

Coherent Quantum Filtering for Physically Realizable Linear Quantum Plants*

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Abstract—The paper is concerned with a problem of coherent (measurement-free) filtering for physically realizable (PR) linear quantum plants. The state variables of such systems satisfy canonical commutation relations and are governed by linear quantum stochastic differential equations, dynamically equivalent to those of an open quantum harmonic oscillator. The problem is to design another PR quantum system, connected unilaterally to the output of the plant and playing the role of a quantum filter, so as to minimize a mean square discrepancy between the dynamic variables of the plant and the output of the filter. This coherent quantum filtering (CQF) formulation is a simplified feedback-free version of the coherent quantum LQG control problem which remains open despite recent studies. The CQF problem is transformed into a constrained covariance control problem which is treated by using the Frechet differentiation of an appropriate Lagrange function with respect to the matrices of the filter.

I. INTRODUCTION

Interconnection of open systems, whose internal dynamics are affected by interaction with the surroundings, is often engineered so as to stabilize the resulting network of such systems via redistribution and dissipation of energy generated by active nodes or coming from the environment. In addition to its role in redirecting the energy flow, interaction provides a universal mechanism for creating correlations whereby current states of different subsystems acquire and store dynamic “footprints” of each other and of the past history of the whole system. This informational aspect of interaction is directly employed in the classical Kalman filter whose state is continuously updated by the measurement process from a stochastic system, thus enabling the filter to develop and maintain relatively strong correlation with the unknown state of the system. The ability of such a filter to track a classical linear system with finitely many degrees of freedom and known dynamics by extracting as much information from the noisy observations as possible is, in principle, limited only by digital implementation. The situation is qualitatively different in regard to estimating the dynamic variables of a quantum stochastic system which are noncommutative operators on a Hilbert space evolving in time according to the laws of quantum mechanics [11]. The measurements, which result from the interaction of the quantum system with a relatively invasive classical device, are accompanied by irreversible loss of quantum information as a consequence of the projection postulate of quantum mechanics [5]. It is the idea of using a system of the same kind, that is,

another quantum system, weakly coupled to the quantum mechanical object of interest (a quantum plant), which underlies the coherent (that is, measurement-free) quantum filtering/control paradigm. This approach replaces measurement with interaction of quantum systems, possibly mediated by light fields, where the energy flow can be employed for stabilization/control [7] and the quantum information manifests itself through quantum correlations between the dynamic variables of the systems in the course of time. An important class of quantum stochastic systems is provided by open quantum harmonic oscillators [2], [3] whose variables satisfy canonical commutation relations (CCRs) and are governed by linear quantum stochastic differential equations (QSDEs) driven by boson fields [14]. In combination with the preservation of CCRs under unitary evolutions in the Heisenberg picture of quantum dynamics, the specific energetics of such systems (quadratic Hamiltonian and linear coupling to the external fields) imposes certain constraints [8], [16] on the coefficients of the governing linear QSDEs in order for them to be physically realizable (PR) as open quantum harmonic oscillators. Such systems can be implemented in practice by using quantum-optical components [3]. Being quadratic with respect to the state-space matrices, the PR constraints make the coherent quantum counterpart [13] of the classical LQG control problem [9] substantially harder to solve than its classical predecessor. In fact, the coherent quantum LQG (CQLQG) feedback design problem remains open despite recent studies [18], [20] which explore different approaches to its solution. In the present paper, we consider an infinite-horizon coherent quantum filtering (CQF) problem for PR linear quantum plants. The question of interest is to design another PR quantum system, connected unilaterally to the output of the plant and playing the role of a quantum filter, so as to minimize a steady-state mean square discrepancy between the dynamic variables of the plant and the output of the filter. In the absence of measurements and in the presence of PR constraints, the machinery of recursive Bayesian estimation (including conditional expectations), so useful for Kalman filtering, is inapplicable to the CQF problem. Following [18] based on algebraic ideas from [1], [17], we transform the CQF problem into a constrained covariance control problem which is treated by using the Frechet differentiation of an appropriate Lagrange function with respect to the matrices of the filter. Since the CQF setting is a simplified feedback-free version of the CQLQG control problem mentioned above, this leads to a more explicit set of algebraic equations for the state-space matrices of an optimal PR quantum filter which are amenable to further analysis to be published elsewhere. Note that the

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recognition of the need to take into account PR constraints in coherent quantum filtering problems dates back to [4], although that work was mainly concerned with measurement-based mean square optimal filtering. A coherent quantum filtering problem has recently been discussed in [12], where, unlike the present paper, optimization of the filter was not considered. Complete proofs are given in the preprint [19].

II. PHYSICALLY REALIZABLE QUANTUM PLANT

As in the linear quantum control settings [8], [13], [18] mentioned above, the quantum plant considered below is an open quantum stochastic system with canonically commuting variables whose internal dynamics are affected by the quantum noise from the environment. More precisely, the plant has n dynamic variables $x_1(t), \dots, x_n(t)$ which are self-adjoint operators on the tensor product Hilbert space¹ $\mathcal{H}_1 \otimes \mathcal{F}_1$ evolving in time t and satisfying the canonical commutation relations (CCRs)

$$[x, x^T] := ([x_j, x_k])_{1 \leq j, k \leq n} = xx^T - (xx^T)^T = 2i\Theta_1. \quad (1)$$

