equation is computationally expensive and limited to low dimensional problems.

This paper develops a more efficient and more accurate method for computing the value function of a linear regulator problem with non-quadratic terminal payoff functions. The method is inspired by the idea of max-plus fundamental solutions introduced in [13]. The development of numerical methods for optimal control problems based on max-plus techniques is relatively recent [1], [3], [5], [9], [11], [12], [13]. A key property that makes the max-plus algebra an attractive tool in solving optimal control problems is that the dynamic programming evolution operator associated with an optimal control problem is a linear operator over max-plus algebra. As a consequence, the value function of an optimal control problem satisfies a max-plus linear operator equation derived from the dynamic programming principle (DPP). Thus linear methods can be applied to solve these equations. In particular, they admit max-plus fundamental solutions. In general, these fundamental solutions can be complicated and difficult to compute. However, in the LQR situation, they are quadratic functions with the set of basis functions chosen to be quadratic functions as shown in [13]. The Hessian matrices defining these max-plus fundamental solutions can be computed efficiently and accurately. The fundamental solutions are independent of the terminal payoff. The value functions for a linear regulator problem with any non-quadratic terminal payoff can be conveniently obtained by performing a max-plus integration on the max-plus dual of the terminal payoff with the fundamental solution as the kernel.

The fundamental solutions are determined by the linear dynamics, the quadratic running payoff and the max-plus basis used. Besides the quadratic basis functions in [13], this paper identifies two other max-plus basis function sets for the linear regulator problem. It is shown that these two new fundamental solutions are also quadratic so that they can be computed efficiently. One consists of linear functions which are max-plus basis functions for the max-plus vector space of convex functions. The other consists of impulse like functions which can be regarded as the limit situation of the concave quadratic max-plus basis function of [9], [11], [13]. The max-plus fundamental solutions in these three spaces are developed in a unified fashion. Examples are presented to show the computational advantage of these max-plus based computational method over the direct DPP based iteration algorithm.

II. OPTIMAL CONTROL PROBLEMS AND MAX-PLUS VECTOR SPACES

Consider a discrete time linear system

$$x_{k+1} = Ax_k + Bw_k, \qquad x_0 = x. \tag{1}$$

Here, $x_k \in \mathbb{R}^n, w_k \in \mathbb{R}^m$ denote the state and input variables, all defined for $k \in \mathbb{Z}_{\geq 0}$. Here $\mathbb{Z}_{\geq 0}$ denotes the set of all non-negative integers. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are constant real matrices. Assume rank$(B) = m \leq n$ and $[A, B]$ is controllable.

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Define an optimal control problem over time $[0, K]$

$$W_K(x) = \sup_{w_{[0,K-1]} \in (\mathbb{R}^m)^K} J_K(x; w_{[0,K-1]}) \tag{2}$$

with total payoff given by

$$J_K(x; w_{[0,K-1]}) = \sum_{k=0}^{K-1} \left(x_k^T \Phi x_k - \frac{1}{2}\gamma^2 w_k^T w_k \right) + \Psi(x_K). \tag{3}$$

Here $\Phi \in \mathbb{R}^{n\times n}$ is symmetric and semi-positive definite, i.e. $\Phi = \Phi^T \geq 0$, $\gamma > 0$ and $\Psi : \mathbb{R}^n \to \mathbb{R}$ is the termimal payoff.

Application of standard dynamic programming leads to the following iteration for the value functions W_k

$$W_{k+1}(x) = \sup_{w\in\mathbb{R}^m} \left\{ \; x^T\Phi x - \tfrac{1}{2}\gamma^2 w^T w + W_k(Ax + Bw) \; \right\}, \tag{4}$$

$$W_0(x) = \Psi(x).$$

Where the terminal payoff Ψ is quadratic of the form $\Psi(x) = \frac{1}{2}x^T \Lambda x,\; x \in \mathbb{R}^n$ with $\Lambda = \Lambda^T \geq 0$, the value functions are quadratic $W_k(x) = \frac{1}{2}x^T P_k x, x \in \mathbb{R}^n$. This allows to reduce the one-step dynamic programming equation (4) to the matrix Riccati difference equation for P_k

$$P_{k+1} = 2\Phi + A^T P_k A + A^T P_k B(\gamma^2 I - B^T P_k B)^{-1} B^T P_k A \tag{5}$$

$$P_0 = \Lambda.$$

This paper is focused on the situation where the terminal payoff function is not necessarily quadratic. In formalizing the restrictions on the terminal payoffs, define a hierarchy of three function spaces $\mathscr{B}_r^1 \subset \mathscr{B}_r^2 \subset \mathscr{B}_r^3$ as

$$\begin{aligned}
\mathscr{B}_r^1 &\doteq \left\{ \phi \in \mathscr{B}_r^2 \;\middle|\; \phi \text{ is convex} \right\}, \\
\mathscr{B}_r^2 &\doteq \left\{ \phi \in \mathscr{B}_r^3 \;\middle|\; \phi \text{ is semi-convex} \right\}, \\
\mathscr{B}_r^3 &\doteq \left\{ \phi : \mathbb{R}^n \to \mathbb{R}^- \;\middle|\; \begin{array}{c} \exists\; c \in \mathbb{R} \text{ s.t.} \\ \phi(x) \leq \frac{r}{2}|x|^2 + c \\ \text{for all } x \in \mathbb{R}^n \end{array} \right\}.
\end{aligned} \tag{6}$$

