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Max-plus based computation of nonlinear $\mathcal{L}_2$-gain performance bounds using a piecewise affine-quadratic basis

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***Abstract*— Nonlinear $\mathcal{L}_2$-gain is a generalization of the well-known finite $\mathcal{L}_2$-gain robust stability property for nonlinear systems. Computation of tight performance bounds associated with this nonlinear $\mathcal{L}_2$-gain property is key to avoiding conservatism in its application, for example in small-gain based design. In previous work, a number of max-plus eigenvector methods have been proposed to facilitate this computation. Those methods have each employed quadratic basis functions, which have been shown to lead to a specific computational issue concerning continuity of the associated Hamiltonian. In this paper, an alternative piecewise affine-quadratic basis is proposed that allows the development of a refined max-plus eigenvector method that avoids this computational issue.**

I. Introduction

Recently, the notion of extensively investigated finite $\mathcal{L}_2$-gain property [10], [16], [17], for nonlinear systems has been generalized to that of nonlinear $\mathcal{L}_2$-gain property [5], [7], [8], [18], [21], [22]. This nonlinear $\mathcal{L}_2$-gain notion captures more naturally the worst-case energy transfer property from input to output of a nonlinear system. This notion also arises naturally from the framework of input-to-state stability (ISS) properties [15]. As is known that finite $\mathcal{L}_2$-gain is qualitatively equivalent to the ISS property [9], it has been established in [6] that the nonlinear $\mathcal{L}_2$-gain is qualitatively equivalent to the integral input-to-state stability (iISS) property. One important issue in the research of nonlinear $\mathcal{L}_2$-gain property is the characterization and computation of tight transient / gain bound, which is particularly important in the application of nonlinear $\mathcal{L}_2$-gain as a system analysis and control design tool using small gain based design[6]. Various computational methods [5], [7], [18], [19], [21] have been developed to compute these nonlinear $\mathcal{L}_2$-gain tight performance bounds. In these methods, the essential point is that the various tight performance bounds can be characterized via the value function of an associated nonlinear constrained optimal control problem. Thus, the issue of computing tight bounds is transformed to one of approximating the solution of the associate optimal control problem using numerical method such as the approximating Markov chain method [11].

Methods based on max-plus algebra have recently been developed as a useful mathematical tool for solving nonlinear optimal control problems [1], [4], [12], [13], [14]. The max-plus eigenvector method [12], [13], [14] has been developed for solving the nonlinear optimal control problem associated with the nonlinear $\mathcal{H}_\infty$-control (linear finite $\mathcal{L}_2$-gain) problem. There, applying the method of dynamic programming, the value function is shown to be a fixed point of the dynamic programming evolution operator which is a linear operator over max-plus algebra in a max-plus vector space of semi-convex functions. By exploring a suitable max-plus basis for this vector space, solving the fixed point problem becomes a specific max-plus eigenvector problem of an associated matrix. It has been demonstrated that the max-plus eigenvector method is computational more efficient than the approximating Markov chain method for specific problems [14].

In [18], [19], the max-plus eigenvector method has been applied to solve the constrained nonlinear optimal control problem associated with the tight nonlinear $\mathcal{L}_2$-gain computation problem. In [18], a specific technical issue has been shown to arise in the application of the method to this particular class of nonlinear optimal control problem. This issue has been further explored in [19] where a new auxiliary problem formulation has been developed to alleviate the difficulty. The focus of this paper is to propose a new solution to this technical issue. Here, exploitation of additional property of value function leads to the use of a set of piecewise affine-quadratic basis in approximating the associated max-plus spectrum problem. It is shown that the technical issue as identified in [18] and [19] using quadratic basis functions can be avoided under the new set of max-plus basis.

In terms of organization of the paper, the nonlinear $\mathcal{L}_2$-gain property of interest is recalled in Section II, along with an associated optimal control problem from which gain and transient bound estimates may be recovered. Section III summarizes the application of max-plus eigenvector method in the tight nonlinear $\mathcal{L}_2$-gain bound computation in [18], [19] with the intention to illustrating the implementation issue there. Section IV formulates the new max-plus basis. Section V discusses the implementation under the new set of max-plus basis, where it is shown that the technical issue does not arise. A simple example is presented in Section VI to illustrate the application of this method. Some brief conclusions are provided in Section VII. Throughout, $\mathbb{R}$ (respectively $\mathbb{R}_{\geq 0}$) is used to denote the set of all (non-negative) real numbers and $\mathbb{R}^m$ the m-dimensional Euclidean space. A function $\gamma : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is of class $\mathscr{K}$ if it is continuous, non-decreasing, and satisfies $\gamma(0) = 0$.