Here, $x := (x_k)_{1 \leq k \leq n}$ is the vector² of the plant variables (the time argument is often omitted for the sake of brevity), $[\eta, \zeta] := \eta\zeta - \zeta\eta$ is the commutator of operators, $i := \sqrt{-1}$ is the imaginary unit, and Θ_1 is a constant real antisymmetric matrix of order n (the subspace of such matrices is denoted by \mathbb{A}_n) which is assumed to be nonsingular, and hence, n is even. The operators $x_1(0), \dots, x_n(0)$ act on the initial complex separable Hilbert space \mathcal{H}_1 of the system, and \mathcal{F}_1 is the boson Fock space [14] which provides a domain for the action of the quantum Wiener processes $w_1(t), \dots, w_{m_1}(t)$. The latter are self-adjoint operators on \mathcal{F}_1 (which are obtained from pairs of field annihilation and creation operators by using complex unitary (2×2) -matrices) and represent the quantum noise from the environment with the quantum Ito table $dw dw^T = \Omega_1 dt$. Here, $w := (w_k)_{1 \leq k \leq m_1}$, and Ω_1 is a constant complex positive semi-definite Hermitian matrix of order m_1 . Similarly to (1), the imaginary part $J_1 := \text{Im}\Omega_1 \in \mathbb{A}_{m_1}$ of the quantum Ito matrix Ω_1 specifies the CCRs between the quantum Wiener processes as

$$[dw, dw^T] = 2iJ_1 dt. \quad (2)$$

In what follows, the real part $\text{Re}\Omega_1$ is the identity matrix I_{m_1} of order m_1 , that is,

$$\Omega_1 = I_{m_1} + iJ_1. \quad (3)$$

Also, it is assumed that the noise dimension m_1 is even, and the CCR matrix J_1 has a canonical form

$$J_1 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_{\mu_1} = \begin{bmatrix} 0 & I_{\mu_1} \\ -I_{\mu_1} & 0 \end{bmatrix}, \quad (4)$$

where \otimes is the Kronecker product of matrices, and $\mu_1 := m_1/2$. The plant state vector x evolves in time and contributes to a p -dimensional output of the plant y (whose entries are also self-adjoint operators on $\mathcal{H}_1 \otimes \mathcal{F}_1$) according to QSDEs

¹Some of the spaces and parameters associated with the plant are equipped with the subscript "1", whereas the subscript "2" is used for analogous objects pertaining to the quantum filter introduced in Section III.

²Vectors are organized as columns unless specified otherwise, and the transpose $(\cdot)^T$ acts on matrices with operator-valued entries as if the latter were scalars.

$$dx = Axdt + Bdw, \quad dy = Cxdt + Ddw. \quad (5)$$

Here, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m_1}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m_1}$ are given constant matrices, with A Hurwitz. In addition to the asymptotic stability, the plant is assumed to be physically realizable (PR) as an open quantum harmonic oscillator [2], [3] whose Hamiltonian is quadratic and the coupling operators are linear with respect to the plant variables. By the results of [8], [13], [16], in the case of linear quantum dynamics being considered, the PR property is equivalent to the algebraic relations

$$A\Theta_1 + \Theta_1 A^T + BJ_1 B^T = 0, \quad (6)$$

$$\Theta_1 C^T + BJ_1 D^T = 0 \quad (7)$$

which describe the preservation in time of the CCRs between the state and output variables of the plant described by (1) and $[x, y^T] = 0$. Indeed, (6), (7) are obtained from the relationships

$$d[x, x^T] = (A[x, x^T] + [x, x^T]A^T + 2iBJ_1 B^T)dt, \quad (8)$$

$$d[x, y^T] = (A[x, y^T] + [x, x^T]C^T + 2iBJ_1 D^T)dt \quad (9)$$

which follow from the bilinearity of the commutator [11], the quantum Ito rule, and the commutativity between adapted processes and forward increments of the quantum Wiener process [14], in view of (2), (5).

III. PHYSICALLY REALIZABLE COHERENT QUANTUM FILTER

Consider another PR open quantum stochastic system with q canonically commuting dynamic variables which are influenced by the environment and the plant in a unilateral fashion. Due to this cascade connection which is shown in Fig. 1, the state and output variables of the second system

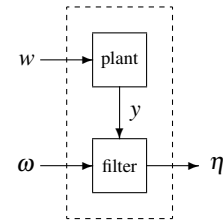


Fig. 1. The dynamics of the filter are influenced by the quantum Wiener process ω and the plant output y through a unilateral cascade connection described by (5), (11).

under consideration acquire quantum correlation with the plant variables in the course of time, which enables this system to be regarded as a coherent (that is, measurement-free) quantum filter. A performance criterion for such a filter will be specified in Section IV. Now, the filter state variables $\xi_1(t), \dots, \xi_q(t)$ are assumed to be self-adjoint operators on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{F}_1 \otimes \mathcal{F}_2$ satisfying the CCRs

$$[\xi, \xi^T] = 2i\Theta_2, \quad (10)$$

where $\xi := (\xi_k)_{1 \leq k \leq q}$ and $\Theta_2 \in \mathbb{A}_q$, with q even and $\det \Theta_2 \neq 0$. Here, \mathcal{H}_2 is the initial space of the filter and \mathcal{F}_2 is the boson Fock space which provides a domain for the action of a quantum Wiener process $\omega := (\omega_k)_{1 \leq k \leq m_2}$. The latter commutes with the plant noise w and drives the filter variables according to the QSDE

$$d\xi = a\xi dt + b d\omega + e dy, \quad d\eta = c\xi dt + d d\omega, \quad (11)$$

where $a \in \mathbb{R}^{q \times q}$, $b \in \mathbb{R}^{q \times m_2}$, $c \in \mathbb{R}^{r \times q}$, $d \in \mathbb{R}^{r \times m_2}$, $e \in \mathbb{R}^{q \times p}$ are constant matrices, and a is Hurwitz. The quantum Wiener process ω is assumed to have even dimension m_2 and a canonical quantum Ito table

$$d\omega d\omega^T = \Omega_2 dt, \quad \Omega_2 := I_{m_2} + iJ_2, \quad (12)$$