It can be shown that all three spaces are max-plus vector spaces in the sense

$$\forall\; a \in \mathbb{R}^-, \phi_1, \phi_2 \in \mathscr{B}_r^i \;\Rightarrow\; a \otimes \phi_1 \oplus \phi_2 \in \mathscr{B}_r^i, i = 1, 2, 3.$$

Here the max-plus additioin $\oplus$ and max-plus multiplication $\otimes$ are defined to be $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}^-$.

Assumption 2.1: There exists an $r \in \mathbb{R}$ and at least one $i \in \{1, 2, 3\}$ such that the terminal payoff Ψ satisfies $\Psi \in \mathscr{B}_r^i$.

For non-quadratic terminal payoff functions, the value functions W_k are not necessarily quadratic. So (4) cannot be reduced to the matrix iteration (5). This implies an expensive computation for W_k using direct iteration (4) for large k. This paper develops a characterization and computation of W_k using the method of max-plus fundamental solution [13]. Three fundamental solutions will be developed for each space $\mathscr{B}_r^i, i = 1, 2, 3$. The max-plus fundamental solutions in space $\mathscr{B}_r^2$ was formulated and studied in [13] for continuous time LQR problem.

In each max-plus vector space $\mathscr{B}_r^i, i = 1, 2, 3$, a max-plus duality relation exists in the form

$$\phi(x) = \max_{z\in\mathbb{R}^n} \left\{ \hat{\phi}(z) + \psi_z^i(x) \right\} \doteq \int_{\mathbb{R}^n}^{\oplus} \hat{\phi}(z) \otimes \psi_z^i(x) dz \tag{7}$$

for any $\phi \in \mathscr{B}_r^i$ and the max-plus dual $\hat{\phi} : \mathbb{R}^n \to \mathbb{R}^-$ is

$$\begin{aligned}
\hat{\phi}(z) &= -\max_{x\in\mathbb{R}^n} \left\{ \psi_z^i(x) - \phi(x) \right\} \qquad (8) \\
&= -\int_{\mathbb{R}^n}^{\oplus} \psi_z^i(x) \otimes (-\phi(x)) dx.
\end{aligned}$$

Here, the parametrization functions $\psi_z^i : \mathbb{R}^n \to \mathbb{R}^-$ for spaces $\mathscr{B}_r^i, i = 1, 2, 3$ are given by

$$\begin{aligned}
&\psi_z^1(x) \doteq z^T x, \\
&\psi_z^2(x) \doteq -\frac{1}{2}(x - z)^T M (x - z),\; M = M^T > 0, \\
&\psi_z^3(x) \doteq \delta(x - z),
\end{aligned} \tag{9}$$

where

$$\delta(y) \doteq \begin{cases} 0, & y = 0, \\ -\infty, & y \neq 0. \end{cases}$$

The duality in $\mathscr{B}_r^1$ is the well-known convex duality. The semi-convexity duality in $\mathscr{B}_r^2$ was first proved in [8] and was used in a series of papers [9], [10], [11], [12], [13] in developing max-plus based numerical methods for optimal control problems. The duality in $\mathscr{B}_r^3$ can be verified directly

$$\hat{\phi}(z) = -\max_{x\in\mathbb{R}^n} \left\{ \delta(x - z) - \phi(x) \right\} = \phi(z).$$

That is, the max-plus dual of $\phi \in \mathscr{B}_r^3$ is ϕ itself.

III. Max-plus fundamental solutions

A. Max-plus linear operators

Define operators $\mathcal{S}_k$ for $k \in \mathbb{Z}_{>0}$ recursively

$$\mathcal{S}_{k+1}\phi \doteq \mathcal{S}_1(\mathcal{S}_k \phi) \tag{10}$$

where $\mathcal{S}_1$ is defined explicitly for $\phi \in \mathscr{B}_r^i, i = 1, 2, 3$

$$(\mathcal{S}_1\phi)(x) \doteq \sup_{w\in\mathbb{R}^m} \left\{ x^T\Phi x - \frac{1}{2}\gamma^2 w^T w + \phi(Ax + Bw) \right\}. \tag{11}$$

In terms of $\mathcal{S}_k$, the value function W_k of (2) is given by $W_k(x) = \mathcal{S}_k[\Psi](x), \forall\; x \in \mathbb{R}^n$. The next result shows the conditions under which the spaces $\mathscr{B}_r^i, i = 1, 2, 3$ are invariant with respect to the family of evolution operators $\mathcal{S}_k, k \in \mathbb{Z}_{\geq 0}$.