II. Tight bounds of nonlinear $\mathcal{L}_2$-gain and optimal control formulation of [21]

Consider a nonlinear dynamical system of the form

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))\, w(t)\,, \quad x(0) = x, \\ z(t) = h(x(t))\,, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^s$ and $z(t) \in \mathbb{R}^l$ denote the state, input and output respectively, all at time $t \in \mathbb{R}_{\geq 0}$. The input space, denoted by $\mathscr{W}$ is restricted to locally square-integrable mappings from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^s$. The output space $\mathscr{Z}$ is similarly defined to be locally square-integrable mappings from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^l$. Let $\|\cdot\|_{\mathscr{W}[0,t]}$ and $\|\cdot\|_{\mathscr{Z}[0,t]}$ denote the $\mathcal{L}_2$-norm on $\mathscr{W}[0,t]$ and $\mathscr{Z}[0,t]$ respectively, where $\mathscr{W}[0,t] \doteq \mathcal{L}_2([0,t];\mathbb{R}^s)$ and $\mathscr{Z}[0,t] \doteq \mathcal{L}_2([0,t];\mathbb{R}^l)$. Standard assumptions are imposed on functions f and g in (1) in order to guarantee existence and uniqueness of solutions given the initial condition $x \in \mathbb{R}^n$ and input $w \in \mathscr{W}$.

Recalling [5], [7], [8], system (1) has nonlinear $\mathcal{L}_2$-gain with transient / gain bound pair $(\beta, \gamma) \in \mathscr{K} \times \mathscr{K}$ if

$$\|z\|^2_{\mathscr{Z}[0,T]} \leq \beta(|x|) + \gamma\left(\|w\|^2_{\mathscr{W}[0,T]}\right) \tag{2}$$

for all $x \in \mathbb{R}^n$, $w \in \mathscr{W}[0,T]$ and $T \geq 0$. This reduces to the conventional notion of linear $\mathcal{L}_2$-gain that arises in $\mathcal{H}_\infty$-control (e.g. [10], [16]) when $\gamma(s) = \bar{\gamma}^2 s$ for some $\bar{\gamma} \in \mathbb{R}_{\geq 0}$.

For convenience, let $\Pi^\Sigma \subset \mathscr{K} \times \mathscr{K}$ denote the set of all transient / gain bound pairs (β, γ) for which system (1) satisfies the nonlinear $\mathcal{L}_2$-gain property (2). Denote the set of all transient bounds defined with respect to a gain bound $\gamma \in \mathscr{K}$ by $\mathcal{B}^\Sigma(\gamma)$. Similarly, denote the set of all gain bounds given a transient bound $\beta \in \mathscr{K}$ by $\mathcal{G}^\Sigma(\beta)$. That is,

$$\mathcal{B}^\Sigma(\gamma) \doteq \left\{ \beta \in \mathscr{K} \;\middle|\; (\beta,\gamma) \in \Pi^\Sigma \right\},$$
$$\mathcal{G}^\Sigma(\beta) \doteq \left\{ \gamma \in \mathscr{K} \;\middle|\; (\beta,\gamma) \in \Pi^\Sigma \right\}.$$

Three specific quantities are of interest [21] when considering systems of the form (1) satisfying the nonlinear $\mathcal{L}_2$-gain property (2), namely,

$$\gamma_*(\xi) \doteq \sup_{T\geq 0} \sup_{\|w\|^2_{\mathscr{W}[0,T]}=\xi} \left\{ \|z\|^2_{\mathscr{Z}[0,T]} \;\middle|\; \begin{array}{c}\text{(1) holds with}\\ x(0)=0\end{array} \right\}, \tag{3}$$

$$\gamma_*^\beta(\xi) \doteq \inf\left\{ \gamma(\xi) \;\middle|\; \gamma \in \mathcal{G}^\Sigma(\beta) \right\},\ \beta \in \mathscr{K} \text{ s.t. } \mathcal{G}^\Sigma(\beta) \neq \emptyset, \tag{4}$$

$$\beta_*^\gamma(s) \doteq \inf\left\{ \beta(s) \;\middle|\; \beta \in \mathcal{B}^\Sigma(\gamma) \right\},\ \gamma \in \mathscr{K} \text{ s.t. } \mathcal{B}^\Sigma(\gamma) \neq \emptyset. \tag{5}$$

Here, $\gamma_* : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ of (3) captures a tight lower bound for all gain bounds for which (2) holds, $\gamma_*^\beta : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ of (4) defines the minimum gain bound compatible with transient bound $\beta \in \mathscr{K}$, while $\beta_*^\gamma : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ of (5) is the minimum transient bound compatible with gain bound $\gamma \in \mathscr{K}$ [21].

It may be shown [21] that these quantities can be characterized via the value function $W : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ of an optimal control problem, where

$$W(x,\xi) \doteq \sup_{T\geq 0} \sup_{\|w\|^2_{\mathscr{W}[0,T]}=\xi} \left\{ \|z\|^2_{\mathscr{Z}[0,T]} \;\middle|\; \begin{array}{c}\text{(1) holds with}\\ x(0)=x\end{array} \right\} \tag{6}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$. In particular, it has been proved in [21] that

$$\gamma_*(s) = W(0,s), \tag{7}$$
$$\gamma_*^\beta(s) = \sup_{x\in\mathbb{R}^n} \{W(x,s) - \beta(|x|)\}, \tag{8}$$
$$\beta_*^\gamma(s) = \sup_{|x|\leq s} \sup_{\xi\geq 0} \{W(x,\xi) - \gamma(\xi)\} \tag{9}$$

for all $s \in \mathbb{R}_{\geq 0}$.