analogous to (3), (4). Here, $J_2 \in \mathbb{A}_{m_2}$ is a CCR matrix of the filter noise in the sense that

$$[d\omega, d\omega^T] = 2iJ_2 dt, \quad J_2 := \begin{bmatrix} 0 & I_{\mu_2} \\ -I_{\mu_2} & 0 \end{bmatrix}, \quad (13)$$

with $\mu_2 := m_2/2$. The matrices b , e in (11) play the role of gain matrices of the quantum filter with respect to the filter noise ω and the plant output y . The plant, filter and the environment form a closed quantum system which is governed by (5), (11). Therefore, the $(n+q)$ -dimensional vector

$$\mathcal{X} := \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad (14)$$

formed by the plant and filter state variables, is driven by the combined quantum Wiener process

$$\mathcal{W} := \begin{bmatrix} w \\ \omega \end{bmatrix} \quad (15)$$

of dimension $m := m_1 + m_2$ according to the QSDE

$$d\mathcal{X} = \mathcal{A} \mathcal{X} dt + \mathcal{B} d\mathcal{W}, \quad (16)$$

with

$$\mathcal{A} := \begin{bmatrix} A & 0 \\ eC & a \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} B & 0 \\ eD & b \end{bmatrix}. \quad (17)$$

The matrices \mathcal{A} , \mathcal{B} have a block lower triangular structure due to the absence of feedback, as opposed to the closed-loop quantum control settings [8], [13], [18]. The plant and filter noises w , ω result from interaction of the systems with external boson fields which are assumed to be in the product vacuum state $v := v_1 \otimes v_2$ on the space $\mathcal{F}_1 \otimes \mathcal{F}_2$. Since the noises commute with each other and are uncorrelated, then, in view of (2)–(4) and (12), (13), the combined Wiener process \mathcal{W} in (15) has a block diagonal quantum Ito table

$$d\mathcal{W} d\mathcal{W}^T = \Omega dt, \quad \Omega := \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} = I_m + iJ. \quad (18)$$

Here, $J \in \mathbb{A}_m$ denotes the corresponding CCR matrix of \mathcal{W} :

$$[d\mathcal{W}, d\mathcal{W}^T] = 2iJ dt, \quad J := \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}. \quad (19)$$

In a similar vein, the plant and filter variables are assumed to commute, and hence, the combined vector (14) has a block-diagonal CCR matrix:

$$[\mathcal{X}, \mathcal{X}^T] = 2i\Theta, \quad \Theta := \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{bmatrix}, \quad (20)$$

where use is made of (1), (10). Due to the unitary evolution of the isolated system formed by the plant, filter and their environment, the CCR matrix Θ is also preserved in time, which is equivalent to the algebraic Lyapunov equation

$$\mathcal{A}\Theta + \Theta\mathcal{A}^T + \mathcal{B}J\mathcal{B}^T = 0. \quad (21)$$

The left-hand side of (21) is a real antisymmetric matrix whose respective blocks are computed as

$$A\Theta_1 + \Theta_1 A^T + BJ_1 B^T = 0, \quad (22)$$

$$(\Theta_1 C^T + BJ_1 D^T)e^T = 0, \quad (23)$$

$$a\Theta_2 + \Theta_2 a^T + eDJ_1 D^T e^T + bJ_2 b^T = 0 \quad (24)$$

in view of (17), (19), (20). The fulfillment of (22), (23) is guaranteed by the PR properties (6), (7) of the quantum plant for an arbitrary coherent quantum filter (11), whilst (24) and

$$\Theta_2 c^T + bJ_2 d^T = 0 \quad (25)$$

describe PR conditions for the filter which correspond to the preservation of the CCRs (10) and $[\xi, \eta^T] = 0$, respectively. The derivation of the PR conditions (24), (25) is similar to that of (6), (7) in (8), (9), except that the dynamic variables of the filter are driven not only by the quantum noise ω , but also by the plant noise w through the plant output y according to the QSDE (16), which explains the presence of the additional term $eDJ_1 D^T e^T$ in (24). Since $\det \Theta_2 \neq 0$, the general solution of (24), as a linear equation with respect to the matrix a , is

$$a = 2\Theta_2 R - \frac{1}{2}(eDJ_1 D^T e^T + bJ_2 b^T)\Theta_2^{-1}. \quad (26)$$

Here, R is an arbitrary real symmetric matrix of order q (the subspace of such matrices is denoted by \mathbb{S}_q) which specifies the quadratic Hamiltonian $\xi^T R \xi / 2$ of an equivalent representation of the PR filter as an open quantum harmonic oscillator. Furthermore, (25) allows the matrix c to be expressed in terms of b as

$$c = -dJ_2 b^T \Theta_2^{-1}. \quad (27)$$

The coupling of the state-output matrix c to the noise gain matrix b makes the optimization of the coherent quantum filter (11) qualitatively different from that of the classical controllers and filters, regardless of the performance criterion. Indeed, (27) shows that the PR quantum filter requires an ‘‘intake’’ of the additional quantum noise ω (through $b \neq 0$) in order to produce a useful output η with a nonzero ‘‘signal’’ component

$$\zeta := c\xi. \quad (28)$$

In their original form [8], [13], [16], the PR conditions also involve a specification of the noise feedthrough matrices D , d in (5), (11) as those formed from rows of orthogonal matrices, whereby

$$DD^T = I_p, \quad dd^T = I_r. \quad (29)$$

Therefore, $p \leq m_1$, $r \leq m_2$, and both D and d have full row rank. If y in (5) were a classical observation process, the full row rank property of D would correspond to nondegeneracy of the measurements. Also, since d is of full row rank, the map $\mathbb{R}^{q \times m_2} \ni b \mapsto c \in \mathbb{R}^{r \times q}$, described by (27), is surjective, and hence, the matrix c can be assigned any value by an appropriate choice of b . Although (29) will simplify algebraic manipulations, it is the rank properties of D , d that are principal for what follows.