Theorem 3.1: There exist $\Phi_0 \geq 0$ and $\gamma_0 > 0$ such that $W_k = \mathcal{S}_k\Psi \in \mathscr{B}_r^i, k \in \mathbb{Z}_{\geq 0}, i = 1, 2, 3$ when $\Psi \in \mathscr{B}_r^i, i = 1, 2, 3$ for all $\Phi \leq \Phi_0$ and $\gamma \geq \gamma_0$.

It can be shown as in [11] that $\mathcal{S}_k, k \in \mathbb{Z}_{>0}$ are max-plus linear operators

$$\mathcal{S}_k(a \otimes \phi_1 \oplus \phi_2) = a \otimes \mathcal{S}_k\phi_1 \oplus \mathcal{S}_k\phi_2$$

for all $a \in \mathbb{R}^-, \phi_1, \phi_2 \in \mathscr{B}_r^i, i = 1, 2, 3$.

B. Max-plus fundamental solutions

The max-plus linearity of the operator $\mathcal{S}_k$ implies a representation and computation of W_k of (2)

$$W_k(x) = (\mathcal{S}_k\Psi)(x) = \left(\mathcal{S}_k\left(\int_{\mathbb{R}^n}^{\oplus} \widehat{\Psi}^i(z)\otimes\psi_z^i(\cdot)dz\right)\right)(x)$$
$$= \int_{\mathbb{R}^n}^{\oplus} \widehat{\Psi}^i(z)\otimes\left(\mathcal{S}_k\psi_z^i\right)(x)dz. \tag{12}$$

Here, $\widehat{\Psi}^i(z) = -\int_{\mathbb{R}^n}^{\oplus}\psi_z^i(x)\otimes(-\Psi(x))\,dx$ is the max-plus dual of the terminal payoff (initial condition of the iteration (4)) in space $\mathscr{B}_r^i, i=1,2,3$. Hence, given the parameterized function

$$\mathrm{S}_k^i(x,z) \doteq (\mathcal{S}_k\psi_z^i)(x), \tag{13}$$

the value functions $W_k = \mathcal{S}_k\Psi, k\in\mathbb{Z}_{>0}$ can be computed from the dual of the terminal payoff $\Psi\in\mathscr{B}_r^i$ via performing a max-plus integration (12).

Fast computation of the function $\mathrm{S}_k^i, k\in\mathbb{Z}_{>0}$ (13) is critical to the computation of the value function W_k via (12). It has been shown that $\mathrm{S}_k^2, k\in\mathbb{Z}_{>0}$ are qudratic functions in [13]. A an observation is that the functions $\mathrm{S}_k^i, i=1,3$ are also qudratic as a consequence of the linear dynamics (1), the qudratic running payoff in (3) and the specific choices of the max-plus basis functions (9). It has also been shown in [6], [7], [13] that it is more efficient to compute S_k^i via its max-plus dual

$$\mathrm{B}_k^i(y,z) = -\int_{\mathbb{R}^n}^{\oplus}\psi_y^i(x)\otimes(-\mathrm{S}_k^i(x,z))dx \tag{14}$$

since a particular doubling algorithm can be developed for the propagation of B_k^i in time horizaon $k\in\mathbb{Z}_{>0}$. When B_k^i is computed, the function S_k^i is recovered by the

$$\mathrm{S}_k^i(x,z) = \int_{\mathbb{R}^n}^{\oplus}\mathrm{B}_k^i(y,z)\otimes\psi_y^i(x)dy. \tag{15}$$

The functions $\mathrm{B}_{k,i}$ of (14) can be used to define a fundamental solution semi-group

$$(\mathcal{B}_{k,i}a)(z) \doteq \int_{\mathbb{R}^n}^{\oplus}\mathrm{B}_{k,i}(y,z)\otimes a(y)\,dy, \tag{16}$$

which propagates the dual of the value functions (2) [6], [7], [13].

It can be shown that if the functions $\mathrm{S}_k^i(x,z)$ are quadratic in $\mathbb{R}^n\times\mathbb{R}^n$, then the corresponding max-plus dual $\mathrm{B}_k^i(y,z)$ is also a quadratic function in $\mathbb{R}^n\times\mathbb{R}^n$ for all $i=1,2,3$, and vice versa. In this case, the computation of the max-plus duality (15) and (14) can be reduced to transformations on their Hessian.

Define maps $\Gamma^i:\mathbb{R}^{2n\times 2n}\to\mathbb{R}^{2n\times 2n}$ for $i=1,2,3$

$$\Gamma^1(\Theta) \doteq \begin{bmatrix} (\Theta^{11})^{-1} & -(\Theta^{11})^{-1}\Theta^{12} \\ -\Theta^{21}(\Theta^{11})^{-1} & \Theta^{21}(\Theta^{11})^{-1}\Theta^{12}-\Theta^{22}\end{bmatrix},$$
$$\Gamma^2(\Theta) = \begin{bmatrix} M(\Theta^{11}+M)^{-1}M-M & -M(\Theta^{11}+M)^{-1}\Theta^{12} \\ -\Theta^{21}(\Theta^{11}+M)^{-1}M & \Theta^{21}(\Theta^{11}+M)^{-1}\Theta^{12}-\Theta^{22}\end{bmatrix}, \tag{17}$$
$$\Gamma^3(\Theta) \doteq -\Theta.$$

Here, $M>0$ is the maxtrix defining the basis ψ_z^2 in (9). It is required that Θ^{11} is invertible for Γ^1 and $\Theta^{11}+M$ is invertible for Γ^2. It can be verified directly that

$$\Theta = \Gamma^i(\Gamma^i(\Theta)) \doteq \Gamma^i\circ\Gamma^i(\Theta)$$

for any allowed Θ, that is, $\Gamma^i\circ\Gamma^i = I$.