In further exploring a solution path for the optimal control problem of (6), it is useful to consider a specific growth property of $W(0,\cdot)$, i.e. the value function W evaluated at $x=0$. In particular, system (1) is deemed to be *zero-state small-input sensitive (0-SIS)* with sensitivity pair $(\varepsilon_0, \sigma_0) \in \mathbb{R}^2_{>0}$ if for all $\xi \in (0, \sigma_0) \subset \mathbb{R}_{>0}$,

$$W(0,\xi) \geq \varepsilon_0\, \xi. \tag{10}$$

The following lemma will be useful , the proof can be found in [21].

Lemma 2.1: Suppose system (1) is 0-SIS with sensitivity pair $(\varepsilon_0, \sigma_0) \in \mathbb{R}^2_{>0}$. Then,

$$W(x,\xi+\eta) - W(x,\xi) \geq \varepsilon_0\, \eta \tag{11}$$

for all $\eta \in [0,\sigma_0)$, $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$.

To facilitate the application of max-plus methods in the computation of the value function of (6), various techniques have been proposed to lift the integral constraints in (6) so that it is transformed into an equivalent unconstrained optimal control problem [5], [7], [19]. In this paper, a problem reformulation similar to that of [5] is used. Consider the augmented system

$$\tilde{\Sigma}^a : \begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} f(x(t)) + g(x(t))\, w(t) \\ -|w(t)|^2 \end{bmatrix}, \\ y(t) = \sigma(\xi(t))\, h(x(t)), \end{cases} \tag{12}$$

with initial state $\begin{bmatrix} x(0) \\ \xi(0) \end{bmatrix} = \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}$, where $\sigma : \mathbb{R} \mapsto \{0,1\}$ is the sign indicator function

$$\sigma(\xi) \doteq \begin{cases} 1 & \xi \geq 0, \\ 0 & \xi < 0. \end{cases}$$

Using these dynamics, define the value function $V : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}_{\geq 0}$ of the auxiliary optimal control problem by

$$V(x,\xi) \doteq \sup_{T\geq 0} \sup_{w\in\mathscr{W}[0,T]} \left\{ \|y\|^2_{\mathscr{Z}[0,T]} \;\middle|\; \begin{array}{c}\text{(12) holds,}\\ x(0)=x,\\ \xi(0)=\xi\end{array} \right\}. \tag{13}$$

It is observed from the definition that

$$V(x,\xi) = 0,\ \forall\ (x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{<0}.$$

For $\xi \in \mathbb{R}_{\geq 0}$, it is shown in [19] that W of (6) and V of (13) are equivalent

$$W(x,\xi) = V(x,\xi),\ (x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}. \tag{14}$$

Applying dynamic programming to (13) yields the dynamic programming principle (DPP)

$$V = \mathcal{R}_\tau[V], \tag{15}$$

where $\mathcal{R}_\tau$ denotes the DPP evolution operator defined by

$$(\mathcal{R}_\tau\, \phi)(x,\xi) \doteq \sup_{w\in\mathscr{W}[0,\tau]} \left\{ \begin{array}{l} \|y\|^2_{\mathscr{L}[0,\tau]} \\ +\phi(x(\tau),\xi(\tau)) \end{array} \middle| \begin{array}{c} \text{(12) holds,} \\ x(0)=x, \\ \xi(0)=\xi \end{array} \right\}. \tag{16}$$

III. Max-plus eigenvector method with quadratic basis and an implementation issue

The max-plus algebra (e.g. [2], [14]) is a commutative semifield over $\mathbb{R}^- \doteq \mathbb{R}\cup\{-\infty\}$ equipped with addition and multiplication operations $\oplus$ and $\otimes$ defined by $a\oplus b \doteq \max(a,\, b)$ and $a \otimes b \doteq a+b$. Vector spaces of functions mapping the state space $\mathbb{R}^{n+1}$ to $\mathbb{R}^-$ may be defined with respect to such algebras. The selection of such a vector space $\mathscr{V}$ is of key importance to the development of max-plus methods. For the purpose of applying the max-plus method to solution of the operator equation (15), it is required that the max-plus vector space $\mathscr{V}$ to be invariant for the dynamic programming operator $\mathcal{R}_\tau$ of (16). That is, $\mathcal{R}_\tau[\phi]\in\mathscr{V}$ when $\phi\in\mathscr{V}$. One such max-plus vector space, denoted as $\mathscr{V}_{sc}$, is the space consisting of semi-convex functions mapping $\mathbb{R}^{n+1}$ to $\mathbb{R}^-$ [12], [14], [18], [19]. It is proved in [12], [14] that a set of (countable) basis functions for such a vector space can be taken as quadratic functions of the form

$$\psi_{(z,\eta)}(x,\xi) \doteq -\frac{1}{2}(x-z)^T C(x-z) - \frac{1}{2}q(\xi-\eta)^2 \tag{17}$$

parameterized by $(z,\eta)\in\mathbb{R}^{n+1}$. Here $C\in\mathbb{R}^{n\times n}$ is positive definite and $q>0$. Any semi-convex function $\phi\in\mathscr{V}_{sc}$ has a representation under the max-plus basis (17) in the form of the max-plus semi-convex duality given by [12], [14].