IV. COHERENT QUANTUM FILTERING PROBLEM

Consider the problem of constructing a PR coherent quantum filter (11) of fixed dimensions (and with a fixed noise feedthrough matrix d), described in Section III, so as to minimize a steady-state mean square discrepancy between the filter output variables and the state variables of a given PR quantum plant (5) specified in Section II. More precisely, let $\mathcal{Z}(t) := (\mathcal{Z}_k(t))_{1 \leq k \leq s}$ denote an s -dimensional quantum process defined by

$$\mathcal{Z} := Fx - G\zeta = \mathcal{C}\mathcal{X}, \quad (30)$$

where $F \in \mathbb{R}^{s \times n}$, $G \in \mathbb{R}^{s \times r}$ are given matrices, with $s \geq r$ and G having full column rank (the role of this assumption is clarified later), and ζ is the signal part (28) of the filter output η in (11), and hence, in view of (14),

$$\mathcal{C} := [F \quad -Gc]. \quad (31)$$

The coherent quantum filtering (CQF) problem is formulated as the minimization of the quantity

$$\begin{aligned} \mathcal{E} &:= \lim_{t \rightarrow +\infty} \mathbf{E}(\mathcal{Z}^T \mathcal{Z})/2 = \langle \mathcal{C}^T \mathcal{C}, P \rangle / 2 \\ &\rightarrow \min, \quad \text{subject to (24), (25)}. \end{aligned} \quad (32)$$

Here, the $1/2$ factor is introduced for further convenience, $\mathcal{Z}^T \mathcal{Z} = \sum_{k=1}^s \mathcal{Z}_k^2$ is the sum of squared entries of \mathcal{Z} , and $\mathbf{E}(\cdot)$ denotes the quantum expectation over the product state $\varpi \otimes \nu$, where ϖ is the initial quantum state of the plant-filter composite system on $\mathcal{H}_1 \otimes \mathcal{H}_2$, and ν is the vacuum state of the external fields. Also, $\langle M, N \rangle := \text{Tr}(M^* N)$ is the Frobenius inner product of complex or real matrices, with $(\cdot)^* := (\overline{(\cdot)})^T$ the complex conjugate transpose, and P is the real part of the steady-state quantum covariance matrix of the vector \mathcal{X} in (14):

$$K := \lim_{t \rightarrow +\infty} \mathbf{E}(\mathcal{X} \mathcal{X}^T) = P + i\Theta, \quad P := \text{Re}K, \quad (33)$$

with $\lim_{t \rightarrow +\infty} \mathbf{E}\mathcal{X} = 0$, since the matrix \mathcal{A} in (17) is Hurwitz. The latter condition is ensured by the Hurwitz property of the matrix a , since \mathcal{A} is block lower triangular and A is Hurwitz. The matrix K has the imaginary part $\text{Im}K = \Theta$ given by (20) if the coherent quantum filter is also PR. In view of (16), (18), the matrix P in (33) is a unique solution of the algebraic Lyapunov equation

$$\mathcal{A}P + P\mathcal{A}^T + \mathcal{B}\mathcal{B}^T = 0 \quad (34)$$

and coincides with the controllability Gramian [9] of the pair $(\mathcal{A}, \mathcal{B})$. The minimum in the CQF problem (32) is taken over the quadruple $(a, b, c, e) \in \mathbb{R}^{q \times q} \times \mathbb{R}^{q \times m_2} \times \mathbb{R}^{r \times q} \times \mathbb{R}^{q \times p}$ of the state-space matrices of the filter (11) subject to the PR constraints (24), (25), with a fixed noise feedthrough matrix $d \in \mathbb{R}^{r \times m_2}$ satisfying (29). In particular, if $F = G = I_n$, with $r = s = n$, the CQF problem (32) consists in approximating the plant state vector x by the signal part ζ of the filter output η from (28) so as to minimize the ‘‘estimation error’’ $x - \zeta$ in the mean square sense.

V. QUALITATIVE DEPENDENCE ON FILTER MATRICES

The performance index in the CQF problem (32) is a composite function $(a, b, c, e) \mapsto (\mathcal{A}, \mathcal{B}, \mathcal{C}) \mapsto \mathcal{E}$, where the triple $(\mathcal{A}, \mathcal{B}, \mathcal{C}) \in \mathbb{R}^{(n+q) \times (n+q)} \times \mathbb{R}^{(n+q) \times m} \times \mathbb{R}^{s \times (n+q)}$ of matrices from (17), (31) depends affinely on the matrix quadruple (a, b, c, e) (with both being regarded as elements of appropriate direct sum Hilbert spaces). The smoothness of \mathcal{E} with respect to $\mathcal{A}, \mathcal{B}, \mathcal{C}$ (regardless of the specific structure of these matrices) is ensured by the smooth dependence of the controllability Gramian

$$P = \Lambda_{\mathcal{A}}(\mathcal{B}\mathcal{B}^T) \quad (35)$$

from (34) on \mathcal{A}, \mathcal{B} over the open subset of Hurwitz matrices \mathcal{A} . Here, Λ_{α} denotes the inverse Lyapunov operator, that is, a particular case of the inverse Sylvester operator $\Sigma_{\alpha, \beta}$ which is associated with Hurwitz matrices α, β and maps an appropriately dimensioned matrix X to the unique solution $Y := \Sigma_{\alpha, \beta}(X)$ of the algebraic Sylvester equation $\alpha Y + Y\beta + X = 0$:

$$\Lambda_{\alpha} := \Sigma_{\alpha, \alpha^T}, \quad \Sigma_{\alpha, \beta}(X) := \int_0^{+\infty} e^{\alpha t} X e^{\beta t} dt. \quad (36)$$