Theorem 3.2: Suppose $\mathrm{S}_k^i(x,z)$ and its max-plus dual $\mathrm{B}_k^i(y,z)$ are quadratic in the form

$$\mathrm{S}_k^i(x,z) = \frac{1}{2}\begin{bmatrix} x \\ z\end{bmatrix}^T Q_{k,i}\begin{bmatrix} x \\ z\end{bmatrix} \tag{18}$$

and

$$\mathrm{B}_k^i(y,z) = -\frac{1}{2}\begin{bmatrix} y \\ z\end{bmatrix}^T \Theta_{k,i}\begin{bmatrix} y \\ z\end{bmatrix}, \tag{19}$$

in which $Q_{k,i}\in\mathbb{R}^{2n\times 2n}, \Theta_{k,i}\in\mathbb{R}^{2n\times 2n}, k\in\mathbb{Z}_{>0}, i=1,2,3$. Then, $Q_{k,i}$ and $\Theta_{k,i}$ are related

$$\Theta_{k,i} = \Gamma^i(Q_{k,i}),\ Q_{k,i} = \Gamma^i(\Theta_{k,i}). \tag{20}$$

To obtain the dynamics for $\Theta_k, k\in\mathbb{Z}_{>0}$, define a matrix operation [13] $\Omega = \Omega_1\circledast\Omega_2$ for $\Omega_j\in\mathbb{R}^{2n\times 2n}, \Omega_j = \Omega_j^T, j=1,2$ with

$$\Omega_j = \begin{bmatrix} \Omega_j^{11} & \Omega_j^{12} \\ \Omega_j^{21} & \Omega_j^{22}\end{bmatrix},\quad \Omega_j^{22}+\Omega_j^{11}>0$$

to be

$$\Omega = \Omega_1\circledast\Omega_2 \tag{21}$$
$$\doteq \begin{bmatrix} \Omega_1^{11} & 0 \\ 0 & \Omega_2^{22}\end{bmatrix} - \begin{bmatrix} \Omega_1^{12} \\ \Omega_2^{21}\end{bmatrix}(\Omega_1^{22}+\Omega_2^{11})^{-1}[\Omega_1^{21}\quad \Omega_2^{12}].$$

The following theorem gives the dynamics of the max-plus dual B_k^i which has the same form for $i=1,2,3$. The case for $i=2$ is proved in [13].

Theorem 3.3: Suppose $\mathrm{B}_{k_j}^i$ for $j=1,2$ and $i=1,2,3$ are quadratic functions in the form

$$\mathrm{B}_{k_j}^i(y,z) = -\frac{1}{2}\begin{bmatrix} y \\ z\end{bmatrix}^T\begin{bmatrix} \Theta_{k_j,i}^{11} & \Theta_{k_j,i}^{12} \\ \Theta_{k_j,i}^{21} & \Theta_{k_j,i}^{22}\end{bmatrix}\begin{bmatrix} y \\ z\end{bmatrix}, \tag{22}$$

for $j=1,2, i=1,2,3$. Then, $\mathrm{B}_{k_1+k_2}^i$ are quadratic

$$\mathrm{B}_{k_1+k_2}^i(y,z) = -\frac{1}{2}\begin{bmatrix} y \\ z\end{bmatrix}^T\begin{bmatrix} \Theta_{k_1+k_2,i}^{11} & \Theta_{k_1+k_2,i}^{12} \\ \Theta_{k_1+k_2,i}^{21} & \Theta_{k_1+k_2,i}^{22}\end{bmatrix}\begin{bmatrix} y \\ z\end{bmatrix}$$

where

$$\Theta_{k_1+k_2,i} = \begin{bmatrix} \Theta_{k_1+k_2,i}^{11} & \Theta_{k_1+k_2,i}^{12} \\ \Theta_{k_1+k_2,i}^{21} & \Theta_{k_1+k_2,i}^{22}\end{bmatrix}$$

is given by

$$\Theta_{k_1+k_2,i} = \Theta_{k_1,i}\circledast\Theta_{k_2,i}. \tag{23}$$

Theorem 3.3 implies $\Theta_{2k,i} = \Theta_{k,i}\circledast\Theta_{k,i}$ for any $k\in\mathbb{Z}_{\geq 0}$. Thus, it takes k matrix $\circledast$ operations to obtain $\Theta_{2^k,i}$ from initial condition $\Theta_{1,i}$. When $\Theta_{2^k,i}$ is available, the Hessian matrix $Q_{2^k,i}$ characterizing the fundamental solution is obtained $Q_{2^k,i} = \Gamma^i(\Theta_{2^k,i})$. It is left to obtain the initial condition $\Theta_{1,i}$ for the iteration (23) for different spaces which is the content of the next section.

