2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Finite time robust filtering for time-varying uncertain polytopic linear systems

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Abstract— **In this paper, the problem of robust filter design for uncertain continuous-time systems is investigated in the context of finite time stability. The filter is obtained in order to guarantee that the augmented system is finite time stable. The system is considered time varying with the parameters modeled by a polytope. The design conditions obtained by means of Lyapunov functions are expressed as linear matrix inequalities. A complete order filter is obtained by the solution of a factibility problem. A numerical example is provided.**

I. Introduction

The concept of finite time stability has been used in the literature to describe two main situations in system analysis: non-linear systems that can reach an equilibrium point within a finite time interval while still maintaining the classical characteristics of stability [1], and dynamical systems that present bounding constraints on their state space variables over a finite time interval without any other requirement with respect to the classical stability [2]. No matter which framework one is dealing with, the motivation of such investigation comes from the importance of knowing a system behavior during a short and/or transient time. As presented in [3], the characterization of a system trajectories during its transient time provides a good framework to set control strategies in order to avoid the excitation of nonlinear dynamics, deal with systems with saturation, and so on, what has also motivated the present work.

The finite time stability definition considered in this paper states that a time-varying linear system

$$\dot{x}(t) = A(t)x(t), \quad t \in [0,\ T]$$

is finite time stable with respect to (c_1, c_2, T, R), with $c_2 > c_1$ and $R > 0$, if

$$x(0)'Rx(0) \leq c_1 \Rightarrow x(t)'Rx(t) \leq c_2, \quad \forall t \in [0,\ T].$$

The above definition, presented in [4], clearly reveals the bounding constraints mentioned above, and reduces the problem of designing a controller that assures finite time stabilization to finding a control law that bounds the magnitude of the closed-loop system states over a finite time interval. If filter design is at issue, the estimation error dynamic, or the augmented system, should be guaranteed finite time stable.

The works appeared so far present solutions to the problem of filter and controller design for finite time stabilization, as well as methods for finite time stability analysis considering different scenarios faced by control system specialists. Variants of the bounding constraints, uncertainty models, time-varying or time-invariant structures have driven some of the recent research in this area. It is worth mentioning the works [5] where the finite time stability condition of [4] is used to design an observer based dynamic output feedback control, and [6] that provides sufficient design conditions of state-feedback controllers. In [6], the system is considered precisely known and affected by norm-bounded disturbances. The main difference from [4] is the use of a norm-bounded class of uncertainties and the $\mathscr{H}_\infty$ index of performance. An extension of [6] was presented in [7] considering linear parameter varying systems and gain scheduling state feedback controllers assuring finite time stability of the closed-loop system. In [3], necessary and sufficient conditions for finite time stability are presented in terms of differential linear matrix inequalities.

In the filter context, recent works include [8] that deals with the problem of filtering design for a Markov jump system. The filter is obtained in such a way that the estimation error is guaranteed finite time stable using for this end a modified version of the analysis condition presented in [4]. Similar filtering problems are investigated in [9], where an L_2-L_∞ index of performance is applied, and in [10] that considers Markov jump singular systems. In both cases, the estimation error is guaranteed finite time stable.

Despite all methods proposed to date, improvements are always required. Even if necessary and sufficient conditions are provided, as the one in [3], they may not be useful from a practical point of view, or even computationally prohibitive. In this sense, the investigation of different techniques and their combinations in the solution of complex scenarios is necessary. In the filtering context, one of the difficulties, for example, concerns the choice of the Lyapunov function to be used. In all papers about filtering mentioned above, the Lyapunov matrix is a block diagonal type, a conservative structure by not considering the effect of cross terms between the state variables of the plant and the filter. This choice, however, is related to the condition of analysis used, [4], which imposes constraints on the Lyapunov matrix eigenvalues, which in turn complicates the algebraic manipulations involved in obtaining the design conditions.

In the context of time-varying systems, linear parameter varying (LPV) models have been extensively applied in the solution of many problems [11]. These models belong to a class of linear systems whose dynamics are described as a function of time-varying parameters, and it is useful, as for

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instance, to describe nonlinear systems in different operation points [12].

In this paper, the problem of finite time robust filtering of LPV systems is investigated. The time-varying parameters are described using a polytope, and it is assumed to be uncertain with no information in their variation rates. Different from some results seen in the literature for finite time filter design, Lyapunov functions with less conservative structures is employed, using for this the end the lemma of N. Aronszajn on Hermitian positive semidefinite matrices found in [13]. A numerical example illustrates the main result.