$$\begin{aligned}\phi(x,\xi) &= \int^{\oplus}_{\mathbb{R}^{n+1}} \hat{\phi}(z,\eta)\otimes\psi_{(z,\eta)}(x,\xi)\, dz d\eta \qquad (18)\\ &\doteq \max_{(z,\eta)\in\mathbb{R}^{n+1}} \left\{\hat{\phi}(z,\eta)+\psi_{(z,\eta)}(x,\xi)\right\}\end{aligned}$$

with the max-plus coordinates (max-plus semi-convex dual) $\hat{\phi}:\mathbb{R}^{n+1}\mapsto\mathbb{R}^-$

$$\hat{\phi}(z,\eta) = -\int^{\oplus}_{\mathbb{R}^{n+1}} \psi_{(z,\eta)}(x,\xi)\otimes(-\phi(x,\xi))\, dx d\xi. \tag{19}$$

Suppose V of (13) is semi-convex on $\mathbb{R}^n\times\mathbb{R}$, then V has a max-plus representation from (18) and (19) with max-plus dual denoted as $\widehat{V}(z,\eta)$. In seeking a computational scheme to compute the value function V of (13) on a domain $D\subset\mathbb{R}^n\times\mathbb{R}$, discretize the basis parameter space $(z,\eta)\in\mathbb{R}^{n+1}$ and keep finite number of basis

$$\psi_{i,j}(x,\xi) \doteq \psi_{(z_i,\eta_j)}(x,\xi), \quad 1\le i\le\nu_x, 1\le j\le\nu_\xi$$

in the representation (18) to obtain an approximation

$$V(x,\xi)\approx\overline{V}(x,\xi) = \bigoplus_{i=1,j=1}^{\nu_x,\nu_\xi} a_{i,j}\otimes\psi_{i,j}(x,\xi) \tag{20}$$

with

$$a_{i,j} = -\max_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}} \left\{\psi_{i,j}(x,\xi)\otimes(-\overline{V}(x,\xi))\right\}. \tag{21}$$

Suppose V of (13) is approximated by $\overline{V}$ which has a max-plus expansion of the form (20) and (21). Then, using the max-plus linearity of the operator (16) [14], [19], the coefficients $a_{i,j}$ satisfies

$$a_{i,j} = \bigoplus_{k=1,l=1}^{\nu_x,\nu_\xi} B^{k,l}_{i,j}\otimes a_{k,l} \tag{22}$$

with

$$B^{k,l}_{i,j} = -\max_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}} \left\{\psi_{i,j}(x,\xi)\otimes(-\mathcal{R}_\tau[\psi_{k,l}](x,\xi))\right\}. \tag{23}$$

Under certain conditions, (22) can be solved easily by a max-plus power method [14] or policy iteration [3] when the $B^{k,l}_{i,j}$ is known. With $a_{i,j}$ computed, the approximation $\overline{V}$ of the value function V of (13) can be obtained via (20).

The computational advantage of the max-plus eigenvector method can only be realized when $B^{k,l}_{i,j}$ can be computed efficiently. From (23), the computation of $B^{k,l}_{i,j}$ is essentially focused on the computation of $\mathcal{R}_\tau[\psi_{(z_k,\eta_l)}](x,\xi)$ which is a finite horizon optimal control problem on $[0,\tau]$ for the same system (12) with the quadratic functions $\psi_{(z_k,\eta_l)}$ as the terminal costs. When τ is chosen to be small, a Taylor series approximation can be applied as in [12], [14], [18], [19]. To this end, define the finite horizon optimal control value function $V_{i,j}:\mathbb{R}_{\ge 0}\times\mathbb{R}^n\times\mathbb{R}_{\ge 0}\mapsto\mathbb{R}$ to be

$$V_{i,j}(\tau,x,\xi) \doteq \mathcal{R}_\tau[\psi_{i,j}](x,\xi). \tag{24}$$

Following the standard dynamic programming development, $V_{i,j}$ satisfies a non-stationary Hamilton-Jacobi-Bellman (HJB) partial differential equation

$$\frac{\partial V_{i,j}}{\partial\tau}(\tau,x,\xi) = H(x,\xi,\nabla_x V_{i,j}(\tau,x,\xi),\nabla_\xi V_{i,j}(\tau,x,\xi)) \tag{25}$$

subject to initialization $V_{i,j}(0,x,\xi)=\psi_{i,j}(x,\xi)$. Here the Hamiltonian $H:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\mapsto\mathbb{R}^-$ is given by