C. Initializations for max-plus fundamental solutions

This section presents the initial conditions for the iterations for the matrix sequence $\Theta_{k,i}$ in (23) of Theorem 3.3.

Theorem 3.4: The matrix $\Theta_{1,1}$ that defines the function B_1^1 in (14) is given by

$$\Theta_{1,1} = \begin{bmatrix} \frac{1}{2}\Phi^{-1} & -\frac{1}{2}\Phi^{-1}A^T \\ -\frac{1}{2}A\Phi^{-1} & \frac{1}{2}A\Phi^{-1}A^T - \gamma^{-2}BB^T \end{bmatrix}. \quad (24)$$

Theorem 3.5: The matrix $\Theta_{1,2}$ that defines the function B_1^2 in (14) is given by

$$\Theta_{1,2} = \Gamma^2(Q_{1,2}) \quad (25)$$

with

$$Q_{1,2} = \begin{bmatrix} A^T\Delta A + 2\Phi & -A^T\Delta \\ -\Delta A & \Delta \end{bmatrix}, \quad (26)$$

$$\Delta = MB(\gamma^2 I + B^T MB)^{-1}B^T M - M.$$

For space $\mathscr{B}_r^3$,

$$\mathrm{B}_k^3(y,z) = \mathrm{S}_k^3(y,z) = \sup_{w_{[0,k-1]}} \left\{ \sum_{j=0}^{k-1} \left(x_j^T \Phi x_j - \frac{1}{2}\gamma^2 w_j^T w_j \right) \Big| x_k = z \right\}.$$

That is, $\mathrm{B}_k^3(y,z)$ is the optimal control problem (2) with constraints $x_0 = y$ and $x_k = z$. Denote

$$\Lambda_k(y,z) \doteq \left\{ w_{[0,k-1]} | x_0 = y, x_k = z \text{ subject to (1)} \right\} \quad (27)$$

the set of controls $w_{[0,k-1]} = (w_0, w_1, \cdots, w_{k-1})$ that steer the initial state from $x_0 = y$ to final state $x_k = z$. To obtain a quadratic function $\mathrm{B}_k^3(y,z)$ on $\mathbb{R}^n \times \mathbb{R}^n$, it is necessary that $\Lambda_k(y,z) \neq \emptyset$ for any $(y,z) \in \mathbb{R}^n \times \mathbb{R}^n$. Different cases are considered.

1) Case 1: $m = n$: In this case, $B \in \mathbb{R}^{n\times n}$ invertible under the assumption that B has full collum rank, then

$$\Lambda_1(y,z) = \left\{ w_0 \in \mathbb{R}^m \, | w_0 = B^{-1}(Ay - z) \right\}. \quad (28)$$

Theorem 3.6: Suppose B is invertible, the matrix $\Theta_{1,3}$ defining B_1^3 of (14) is given by

$$\Theta_{1,3} = \begin{bmatrix} -2\Phi + \gamma^2 A^T(BB^T)^{-1}A & -\gamma^2 A^T(BB^T)^{-1} \\ -\gamma^2(BB^T)^{-1}A & \gamma^2(BB^T)^{-1} \end{bmatrix}. \quad (29)$$

2) Case 2: $n > m, m = 1$: In this case, $\Lambda_k(y,z) \neq \emptyset$, $k \geq n$ for all $(y,z) \in \mathbb{R}^n \times \mathbb{R}^n$ since system (1) is assumed to be controllable. Denote

$$\bar{x} \doteq \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}, \quad \bar{w} \doteq \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \end{bmatrix}, \quad \bar{A} \doteq \begin{bmatrix} I \\ A \\ \vdots \\ A^{n-1} \end{bmatrix},$$

$$\bar{B} \doteq \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ B & 0 & \cdots & 0 & 0 \\ AB & B & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{n-2}B & A^{n-3}B & \cdots & B & 0 \end{bmatrix}, \quad (30)$$

$$\bar{C} \doteq \left[A^{n-1}B, A^{n-1}B, \cdots, AB, B \right],$$

$$\bar{\Phi} \doteq \mathrm{diag}(\Phi, \Phi, \cdots, \Phi).$$

Using these notation,

$$\bar{x} = \bar{A}y + \bar{B}\bar{w}, \;\; z = A^n y + \bar{C}\bar{w}.$$

Here $x_0 = y, x_n = z$.