For what follows, the controllability Gramian P in (35) (and other related matrices of order $n+q$) is split into blocks $P_{11} \in \mathbb{S}_n$, $P_{22} \in \mathbb{S}_q$, $P_{12} = P_{21}^T \in \mathbb{R}^{n \times q}$ and the corresponding block-rows $P_{j\bullet}$ and block-columns $P_{\bullet k}$ in accordance with their association with the plant and filter variables in (14) as

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} P_{1\bullet} \\ P_{2\bullet} \end{bmatrix} = [P_{\bullet 1} \quad P_{\bullet 2}]. \quad (37)$$

The Lyapunov equation (34), whose left-hand side is a symmetric matrix with a similar block partitioning, can be written in terms of (37) as

$$AP_{11} + P_{11}A^T + BB^T = 0, \quad (38)$$

$$AP_{12} + P_{12}a^T + P_{11}C^T e^T + BD^T e^T = 0, \quad (39)$$

$$aP_{22} + P_{22}a^T + eCP_{12} + P_{21}C^T e^T + ee^T + bb^T = 0, \quad (40)$$

where (17), (29) are used. The inverse Lyapunov and Sylvester operators (36) allow the solution of (38)–(40) to be represented as

$$P_{11} = \Lambda_A(BB^T), \quad (41)$$

$$P_{12} = \Sigma_{A, a^T}((P_{11}C^T + BD^T)e^T), \quad (42)$$

$$P_{22} = \Lambda_a(eCP_{12} + P_{21}C^T e^T + ee^T + bb^T). \quad (43)$$

Since the matrix P_{11} in (41) is specified completely by the quantum plant (5) and does not depend on the coherent quantum filter, the matrix $P_{11}C^T + BD^T$ in (42) is also constant. Therefore, for any given Hurwitz matrix a of the filter (11), the matrix P_{12} is a linear function of e . Hence, P_{22} in (43) is a homogeneous quadratic polynomial of the filter matrices b, e from (11), whose coefficients depend on a and which does not contain the $b_{jke}l_z$ cross-terms with mixed entries of b, e . The performance index \mathcal{E} in the CQF problem (32) is representable in terms of the block partitioning (37) as

$$\mathcal{E} = \langle F^T F, P_{11} \rangle / 2 - \langle F^T Gc, P_{12} \rangle + \langle c^T G^T Gc, P_{22} \rangle / 2, \quad (44)$$

where (31) is used together with the symmetry of the matrices $\mathcal{C}^T \mathcal{C}$ and P . In combination with the quadratic

dependence of P_{22} on b , e , and the linear dependence of P_{12} on e discussed above, (44) implies that \mathcal{E} is quadratic in b , c , e for any fixed Hurwitz matrix a . Such dependence of the quadratic performance index \mathcal{E} on the filter matrices also holds in the classical filtering problem. However, in contrast to its classical predecessor, the CQF problem (32) is constrained by the PR conditions (24), (25) which couple the skew-Hamiltonian part of the matrix a in (26) to b , e and the matrix c in (27) to b .

VI. CONDITIONS OF STATIONARITY

If an optimal filter exists for the CQF problem (32), such a filter is among stationary points of the Lagrange function

$$\begin{aligned} \mathcal{L} := & \mathcal{E} + \langle \Xi, a\Theta_2 + \Theta_2 a^T + eDJ_1 D^T e^T + bJ_2 b^T \rangle / 2 \\ & + \langle \Gamma, \Theta_2 c^T + bJ_2 d^T \rangle \\ = & \mathcal{E} - \langle \Xi \Theta_2, a \rangle + \langle \Xi, eDJ_1 D^T e^T + bJ_2 b^T \rangle / 2 \\ & - \langle \Gamma dJ_2, b \rangle + \langle \Gamma^T \Theta_2, c \rangle. \end{aligned} \quad (45)$$

Here, the $1/2$ factor is introduced for further convenience, and $\Xi \in \mathbb{A}_q$, $\Gamma \in \mathbb{R}^{q \times r}$ are Lagrange multipliers associated with the PR constraints (24) (whose left-hand side is a real antisymmetric matrix) and (25), respectively. We will be concerned with a quadruple (a, b, c, e) of *unconstrained* state-space matrices of the coherent quantum filter (11) which is a stationary point of the Lagrange function \mathcal{L} in (45) for given matrices Ξ , Γ , with a Hurwitz. The Lagrange multipliers Ξ , Γ are to be found so as to ensure that the filter (which depends parametrically on Ξ , Γ) satisfies the PR conditions. In order to find stationary points of the Lagrange function \mathcal{L} in (45), we will compute its Frechet derivatives by using the chain rule and the following lemma [18, Lemma 2] based on algebraic techniques from [1], [17].