$$H(x,\xi,p,l) = -|\sigma(\xi)\,h(x)|^2 - \langle p,\, f(x)\rangle - \pi(x,\xi,p,l), \tag{26}$$

with $\pi(x,\xi,p,l) \doteq \sup_{w\in\mathbb{R}^s}\left\{\langle p,\, g(x)\,w\rangle - l\,|w|^2\right\}$. Here, completion of squares is possible when $l>0$. However, in general,

$$\pi(x,\xi,p,l) = \begin{cases} \frac{|g(x)'\,p|^2}{4\,l} & l>0\,, \\ 0 & \begin{pmatrix} l=0 \text{ and} \\ g'(x)\,p=0\end{pmatrix}, \\ \infty & l<0 \text{ or } \begin{pmatrix} l=0 \text{ and} \\ g'(x)\,p\neq 0\end{pmatrix}. \end{cases} \tag{27}$$

The first order Taylor series approximation of the value function $V_{i,j}(\tau,x,\xi)$ of (24) then

$$V_{i,j}(\tau,x,\xi)\approx V_{i,j}(0,x,\xi)+\tau\,\frac{\partial V_{i,j}}{\partial t}(t,x,\xi)|_{t=0}. \tag{28}$$

When $V_{i,j}(t,x,\xi)$ is smooth on $t \in (0,\tau)$, applying (25) leads to

$$\frac{\partial V_{i,j}}{\partial t}(t,x,\xi)|_{t=0} = H(x,\xi,\nabla_x V_{i,j}(0,x,\xi),\nabla_\xi V_{i,j}(0,x,\xi)) = H(x,\xi,-C(x-z_i),-q(\xi-\eta_j)).$$

But from (26) and (27), $H(x,\xi,-C(x-z_i),-q(\xi-\eta_j))$ is unbounded on

$$\widetilde{\mathcal{O}} \doteq \left\{ (x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{c} q(\xi-\eta_j) > 0 \text{ or} \\ \begin{pmatrix} q(\xi-\eta_j) = 0 \text{ and} \\ g(x)' C(x-z_i) \neq 0 \end{pmatrix} \end{array} \right\}. \tag{29}$$

Thus, $V_{i,j}(t,x,\xi)$ is not differentiable with respect to t at $t=0, (x,\xi) \in \widetilde{\mathcal{O}}$. Hence the Taylor series expansion can not be used to approximate $V_{i,j}(\tau,x,\xi)$. This puts a serious limitation on the application of max-plus eigenvector method to the computation of the value function $W(x,\xi)$ of (6). The same implementation issue was encountered in [18] and further discussed in [19] where an alternative problem formulation was developed to alleviate the difficulty.

IV. A MAX-PLUS REPRESENTATION USING PIECEWISE AFFINE-QUADRATIC BASIS

The implementation issue discussed in Section III is caused by the specific dynamics (12) in ξ direction and the cost function (13). In particular, the input is not directly penalized in the cost (13) so that there maybe an impulse in the optimal input. In [19], a new equivalent auxiliary problem is formulated which includes an explicit penalty on the input energy so that the Hamiltonian $\widetilde{H}(x,\xi,-C(x-x_i),-q(\xi-\xi_j))$ is continuous within the domain of computation. Thus this implementation issue is alleviated.

Here, a new method is proposed to solve this implementation issue by employing new max-plus basis functions to represent the value function (13). Note in the development of [18] and [19], only the semi-convexity property of V of (13) is used in the max-plus representation using quadratic basis functions (17). The 0-SIS property in Lemma 2.1 implies that the value function $V(x,\xi)$ is monotonic in ξ direction and increases with a rate bounded from below by a positive constant $\kappa = \varepsilon_0$. The key is to show that functions defined on $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ with such a monotonic increasing property in ξ direction allows to use basis functions which are monotonic increasing in ξ direction with derivative bounded from below by $\frac{\kappa}{2}$ in the max-plus representation. It will be shown that using such basis functions solves the implementation issue discussed in Section III.

To this end, denote for a $\kappa > 0$

$$\rho_z(x) = -\frac{1}{2}(x-z)^T C(x-z), \tag{30}$$
$$\sigma_\eta(\xi) = -\frac{1}{2}q(\xi-\eta)^2,$$
$$\chi_\eta(\xi) = \begin{cases} \frac{1}{2}\kappa(\xi-\alpha), & \xi \geq \eta - \frac{\kappa}{2q} \\ \sigma_\eta(\xi), & \xi \leq \eta - \frac{\kappa}{2q}, \end{cases}$$
$$\alpha = -\frac{\kappa}{4q} + \eta.$$

Note $\chi_\eta(\xi) \geq \sigma_\eta(\xi)$, $\frac{d\chi_\eta}{\xi}(\xi) \geq \frac{1}{2}\kappa$, $\forall\, \xi \in \mathbb{R}_{\geq 0}$.