In the case $m = 1$, $\bar{C} \in \mathbb{R}^{n\times n}$ is invertible. Thus,

$$\Lambda_n(y,z) = \left\{ \bar{w} \in \mathbb{R}^n \, | \bar{w} = \bar{C}^{-1}(z - A^n y) \right\}. \quad (31)$$

Theorem 3.7: Suppose $n > m = 1$, the matrix $\Theta_{n,3}$ that defines the function B_n^3 of (14) is given by

$$\Theta_{n,3} = \quad (32)$$
$$\begin{bmatrix} -2R_1^T\bar{\Phi}R_1 + \gamma^2 R_3^T R_3 & -2R_1^T\bar{\Phi}R_2 + \gamma^2 R_3^T R_4 \\ -2R_2^T\bar{\Phi}R_1 + \gamma^2 R_4^T R_3 & -2R_2^T\bar{\Phi}R_2 + \gamma^2 R_4^T R_4 \end{bmatrix},$$

with

$$R_1 = \bar{A} - \bar{B}\bar{C}^{-1}A^n, \quad R_2 = \bar{B}\bar{C}^{-1},$$
$$R_3 = -\bar{C}^{-1}A^n, \;\; R_4 = \bar{C}^{-1}. \quad (33)$$

3) Case 3: $n > m, m > 1$: In this case, $\mathrm{rank}(\bar{C}) = n$ since (A,B) is controllable. So $\Lambda_n(y,z) \neq \emptyset$ for any $(y,z) \in \mathbb{R}^n \times \mathbb{R}^n$. The following result is needed to obtain a characterization of $\Lambda_n(y,z)$. For a matrix $M \in \mathbb{R}^{p\times q}$, denote

$$\mathrm{Im}(M) \doteq \left\{ Mv | v \in \mathbb{R}^q \right\}. \quad (34)$$

$\mathrm{Im}(M)$ is a subspace in $\mathbb{R}^p$ with dimension $\mathrm{rank}(M)$.

Lemma 3.8: Suppose $M \in \mathbb{R}^{p\times p}$, $\mathrm{rank}(M) = r \leq p$. Then, there exists a matrix $D \in \mathbb{R}^{p\times r}$ with $\mathrm{rank}(D) = r$ such that

$$\mathrm{Im}(M) = \mathrm{Im}(D). \quad (35)$$

Since $\bar{C}$ has full rank, $\Lambda_n(y,z)$ can be characterized

$$\Lambda_n(y,z) = \left\{ \bar{w} \in \mathbb{R}^{mn} \, | z - A^n y = \bar{C}\bar{w} \right\} \quad (36)$$
$$= \left\{ \bar{C}^+(z - A^n y) + (I - \bar{C}^+\bar{C})\, \tilde{w} \, | \tilde{w} \in \mathbb{R}^{nm} \right\}$$

Here $\bar{C}^+ = \bar{C}^T(\bar{C}\bar{C}^T)^{-1} \in \mathbb{R}^{mn\times n}$ is the Moore-Penrose pseudoinverse. Here $I - \bar{C}^+\bar{C} \in \mathbb{R}^{mn\times mn}$ may not invertible. Suppose $\mathrm{rank}(I - \bar{C}^+\bar{C}) = r \leq mn$, from Lemma 3.8, there exists $\bar{D} \in \mathbb{R}^{mn\times r}$ with $\mathrm{rank}(\bar{D}) = r$ such that

$$\left\{ (I - \bar{C}^+\bar{C})\, \tilde{w} \, | \tilde{w} \in \mathbb{R}^{mn} \right\} = \left\{ \bar{D}\, \hat{w} \, | \hat{w} \in \mathbb{R}^r \right\}. \quad (37)$$

Thus $\Lambda_n(y,z)$ is characterized

$$\Lambda_n(y,z) = \left\{ (z - A^n y) + \bar{D}\hat{w} \, | \hat{w} \in \mathbb{R}^r \right\}. \quad (38)$$

Theorem 3.9: Assume $n > m, m > 1$, the matrix $\Theta_{n,3}$ that defines the function B_n^3 of (14) is given by

$$\Theta_{n,3} = \quad (39)$$
$$\begin{bmatrix} -2\bar{R}_1^T\bar{\Phi}\bar{R}_1 + \gamma^2 \bar{R}_3^T \bar{R}_3 & -2\bar{R}_1^T\bar{\Phi}\bar{R}_2 + \gamma^2 \bar{R}_3^T \bar{R}_4 \\ -2\bar{R}_2^T\bar{\Phi}\bar{R}_1 + \gamma^2 \bar{R}_4^T \bar{R}_3 & -2\bar{R}_2^T\bar{\Phi}\bar{R}_2 + \gamma^2 \bar{R}_4^T \bar{R}_4 \end{bmatrix},$$

with

$$\Pi_1 = 2\bar{D}\bar{\Phi}(\bar{A} - \bar{B}\bar{C}^+A^n) + \gamma^2\bar{C}^+A^n,$$
$$\Pi_2 = 2\bar{D}\bar{\Phi}\bar{B}\bar{C}^+ - \gamma^2\bar{D}\bar{C}^+, \;\; \bar{\Omega} = \bar{D}^T(2\bar{B}^T\bar{\Phi}\bar{B} - \gamma^2 I)\bar{D},$$
$$\bar{R}_1 = \bar{A} - \bar{B}\bar{C}^+A^n - \bar{B}\bar{D}\bar{\Omega}^{-1}\Pi_1, \quad (40)$$
$$\bar{R}_2 = \bar{B}\bar{C}^+ - \bar{B}\bar{D}\bar{\Omega}^{-1}\Pi_2,$$
$$\bar{R}_3 = \bar{C}^+A^n + \bar{D}\bar{\Omega}^{-1}\Pi_1, \;\; \bar{R}_4 = -\bar{C}^+ + \bar{D}\bar{\Omega}^{-1}\Pi_2.$$