Lemma 1: The Frechet derivatives of the function \mathcal{E} in (32) with respect to the matrices \mathcal{A} , \mathcal{B} , \mathcal{C} from (17), (31), with \mathcal{A} Hurwitz, are computed as

$$\partial_{\mathcal{A}} \mathcal{E} = QP =: H, \quad \partial_{\mathcal{B}} \mathcal{E} = Q\mathcal{B}, \quad \partial_{\mathcal{C}} \mathcal{E} = \mathcal{C}P, \quad (46)$$

where P is the controllability Gramian from (34), and $Q := \Lambda_{\mathcal{A}^T}(\mathcal{C}^T \mathcal{C})$ is the observability Gramian of the pair $(\mathcal{A}, \mathcal{C})$ satisfying the algebraic Lyapunov equation

$$\mathcal{A}^T Q + Q\mathcal{A} + \mathcal{C}^T \mathcal{C} = 0. \quad (47)$$

The matrix H in (46) will be referred to as the *Hankelian* of the triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ since the spectrum of H is formed by the squared Hankel singular values [9], [17] of an appropriate linear time-invariant system. A block-wise form of (47) is given by

$$A^T Q_{11} + Q_{11} A + C^T e^T Q_{21} + Q_{12} e C + F^T F = 0, \quad (48)$$

$$a^T Q_{21} + Q_{21} A + Q_{22} e C - c^T G^T F = 0, \quad (49)$$

$$a^T Q_{22} + Q_{22} a + c^T G^T G c = 0, \quad (50)$$

which is similar to (38)–(40) except that the lower off-diagonal block is considered instead of the upper one. The solution of (48)–(50) is found by using the inverse Lyapunov and Sylvester operators (36) as

$$Q_{11} = \Lambda_{A^T}(C^T e^T Q_{21} + Q_{21}^T e C + F^T F), \quad (51)$$

$$Q_{21} = \Sigma_{a^T, \mathcal{A}}(Q_{22} e C - c^T G^T F), \quad (52)$$

$$Q_{22} = \Lambda_{a^T}(c^T G^T G c), \quad (53)$$

which corresponds to (41)–(43). The block Q_{22} in (53) is computed first and is then substituted into (52) in order to find Q_{21} , while Q_{11} in (51) is irrelevant for further discussions.

Lemma 2: The Frechet derivatives of the Lagrange function \mathcal{L} from (45) with respect to the state-space matrices a , b , c , e of the quantum filter (11), with a Hurwitz, are computed as

$$\begin{aligned} \partial_a \mathcal{L} &= H_{22} - \Xi \Theta_2, \\ \partial_b \mathcal{L} &= Q_{22} b - \Xi b J_2 - \Gamma d J_2, \\ \partial_c \mathcal{L} &= -G^T F P_{12} + G^T G c P_{22} + \Gamma^T \Theta_2, \\ \partial_e \mathcal{L} &= H_{21} C^T + Q_{21} B D^T + Q_{22} e - \Xi e D J_1 D^T, \end{aligned} \quad (54)$$

where use is made of the block partitioning of the controllability and observability Gramians P , Q and the Hankelian H according to (37).

The following conditions of stationarity of the Lagrange function \mathcal{L} are obtained by equating its Frechet derivatives to zero.

Theorem 1: The coherent quantum filter (11), with a Hurwitz, is a stationary point of the CQF problem (32) if and only if there exist Lagrange multipliers $\Xi \in \mathbb{A}_q$, $\Gamma \in \mathbb{R}^{q \times r}$ in (45) such that the equalities

$$H_{22} = \Xi \Theta_2, \quad (55)$$

$$Q_{22} b = \Xi b J_2 + \Gamma d J_2, \quad (56)$$

$$G^T G c P_{22} - G^T F P_{12} = -\Gamma^T \Theta_2, \quad (57)$$

$$H_{21} C^T + Q_{21} B D^T + Q_{22} e = \Xi e D J_1 D^T \quad (58)$$

are satisfied together with the PR conditions (24), (25).

Together with the Lyapunov equations (34), (47) and the PR conditions (24), (25), the relations (55)–(58) form a complete set of algebraic equations for finding an optimal filter among stationary points $(a, b, c, e, \Xi, \Gamma)$ of the Lagrange function in (45). Since (56), (57) can, in principle, be solved for b , c , the Lagrange multiplier Γ can be found so as to satisfy the PR condition (25). Also, (58) can be solved for e , as discussed below. In view of (55), which describes the stationarity of the Lagrange function with respect to a , and the antisymmetry of Ξ , the matrix $H_{22} = \Theta_2^{-1} \Theta_2 \Xi \Theta_2$ is skew-Hamiltonian in the sense of the symplectic structure specified by Θ_2^{-1} . Note that (55) is implicit in a . In the next section, we will obtain a more explicit equation and discuss its solvability with respect to a along with that of (56)–(58) for b , c , e .

VII. COMBINING THE STATIONARITY AND LYAPUNOV EQUATIONS

We will need two identities for the Gramians P , Q from (34), (47) and the Hankelian H from (46) which hold regardless of whether the QSDEs (5), (11) are PR. To this end, we introduce a matrix

$$\Upsilon := Q_{2\bullet} \mathcal{A} P_{\bullet 2} = Q_{21} A P_{12} + Q_{22} e C P_{12} + Q_{22} a P_{22}, \quad (59)$$

where use is made of (17) along with the block partitioning of P, Q, H in accordance with (37).

Lemma 3: The matrix Υ in (59), with \mathcal{A} Hurwitz, satisfies

$$\Upsilon + (H_{21}C^T + Q_{21}BD^T + Q_{22}e) e^T + H_{22}a^T + Q_{22}bb^T = 0, \quad (60)$$

$$\Upsilon + a^T H_{22} + c^T G^T (GcP_{22} - FP_{12}) = 0. \quad (61)$$

We will now combine the general identities (60), (61) with the equations (55)–(58) of stationarity of the Lagrange function \mathcal{L} from Theorem 1 and the PR conditions (24), (25).