In terms of these notation, the quadratic basis functions (17) can be expressed as

$$\psi_{(z,\eta)}(x,\xi) = \rho_z(x) + \sigma_\eta(\xi).$$

The semi-convexity duality relation (18) and (19) states that a semi-convex function $\phi : \mathbb{R}^{n+1} \mapsto \mathbb{R}$ can be expanded using max-plus basis functions $\psi_{(z,\eta)}$ with the semi-convexity duality denoted as

$$\widehat{\phi}(z,\eta) = -\int^{\oplus}_{\mathbb{R}^{n+1}} \psi_{(z,\eta)}(x,\xi) \otimes (-\phi(x,\xi))dxd\xi. \tag{31}$$

The goal is to show that a function ϕ can be expanded using the following set of parameterized functions

$$\varphi_{(z,\eta)}(x,\xi) \doteq \rho_z(x) + \chi_\eta(\xi) \tag{32}$$

if an incremental property such as the 0-SIS condition holds in ξ. To this end, the proper max-plus vector space is formalized first.

Definition 4.1: Denote $\mathscr{V}^\kappa_{aq}$ for $\kappa \in \mathbb{R}_{>0}$ the space of functions mapping $\phi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ satisfying:

1) For each $\xi \in \mathbb{R}_{\geq 0}$, the function $\phi(\cdot,\xi) : \mathbb{R}^n \mapsto \mathbb{R}$ is semi-convex, i.e. $\phi(\cdot,\xi) \in \mathscr{V}_{sc}$.
2) There exists $\mu \in \mathbb{R}_{>0}$ such that $\phi(x,\xi+\eta) \geq \phi(x,\xi) + \kappa\,\eta$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ and $\eta \in (0,\mu)$.

Theorem 4.2: The set of functions in $\mathscr{V}^\kappa_{aq}$ as per Definition 4.1 form a max-plus vector space.

Proof: To show $\mathscr{V}^\kappa_{aq}$ is a max-plus vector space, it requires to demonstrate $\phi \doteq a \otimes \phi_1 \oplus \phi_2 \in \mathscr{V}^\kappa_{aq}$ for any $a \in \mathbb{R}^-, \phi_1 \in \mathscr{V}^\kappa_{aq}, \phi_2 \in \mathscr{V}^\kappa_{aq}$. For any fixed $x \in \mathbb{R}_{\geq 0}$, the semi-convexity of the projection $\phi(\cdot,\xi)$ follows directly from the fact that semi-convex functions form a max-plus vector space. To verify condition (2) in Definition 4.1 for ϕ, take any $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ and $\eta \in (0, \mu_1 \wedge \mu_2)$,

$$\begin{aligned} \phi(x,\xi+\eta) &= a \otimes \phi_1(x,\xi+\eta) \oplus \phi_2(x,\xi+\eta) \\ &= \max\{a + \phi_1(x,\xi+\eta), \phi_2(x,\xi+\eta)\} \\ &\geq \max\{a + \phi_1(x,\xi) + \kappa\eta, \phi_2(x,\xi) + \kappa\eta\} \\ &\geq \max\{a + \phi_1(x,\xi), \phi_2(x,\xi)\} + \kappa\,\eta \\ &= \phi(x,\xi) + \kappa\eta, \end{aligned}$$

which verifies condition (2) in Definition 4.1 for ϕ. ■

The purpose is to show that the set of functions $\varphi_{(z,\eta)}$ of (32) form a set of basis for the space $\mathscr{V}^\kappa_{aq}$. That is, it is required to show the following max-plus duality for any $\phi \in \mathscr{V}^\kappa_{aq}$

$$\phi(x,\xi) = \int^{\oplus}_{\mathbb{R}^n \times \mathbb{R}_{\geq 0}} \widetilde{\phi}(z,\eta) \otimes \varphi_{(z,\eta)}(x,\xi)dzd\eta, \tag{33}$$

where $\widetilde{\phi} : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ is

$$\widetilde{\phi}(z,\eta) = -\int^{\oplus}_{\mathbb{R}^n \times \mathbb{R}_{\geq 0}} \varphi_{(z,\eta)}(x,\xi) \otimes (-\phi(x,\xi))dxd\xi. \tag{34}$$

To prove this max-plus duality, some properties of the max-plus $\widetilde{\phi}(z,\eta)$ (34) of a function $\phi \in \mathscr{V}^\kappa_{sc}$ are presented first. The proofs are omitted for brevity.

Lemma 4.3: Let $\widehat{\phi}(z,\eta)$ and $\widetilde{\phi}(z,\eta)$ denote the max-plus dual of the function $\phi \in \mathscr{V}_{sc}^{\kappa}$ according to (31) and (34), then

$$\widehat{\phi}(z,\eta) = \widetilde{\phi}(z,\eta), \quad \forall\ (z,\eta) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}. \tag{35}$$

Lemma 4.4: The max-plus dual function $\widetilde{\phi}$ of (34) for a function $\phi \in \mathscr{V}_{sc}^{\kappa}$ satisfies

$$\widetilde{\phi}(z,\eta+\zeta) \geq \widetilde{\phi}(z,\eta) + \frac{1}{2}\kappa\,\zeta \tag{36}$$

for any all $\zeta \in [0, \min\{\mu, \frac{\kappa}{q}\})$, $(z,\eta) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$.