II. Problem statement and preliminary results

Consider a linear uncertain system with $t \in [0,T]$

$$\begin{aligned} \dot{x}(t) &= A(\alpha(t))x(t) + B(\alpha(t))w(t) \\ y(t) &= C_y(\alpha(t))x(t) + D(\alpha(t))w(t) \\ z(t) &= C_z(\alpha(t))x(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state space vector, $y(t) \in \mathbb{R}^q$ is the measured output, $z(t) \in \mathbb{R}^p$ is the signal to be estimated, $w(t) \in \mathbb{R}^r$ is the noise input with bounded L_2 norm. All matrices are real, with appropriate dimensions, belonging to the polytope[1]

$$\mathscr{P} \triangleq \left\{ \left[\begin{array}{c|c} A(\alpha) & B(\alpha) \\ \hline C_y(\alpha) & D(\alpha) \\ \hline Cz(\alpha) & - \end{array} \right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c} A_i & B_i \\ \hline C_{yi} & D_i \\ \hline C_{zi} & - \end{array} \right] \right\} \tag{2}$$

The vector of time-varying parameters $\alpha \in \mathbb{R}^N$ belongs to the unit simplex

$$\mathscr{U} = \left\{ \gamma \in \mathbb{R}^N : \sum_{i=1}^{N} \gamma_i = 1,\ \gamma_i \geq 0\ ,\ i = 1,\ldots,N \right\}$$

and the system matrices are given, for any time $t \geq 0$, by the convex combination of the known vertices of the polytope $\mathscr{P}$. It is also assumed that no assumption is made on the parameter rate of variation.

The robust filter equations investigated here are given by

$$\begin{aligned} \dot{x}_f(t) &= A_f x_f(t) + B_f y(t) \\ z_f(t) &= C_f x_f(t) \end{aligned} \tag{3}$$

where $x_f(t) \in \mathbb{R}^n$ is the filter state space vector and $z_f(t) \in \mathbb{R}^p$ the estimated signal. All filter matrices are real and with appropriate dimensions.

Coupling the filter to the plant, the equations that describe the augmented system dynamic are given by

$$\begin{aligned} \dot{\varsigma}(t) &= \bar{A}(\alpha)\varsigma(t) + \bar{B}(\alpha)w(t) \\ e(t) &= \bar{C}(\alpha)\varsigma(t) \end{aligned} \tag{4}$$

where $\varsigma(t) = [x(t)' \quad x_f(t)']'$, $e(t) = z(t) - z_f(t)$, and

$$\bar{A}(\alpha) = \begin{bmatrix} A(\alpha) & \mathbf{0} \\ B_f C_y(\alpha) & A_f \end{bmatrix}, \quad \bar{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ B_f D(\alpha) \end{bmatrix},$$

$$\bar{C}(\alpha) = \begin{bmatrix} C_z(\alpha) & -C_f \end{bmatrix}.$$

[1]The time dependence of α(t) will be omitted to lighten the notation.

The problem to be solved is to find sufficient design conditions of robust filters given by (3) such that the states of the augmented system (4) are bounded over a finite time interval.

In the presence of external disturbances, the concept of finite time stability is extended to finite time boundedness as presented below.

Definition 1: [5] Given three positive scalars c_1, c_2 and T, with $c_2 > c_1$, a positive definite matrix R and a class of signals $\mathscr{W}$, the time-varying linear system

$$\dot{\varsigma}(t) = \bar{A}(t)\varsigma(t) + \bar{B}(t)w(t) \tag{5}$$

is finite time bounded with respect to $(c_1, c_2, \mathscr{W}, T, R)$, if

$$\begin{gathered} \varsigma(0)'R\varsigma(0) \leq c_1 \Rightarrow \varsigma(t)'R\varsigma(t) \leq c_2, \\ \forall w(t) \in \mathscr{W}, \forall t \in [0,\ T]. \end{gathered}$$

It is important to mention that variants of the class of signals $\mathscr{W}$ give rise to different types of finite time boundedness (FTB) problem formulations. This fact can be explored in accordance with the knowledge of the system designer on the type of disturbances affecting the system. Some cases can be seen in [4], [14], [15], [16].

The class $\mathscr{W}$ to be considered in this work consist of all signals $w(t)$ satisfying the following constraint

$$w(t)'w(t) \leq d, \quad \forall t \in [0,\ T], \quad d \geq 0. \tag{6}$$

It is easy to note that for $w(t) = 0$, FTB implies finite time stability.