Lemma 4: If the PR coherent quantum filter is a stationary point of the Lagrange function in (45), with a Hurwitz, then the matrix $\Xi a + \Gamma c$ is symmetric, and the matrix Υ from (59) satisfies

$$\Upsilon = (\Xi a + \Gamma c) \Theta_2.$$

The lemma will be used to obtain a more explicit equation for a than (55). Now, suppose the “filter blocks” of the controllability and observability Gramians P, Q are both nonsingular:

$$P_{22} \succ 0, \quad Q_{22} \succ 0. \quad (62)$$

The positive semi-definiteness $P_{22} \succcurlyeq 0, Q_{22} \succcurlyeq 0$ is inherited from P, Q . By the general form of Heisenberg’s uncertainty principle [5], the steady-state quantum covariance matrix of the filter state variables satisfies

$$P_{22} + i\Theta_2 = \lim_{t \rightarrow +\infty} \mathbf{E}(\xi \xi^T) \succcurlyeq 0, \quad (63)$$

which is a stronger property than $P_{22} \succcurlyeq 0$. The assumptions (62) enable the following matrices to be defined:

$$L := Q_{22}^{-1} Q_{21}, \quad M := P_{12} P_{22}^{-1}, \quad (64)$$

$$N := Q_{22}^{-1} \Xi, \quad S := \Theta_2 P_{22}^{-1}, \quad (65)$$

$$T := Q_{22}^{-1} \Gamma, \quad U := Q_{22} \Theta_2 P_{22}^{-1}. \quad (66)$$

Here, only N, T involve the Lagrange multipliers Ξ, Γ from (45), while L, M, S, U are completely specified by Θ_2 and the appropriate blocks of the Gramians P, Q , with neither P_{11} nor Q_{11} being involved. The eigenvalues of the matrix S in (65) are purely imaginary and symmetric about the origin, with the spectral radius satisfying

$$\mathbf{r}(S) \leq 1 \quad (67)$$

in view of (63). The matrix “ratios” in (64)–(66) allow the equations of Theorem 1 to be made more explicit in a, b, c, e .

Lemma 5: Under the assumptions (62), the stationarity equations (55)–(58) for the CQF problem (32) can be represented through the matrices L, M, N, S, T, U in (64)–(66) as

$$LM = NS - I_q, \quad (68)$$

$$b = NbJ_2 + TdJ_2, \quad (69)$$

$$c = (G^T G)^{-1} (G^T FM - T^T U), \quad (70)$$

$$e = Ne\Delta - (L(P_{11}C^T + BD^T) + P_{21}C^T), \quad (71)$$

where

$$\Delta := DJ_1 D^T. \quad (72)$$

Furthermore, the matrix a of such a filter satisfies the relation

$$a = (Na + Tc)S - (LA + eC)M. \quad (73)$$

Note that (70) employs the full column rank assumption on the matrix G in (30) to ensure the invertibility of $G^T G$. While (70) is already solved for c , we will show how the stationarity equations (73), (69), (71) can be solved for the other filter matrices a, b, e . To this end, let $\llbracket \alpha, \beta \rrbracket : X \mapsto \alpha X \beta$ denote the linear operator of the left and right multiplication of an appropriately dimensioned matrix X by given real matrices α, β , respectively. More generally [18, Section 7], for any positive integer g and compatibly dimensioned matrices $\alpha_1, \dots, \alpha_g$ and β_1, \dots, β_g , a *special linear operator of grade g* is defined by

$$\llbracket \alpha_1, \beta_1 \mid \dots \mid \alpha_g, \beta_g \rrbracket := \sum_{k=1}^g \llbracket \alpha_k, \beta_k \rrbracket, \quad (74)$$

where the matrix pairs are separated by “ \mid ”s. If, in each of the pairs, the matrices α_k, β_k are either both symmetric or both antisymmetric, then the operator (74) is self-adjoint with respect to the Frobenius inner product of matrices. Now, the solvability of (73) with respect to a depends on whether the grade-two special linear operator

$$\llbracket I_q, I_q \mid -N, S \rrbracket = \llbracket Q_{22}^{-1}, P_{22}^{-1} \rrbracket \circ \llbracket Q_{22}, P_{22} \mid -\Xi, \Theta_2 \rrbracket \quad (75)$$

(with \circ denoting the composition) is invertible on $\mathbb{R}^{q \times q}$. In the case of invertibility, (73) takes the form

$$a = \llbracket I_q, I_q \mid -N, S \rrbracket^{-1} (TcS - (LA + eC)M). \quad (76)$$

In a similar vein, (69), (71) can be solved for b, e , respectively, as

$$b = \llbracket I_q, I_{m_2} \mid -N, J_2 \rrbracket^{-1} (TdJ_2), \quad (77)$$

$$e = -\llbracket I_q, I_p \mid -N, \Delta \rrbracket^{-1} (L(P_{11}C^T + BD^T) + P_{21}C^T), \quad (78)$$

provided the following special linear operators of grade two are invertible:

$$\llbracket I_q, I_{m_2} \mid -N, J_2 \rrbracket = \llbracket Q_{22}^{-1}, I_{m_2} \rrbracket \circ \llbracket Q_{22}, I_{m_2} \mid -\Xi, J_2 \rrbracket, \quad (79)$$

$$\llbracket I_q, I_p \mid -N, \Delta \rrbracket = \llbracket Q_{22}^{-1}, I_p \rrbracket \circ \llbracket Q_{22}, I_p \mid -\Xi, \Delta \rrbracket. \quad (80)$$

By combining the spectral property (67) of the matrix S from (65) with similar properties

$$\mathbf{r}(J_2) = 1, \quad \mathbf{r}(\Delta) \leq 1 \quad (81)$$

of the matrices J_2 in (13) and Δ in (72), it follows that the condition

$$\mathbf{r}(N) < 1 \quad (82)$$

is sufficient for the positive definiteness (and hence, invertibility) of all three operators. This argument is based on applying the following lemma the grade-two operators in (75), (79), (80).