Theorem 4.5: A function $\phi \in \mathscr{V}_{sc}^{\kappa}$ can be represented using the set of functions (32) via (33) and (34).

V. Max-plus eigenvector method using piecewise affine-quadratic basis

This section presents an implementation of the max-plus eigenvector method to approximate the value functions (13) using the piecewise affine-quadratic basis functions $\varphi_{(z,\eta)}$ of (32). The procedure is the same as the implementation for the quadratic basis functions in Section III. The point here is to show that the implementation issue discussed there does not arise under the new basis.

Assume the 0-SIS property (10) holds for the value function W (6) with the pair $(\varepsilon_0, \delta_0) \in \mathbb{R}_{>0}^2$, thus the value function V of (13) satisfies $V \in \mathscr{V}_{sc}^{\varepsilon_0}$ by Lemma 2.1 and (14). Thus, the following max-plus duality relation holds for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$

$$V(x,\xi) = \int_{\mathbb{R}^n \times \mathbb{R}_{\geq 0}}^{\oplus} \widetilde{V}(z,\eta) \otimes \varphi_{(z,\eta)}(x,\xi)dzd\eta, \tag{37}$$

with the max-plus $\widetilde{V}$ given by

$$\widetilde{V}(z,\eta) = -\int_{\mathbb{R}^n \times \mathbb{R}_{\geq 0}}^{\oplus} \varphi_{(z,\eta)}(x,\xi) \otimes (-V(x,\xi))dxd\xi. \tag{38}$$

As indicated for the situation of quadratic basis functions, the max-plus representation (37) of $V(x,\xi)$ of (13) using new basis functions (32) is first discretized and approximated by using finite number of basis functions.

$$V(x,\xi) \approx \overline{V}(x,\xi) = \bigoplus_{i=1,j=1}^{\tilde{\nu}_x,\tilde{\nu}_\xi} \bar{a}_{i,j} \otimes \varphi_{i,j}(x,\xi) \tag{39}$$

with

$$\bar{a}_{i,j} = -\max_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}} \{\varphi_{i,j}(x,\xi) \otimes (-V(x,\xi))\}. \tag{40}$$

Here, $\varphi_{i,j}(x,\xi) \doteq \varphi_{(z_i,\xi_j)}(x,\xi)$.

Under this approximation, the max-plus eigenvector equation approximating the operator equation (15) is

$$\bar{a}_{i,j} = \bigoplus_{k,l=1}^{\tilde{\nu}_x,\tilde{\nu}_\xi} \bar{B}_{i,j}^{k,l} \otimes \bar{a}_{k,l} \tag{41}$$

with

$$\bar{B}_{i,j}^{k,l} = -\max_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}_{\geq 0}} \{\varphi_{i,j}(x,\xi) - \mathcal{R}_\tau[\varphi_{k,l}](x,\xi)\}. \tag{42}$$

Applying the first order Taylor series approximation to compute $\overline{V}_{i,j}(\tau,x,\xi) \doteq \mathcal{R}_\tau[\varphi_{k,l}](x,\xi)$ obtains

$$\begin{aligned}&\overline{V}_{i,j}(\tau,x,\xi) \\ &\approx \overline{V}_{i,j}(0,x,\xi) + \tau\,\frac{\partial \overline{V}_{i,j}}{\partial t}(t,x,\xi)|_{t=0} \\ &= \varphi_{i,j}(x,\xi) + \tau\, H\left(x,\xi,\nabla_x\varphi_{i,j}(x,\xi),\nabla_\xi\varphi_{i,j}(x,\xi)\right).\end{aligned} \tag{43}$$

Here, the Hamiltonian H is as (26) and $\nabla_x\varphi_{i,j}(x,\xi) = -C(x-z_i)$ and

$$\nabla_\xi\varphi_{i,j}(x,\xi) = \begin{cases} -q(\xi-\eta_j), & \xi \leq \eta_j - \frac{\varepsilon_0}{2q} \\ \frac{1}{2}\varepsilon_0, & \xi > \eta_j - \frac{\varepsilon_0}{2q}. \end{cases}$$

Thus, $\nabla_\xi\varphi_{i,j}(x,\xi) \geq \frac{1}{2}\varepsilon_0,\ \forall\ \xi \in \mathbb{R}_{\geq}0$ so that

$$\begin{aligned}&H(x,\xi,\nabla_x\varphi_{i,j}(x,\xi),\nabla_\xi\varphi_{i,j}(x,\xi)) \\ &\qquad = -|\sigma(\xi)\,h(x)|^2 - \langle -C(x-z_i),\, f(x)\rangle \\ &\qquad\qquad - \frac{|g(x)' - C(x-z_i)|^2}{4\,\nabla_\xi\varphi_{i,j}(x,\xi)},\end{aligned} \tag{44}$$

which is finite and continuous for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}_{>0}$.