The filtering problems to be dealt with can be stated as follows.

Problem 1: Find matrices $A_f \in \mathbb{R}^{n \times n}$, $B_f \in \mathbb{R}^{n \times q}$ and $C_f \in \mathbb{R}^{p \times n}$ of the filter (3), such that the augmented dynamic system (4) is FTB with respect to $(c_1, c_2, \mathscr{W}, T, R)$ considering the class of signals (6), and the estimation error satisfies

$$e(t)'e(t) < \varsigma(t)'\Omega^{-1}\varsigma(t), \quad \forall t \in [0,\ T]. \tag{7}$$

for a given $\Omega > 0$.

The following lemma, modified from [4], presents a sufficient condition for the analysis of FTB of a linear time-varying system, and is used in the solution of Problem 1.

Lemma 1: System (5) is FTB with respect to $(c_1, c_2, \mathscr{W}, T, R)$, if there exist positive definite symmetric matrices $Q_1 \in \mathbb{R}^{n \times n}$, $Q_2 \in \mathbb{R}^{r \times r}$, and positive scalars β and μ such that

$$\begin{bmatrix} \bar{A}(t)\tilde{Q}_1 + \tilde{Q}_1\bar{A}'(t) - \beta\tilde{Q}_1 & \bar{B}(t)Q_2 \\ Q_2\bar{B}'(t) & -\mu Q_2 \end{bmatrix} < 0 \tag{8}$$

$$\frac{c_1}{\lambda_{min}(Q_1)} + \frac{d}{\lambda_{min}(Q_2)}\frac{\mu}{\beta} < \frac{c_2 e^{-\beta T}}{\lambda_{max}(Q_1)} \tag{9}$$

in which $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$, and $\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$ indicate, respectively, the maximum and the minimum eigenvalue of the argument.

The proof will be presented for completeness since it differs from the one in [4] by showing that this lemma is also valid in the case where $w(t)$ is any signal, constrained only in its norm as specified by the class $\mathscr{W}$. As shown in [4], the signal $w(t)$ should be constant for all $t \in [0,\ T]$ or, if time varying, must be differentiable within the finite time interval.

Proof: Consider a Lyapunov function $V(\varsigma) = \varsigma' \tilde{Q}_1^{-1} \varsigma$, and assume that the inequality

$$\dot{V}(\varsigma) < \beta V(\varsigma) + \mu w' Q_2^{-1} w$$

is valid for all $t \in [0,\ T]$. Multiplying both sides by $\mathrm{e}^{-\beta t}$ one has

$$\begin{aligned} \mathrm{e}^{-\beta t}\dot{V}(\varsigma) &< \beta \mathrm{e}^{-\beta t} V(\varsigma) + \mu \mathrm{e}^{-\beta t} w' Q_2^{-1} w \\ \mathrm{e}^{-\beta t}\dot{V}(\varsigma) - \beta \mathrm{e}^{-\beta t} V(\varsigma) &< \mu \mathrm{e}^{-\beta t} w' Q_2^{-1} w \\ \frac{d(\mathrm{e}^{-\beta t} V(\varsigma))}{dt} &< \mu \mathrm{e}^{-\beta t} w' Q_2^{-1} w. \end{aligned}$$

Integrating the last inequality it follows that

$$\begin{aligned} \mathrm{e}^{-\beta t} V(\varsigma) - V(\varsigma_0) &< \mu \int_0^t \mathrm{e}^{-\beta \tau} w' Q_2^{-1} w d\tau \\ V(\varsigma) &< \mathrm{e}^{\beta t} V(\varsigma_0) + \mu \mathrm{e}^{\beta t} \int_0^t \mathrm{e}^{-\beta \tau} w' Q_2^{-1} w d\tau. \end{aligned}$$

Knowing that $w(t)$ belongs to the class $\mathscr{W}$ defined in (6), then

$$w' Q_2^{-1} w \le \lambda_{max}(Q_2^{-1}) w' w \le d \lambda_{max}(Q_2^{-1})$$

consequently

$$\begin{aligned} V(\varsigma) &< \mathrm{e}^{\beta t} V(\varsigma_0) + \mu \mathrm{e}^{\beta t} d \lambda_{max}(Q_2^{-1}) \int_0^t \mathrm{e}^{-\beta \tau} d\tau \\ &< \mathrm{e}^{\beta t} \left[V(\varsigma_0) + d\mu \lambda_{max}(Q_2^{-1}) \frac{(1 - \mathrm{e}^{-\beta t})}{\beta} \right]. \end{aligned}$$