Lemma 6: Suppose $\alpha \in \mathbb{S}_r, \beta \in \mathbb{S}_s$ and $\sigma \in \mathbb{A}_r, \tau \in \mathbb{A}_s$, with $\alpha \succ 0, \beta \succ 0$. Then the corresponding special self-adjoint operator $\llbracket \alpha, \beta \mid \sigma, \tau \rrbracket$ of grade two, defined by (74) on the Hilbert space $\mathbb{R}^{r \times s}$, is positive definite if and only if $\mathbf{r}(\alpha^{-1} \sigma) \mathbf{r}(\tau \beta^{-1}) < 1$.

Note that, similarly to [18, Proof of Lemma 5], the second of the spectral relations (81) follows from (4) and (29) which imply that the complex Hermitian matrix $i\Delta = iDJ_1 D^T$ satisfies $-I_p \preccurlyeq i\Delta \preccurlyeq I_p$. Under the assumption $Q_{22} \succ 0$, the condition (82) is equivalent to the strict convexity of the Lagrange function \mathcal{L} in (45) with respect to b . More precisely, \mathcal{L} inherits quadratic dependence on b from \mathcal{E} , as discussed in Section V, and, in view of (54), the second order

Frechet derivative $\partial_b^2 \mathcal{L} = \llbracket Q_{22}, I_{m_2} \mid -\Xi, J_2 \rrbracket$ is a grade-two special self-adjoint operator on the Hilbert space $\mathbb{R}^{q \times m_2}$ whose positive definiteness is indeed equivalent to (82) by Lemma 6. Here, we have again used the spectral property of the matrix J_2 from (81).

VIII. AN ITERATIVE ALGORITHM OUTLINE

A reasoning, similar to that in the previous section, shows that, under the assumptions (62) and (82), the matrices b, c in (69), (70) satisfy the PR constraint (25) if and only if the matrix T in (66) is related to b by

$$T = U^{-T}(M^T F^T G + \Theta_2^{-1} b J_2 d^T G^T F), \quad (83)$$

where $(\cdot)^{-T} := ((\cdot)^{-1})^T$ is the composition of the matrix inverse and transpose. This allows T to be eliminated from (77) as

$$b = \llbracket I_q, I_{m_2} \mid -N, J_2 \mid -U^{-T} \Theta_2^{-1}, J_2 d^T G^T G d J_2 \rrbracket^{-1} (U^{-T} M^T F^T G d J_2), \quad (84)$$

which involves an invertible special operator of grade three. The matrix b from (84) can be substituted into (27) in order to find c . In an iterative algorithm for numerical computation of an optimal quantum filter in the CQF problem (32), the relations (84), (83), (27) can be employed to update the matrices b, T, c for given matrices M, N, U from (64)–(66). In a similar vein, (76), (78) can be used for updating the matrices a, e for given L, M, N, S, T and P_{12} . This is accompanied by updating the matrices L, M, S, U according to (64)–(66) in terms of the blocks of the Gramians P, Q which are computed for given filter matrices a, b, c, e as described by (42), (43) and (52), (53). The algorithm loop can be closed by computing the block H_{22} of the Hankelian H in (46) and updating the Lagrange multiplier Ξ in (65) as $\Xi := (H_{22} \Theta_2^{-1} + \Theta_2^{-1} H_{22}^T)/2$. In the case $n = q$, the algorithm can be initialized with the state-space matrices of a classical Kalman filter from the next section.

IX. REDUCTION TO THE CLASSICAL KALMAN FILTER

If the PR constraints (24), (25) are made inactive by letting $\Sigma = 0, \Gamma = 0$ in (45) (so that the CCRs become irrelevant), the Lagrange function corresponds to a classical filtering problem. In this case, the matrices N, T in (65), (66) vanish, and, under a simplifying assumption that the filter has the same state dimension $q = n$ as the plant, the necessary conditions of optimality (68)–(71), (73) take the form

$$LM = -I_n, \quad (85)$$

$$a = -(LA + eC)M, \quad (86)$$

$$b = 0, \quad (87)$$

$$c = (G^T G)^{-1} G^T F M, \quad (88)$$

$$e = -L(P_{11} C^T + BD^T) - P_{21} C^T = -L(\Pi C^T + BD^T), \quad (89)$$

where, in view of (64), the real positive semi-definite symmetric matrix

$$\Pi := P_{11} - M P_{21} \quad (90)$$

is the Schur complement [6] of P_{22} in (37). In particular, (87) shows that the additional noise ω (uncorrelated with the plant noise w) is redundant in the optimal filter, in conformance

with the classical Kalman filtering theory. Since (85) implies that $L = -M^{-1}$, an appropriate similarity transformation $\xi \mapsto \sigma \xi, a \mapsto \sigma a \sigma^{-1}, b \mapsto \sigma b, c \mapsto c \sigma^{-1}, e \mapsto \sigma e$ of the filter (11) with a nonsingular matrix $\sigma \in \mathbb{R}^{n \times n}$ (which does not have to be symplectic in the sense that $\sigma \Theta_2 \sigma^T = \Theta_2$) leads to $L = -I_n, M = I_n$. The corresponding matrices a, c, e in (86), (88), (89) reduce to

$$a = A - eC, \quad c = (G^T G)^{-1} G^T F, \quad e = \Pi C^T + BD^T, \quad (91)$$

with c being specified completely by the matrices F, G . Accordingly, the Lyapunov equation (38)–(40) takes the form of an algebraic Riccati equation [10] for the matrix Π from (90):

$$A\Pi + \Pi A^T + BB^T - (\Pi C^T + BD^T)(C\Pi + DB^T) = 0. \quad (92)$$

In view of (29), both (91) and (92) indeed correspond to the classical Kalman filter SDE $d\xi = A\xi dt + e(dy - C\xi dt)$, with $dy - C\xi dt$ the innovation.

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