VI. An example

Consider computing the value function V of (13) using the max-plus eigenvector method under the new basis functions for the following scalar linear system

$$\Sigma : \begin{cases} \dot{x}(t) &= -2\,x(t) + w(t), \quad x(0) = x, \\ z(t) &= x(t), \end{cases} \tag{45}$$

The reason to choose such a benchmark problem to test the proposed computational method is because the exact solution of V is available [20]

$$V_0(x,\xi) = \begin{cases} \dfrac{x^2}{2\left(\left(2+\frac{\xi}{x^2}\right)-\sqrt{\left(2+\frac{\xi}{x^2}\right)^2-4}\right)}, & (x,\xi) \in \mathbb{R}_{\neq 0}\times\mathbb{R}_{>0}, \\ \frac{1}{4}\,\xi, & (x,\xi) \in \{0\}\times\mathbb{R}_{\geq 0}, \\ \frac{1}{4}\,x^2, & (x,\xi) \in \mathbb{R}\times\{0\}. \end{cases} \tag{46}$$

The computation is performed to obtain the value function V in the domain $D = [-2 \quad 2] \times [0 \quad 1]$. A discretization of $\mathcal{D} = [-4 \quad 4] \times [0 \quad 1]$ is performed to select a finite number of basis functions to approximate V according to (39). Here, a uniform discretization scheme with grid size 0.2 is used. So the total number of basis functions used in $\nu = \nu_x \times \nu_\xi = 41 \times 6 = 246$. The Hessians are taken as $C = q = 1$. From the value function expression (46), V is 0-SIS with the sensitivity pair $(0.25, 1)$ so that $\varepsilon_0 = 0.25$. Label a basis function with $(z_i,\eta_j), 1 \leq i \leq \nu_x, 1 \leq j \leq \nu_\xi$ to be the l^{th} basis with $l = (j-1)\times\nu_x + i$. So $1 \leq l \leq \nu$. Using this labeling, the max-plus eigenvector equation (41) can be written as

$$\mathbf{a} = \mathbf{a} \otimes \mathbf{B}. \tag{47}$$

Here $\mathbf{B} \in \mathbb{R}^{\nu\times\nu}$ and $\mathbf{B}_{o,p} = \bar{B}_{i,j}^{k,l}$ with $o = (j-1)\times\nu_x + i$ and $p = (k-1)\times\nu_x + l$. Thus $\mathbf{a}$ is the max-plus eigenvector of $\mathbf{B}$ corresponding to the eigenvalue 0 (max-plus multiplicative identity). As in [14], $\mathbf{a}$ is computed from a max-plus power method, that is, $\mathbf{a} = \mathbf{B}^{\otimes K} \otimes \mathbf{0}$ for some finite integer K. The approximated value function $\overline{V}$ is computed via (39) which is shown in Figure 1. Figure 2 is the absolute error with the actual value function V_0 of (46) $E(x,\xi) = |\overline{V}(x,\xi) - V_0(x,\xi)|$. It is shown that the error is less than 0.05.

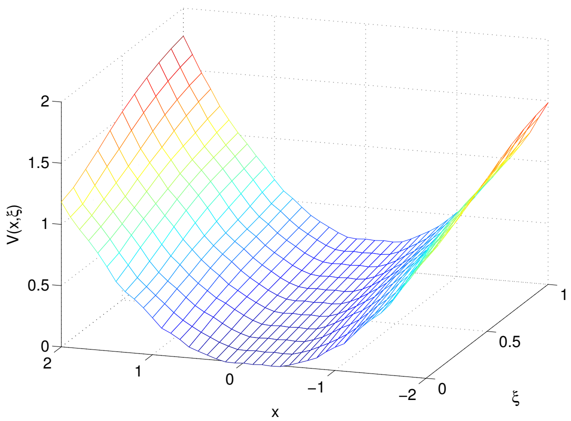

Fig. 1. The computed value function $\overline{V}(x,\xi)$

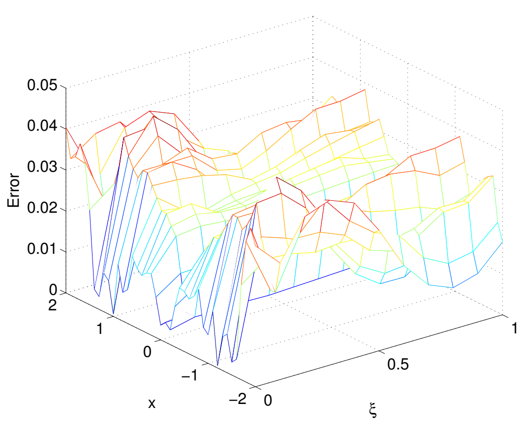

Fig. 2. The error function $E(x,\xi)$

VII. Conclusions

A new method is proposed to overcome a technical issue arising from the application of max-plus eigenvector method to the numerical solution of an optimal control problem. This optimal control problem is formulated to compute the tight performance transient/gain bound associated with a nonlinear $\mathcal{L}_2$-gain property for nonlinear systems. The new method exploits the particular property of the optimal control problem and utilizes new max-plus basis functions which is shown to be able to solve the continuity problem.

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