IV. Examples

Equation (12) gives a formula to compute the value function W_k of (2) using the functions $\mathrm{S}_k^i(x,z)$ and the max-plus dual $\widehat{\Psi}^i(z)$ of the terminal payoff $\Psi(x)$. The functions $\mathrm{S}_k^i, k \in \mathbb{Z}_{>0}$ are quadratic and characterized by the matrix $Q_{k,i}$ which can be obtained from the matrix $\Theta_{k,i}$ via the operators Γ^i of (17). The matrix sequence $\Theta_{k,i}$ is obtained using the matrix operations $\Theta_{k_1+k_2,i} = \Theta_{k_1,i} \circledast \Theta_{k_2,i}$ of (23). Thus the computation of the fundamental solutions is reduced to matrix operations which is fast. The main computation of W_k via (12) is concentrated on the computation of the max-plus dual $\widehat{\Psi}^i$ and the maximization in (12), which is independent of the control horizon. This implies the amount of computation using max-plus fundamental solution method increases very slowly with respect to time k compared to the direct iteration method using dynamic programming equation (4) in which the computation increases linearly with time. It can be expected that significant computational time saving can be achieved using max-plus fundamental solution method over direct iteration on the dynamic programming equation especially when the control horizon is large. This example demonstrates this computation advantage of the max-plus fundamental method in $\mathscr{B}_r^1$.

Consider a control problem with data

$$A = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}, \; B = \begin{bmatrix} 0.1 \\ 0.03 \end{bmatrix}, \; \Phi = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 1 \end{bmatrix}.$$

The terminal payoff is taken as a quadratic function $\Psi(x) = \frac{1}{2}x^T \Lambda x$ with $\Lambda = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 0.5 \end{bmatrix}$. The reason of using a quadratic terminal payoff function to demonstrate the speed advantage of the max-plus fundamental solution is that the actual value function $W_k(x)$ can be approximated rather accurately using the Riccati difference equation (5). Figure 1 is the function $W_{64}(x)$ computed using the DRE (5) which is taken as the actual value function (2) for this optimal control problem.

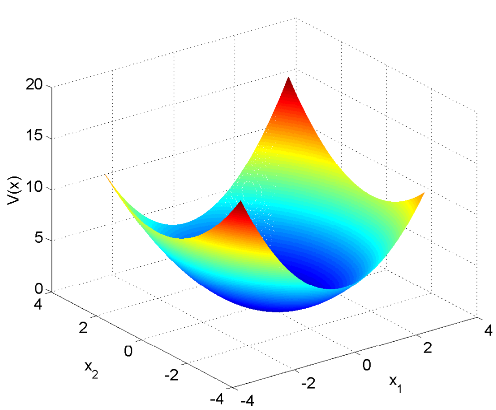


Fig. 1. Value

Now compute the same value function using direct dynamic programming equation (4) and the max-plus fundamental solution (12) on a bounded domain $B = [-3 \;\; 3] \times [-3 \;\; 3]$. The comparison is the time needed to obtain the approximate function using the two methods with the requirement that the obtained functions are within a specified error level with respect to the "actual" value function in Figure 1. Here, the error of a function $\widehat{W}_{64}$ with respect to the value function $W_{64}(x)$ is defined by

$$e = \sup_{x \in \bar{B}} \left\{ |W_{64}(x) - \widehat{W}_{64}(x)| \right\},$$

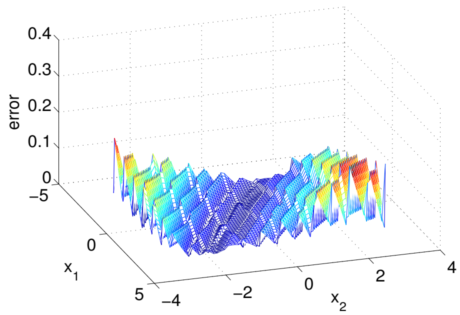
where the set $\bar{B} = [-1 \;\; 1] \times [-1 \;\; 1]$. A smaller region $\bar{B}$ is chosen to compute the error in order to reduce the errors caused by the boundary approximations in using the direct iteration method. The allowable error bound is taken to be 0.1 in this example.

Figure 2 depicts the error function

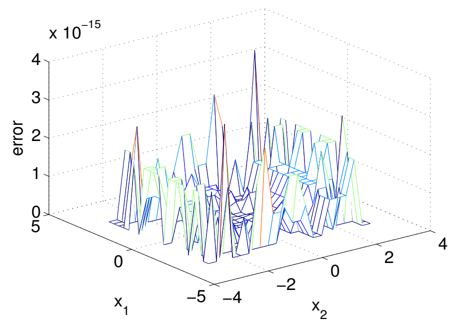
$$E_j(x) \doteq |W_{64}(x) - \widehat{W}_{64}^j(x)|, j = 1,2$$

with W_{64} as shown in Figure 1 and $\widehat{W}_{64}^j, j = 1,2$ denote the approximations using direct DPP iterations and max-plus fundamental solutions methods with grid sizes 0.1 and 0.3 respectively. By observation, the error using direct DPP iteration is much larger than the error with max-plus fundamental solution. Using the definition of the error $e_j = \max_{x \in \bar{B}} E_j(x), j = 1,2,$ the error for the direct DPP iteration method $e_1 = 0.08$ and the error for the max-plus fundamental solutions e_2 is almost zero. An explanation for the smaller error using max-plus method is that the computation of the fundamental solution matrices $Q_{k,1}$ is almost error free since no approximation is involved. So the fundamental solutions can be accurately computed. Errors due to state-space discretization arise only on the computation of the max-plus dual of the terminal payoff and compute the value function using (12). While in the direct DPP iteration method, there are errors caused by state space discretization in every step iteration.

Panel (a) in Figure 3 shows the time used to obtain the approximate solution $\widehat{W}_k^2$ using the max-plus fundamental solution. The time used to compute an approximate solution $\widehat{W}_k^2, k \in [1,128]$ is divided into 2 parts. The first part is the time used to compute the max-plus dual of the terminal payoff and compute the value function $\widehat{W}_k^2$ using the fundamental solution via (12) at the end. The time of this part is fixed for all k and is 2.7961 seconds in this example. The other part is the time used to compute the quadratic fundamental solution Hessian $Q_{k,1}$ by the matrix iteration (23). In using (23) to compute $Q_{k,1}$ for $k \in [1,128]$, first write k in binary form $k = [b_1 b_2 \cdots b_{p_k}]$ with $b_i \in \{0,1\}$ such that $b_1 = 1$ and $k = b_1 \times 2^{p_k - 1} + b_2 \times 2^{p_k - 2} + \cdots + b_{p_k} \times 2^0$. Denote the number of bits which are $b_i = 1$ in the string $k = [b_1 b_2 \cdots b_{p_k}]$ to be n_k. Also denote the time needed to perform the matrix manipulation $\Omega_1 \circledast \Omega_2$ for two matrices $\Omega_1, \Omega_2 \in \mathbb{R}^{4\times4}$ to be τ. Then, the time needed to compute the matrix $Q_{k,1}$ according to iteration (23) is given by $t_k = ((p_k - 1) + (n_k - 1))\tau$. The function t_k is shown in panel (a) of Figure 3 although the variations are very small because τ is very small.

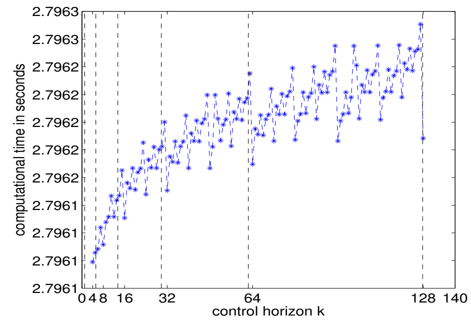


(a) Error using direct DPP iteration.

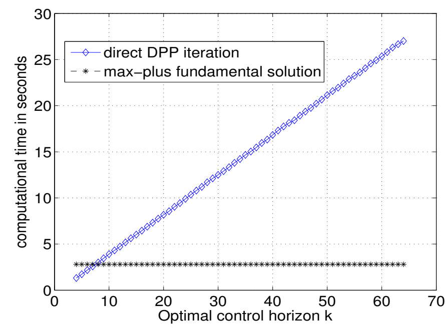


(b) Error using max-plus fundamental solution

Fig. 2. Errors in approximations



(a) Computational time using max-plus fundamental solution.



(b) Computational time comparison: direct DPP iteration vs max-plus fundamental solution.

Fig. 3. Computational times

Panel (b) of Figure 3 depicts the comparison of the time used to compute the approximate value functions using direct DPP iteration and the max-plus fundamental solution. As expected, the time needed for the direct iteration method increases linearly with respect to control horizon k. While the time needed using the max-plus fundamental solution method does not increase appreciably with respect to the horizon of the problem. Thus the max-plus fundamental solution is particularly useful in computing value functions with large horizon.

V. Conclusions

An efficient computational method is developed for solving linear quadratic optimal control problems with non-quadratic payoff functions. The max-plus linearity property of the corresponding dynamic programming evolution operator is exploited to obtain max-plus fundamental solutions through which the value functions can be computed conveniently for any non-quadratic terminal payoff. The computation of the max-plus fundamental solutions are reduced to matrix iterations which can be computed efficiently and accurately. An example is given to demonstrate the performance of the proposed methods.

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