With the choice of the Lyapunov function, one has

$$\lambda_{min}(Q_1^{-1}) \varsigma' R \varsigma \le \varsigma' R^{1/2} Q_1^{-1} R^{1/2} \varsigma \le \lambda_{max}(Q_1^{-1}) \varsigma' R \varsigma$$

then

$$\begin{aligned} \lambda_{min}(Q_1^{-1}) \varsigma' R \varsigma < \mathrm{e}^{\beta t} [&\varsigma_0' R \varsigma_0 \lambda_{max}(Q_1^{-1}) \\ &+ d \frac{\mu}{\beta} \lambda_{max}(Q_2^{-1})(1 - \mathrm{e}^{-\beta t})]. \end{aligned}$$

Knowing that $(1 - \mathrm{e}^{-\beta t}) < 1$, and $\varsigma_0' R \varsigma_0 < c_1$, it follows

$$\varsigma' R \varsigma < \frac{\mathrm{e}^{\beta t}}{\lambda_{min}(Q_1^{-1})} \left[c_1 \lambda_{max}(Q_1^{-1}) + \frac{d\mu}{\beta} \lambda_{max}(Q_2^{-1}) \right].$$

Consequently, the inequality

$$\frac{\mathrm{e}^{\beta t}}{\lambda_{min}(Q_1^{-1})} \left[c_1 \lambda_{max}(Q_1^{-1}) + \frac{d\mu}{\beta} \lambda_{max}(Q_2^{-1}) \right] < c_2$$

assures that $\varsigma' R \varsigma < c_2$. In particular, for $t = T$ one has

$$c_1 \lambda_{max}(Q_1^{-1}) + d \frac{\mu}{\beta} \lambda_{max}(Q_2^{-1}) < c_2 \mathrm{e}^{-\beta T} \lambda_{min}(Q_1^{-1})$$

that is exactly the second inequality of Lemma 1. Moreover, evaluating the inequality

$$\dot{V}(\varsigma) < \beta V(\varsigma) + \mu w' Q_2^{-1} w$$

on the system trajectories, one obtains the first inequality of Lemma 1. ■

The following lemma, due to N. Aronszajn, will be used in the proof of the main result.

Lemma 2: [13] Let

$$C = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

be an n-square Hermitian positive semidefinite matrix, with eigenvalues $\gamma_1 \ge \ldots \ge \gamma_n$. Let A be an a-square matrix with eigenvalues $\alpha_1 \ge \ldots \ge \alpha_a$, and B an b-square matrix with eigenvalues $\beta_1 \ge \ldots \ge \beta_b$. Then, the inequality

$$\gamma_{i+j-1} \le \alpha_i + \beta_j$$

holds for all i and j such that $1 \le i \le a$, and $1 \le j \le b$.

III. Main Results

The following theorem presents sufficient conditions for the synthesis of FTB robust filters.

Theorem 1: (Robust FTB filtering) Given an uncertain time-varying continuous-time system (1) and parameters $(c_1, c_2, d, T, R, \beta, \mu)$, if there exist symmetric positive definite matrices $K \in \mathbb{R}^{n \times n}$, $Z \in \mathbb{R}^{n \times n}$ and $W \in \mathbb{R}^{r \times r}$, matrices $L \in \mathbb{R}^{n \times q}$, $M \in \mathbb{R}^{n \times n}$ and $F \in \mathbb{R}^{p \times n}$, and positive real scalars $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, λ_3 and λ_4, such that[2]

$$\begin{gathered} \begin{bmatrix} \mathscr{M}_{11} & \mathscr{M}_{12} & KB(\alpha) + LD(\alpha) \\ (\star) & A(\alpha)' Z + Z A(\alpha) - \beta Z & Z B(\alpha) \\ (\star) & (\star) & -\mu W \end{bmatrix} < 0 \\ \mathscr{M}_{11} = -(\beta K + M + M') \\ \mathscr{M}_{12} = K A(\alpha) + L C_y(\alpha) + M \end{gathered} \tag{10}$$

$$\begin{bmatrix} c_2 \mathrm{e}^{-\beta T} & \sqrt{c_1} \tilde{\lambda}_1 & \sqrt{d \frac{\mu}{\beta}} \tilde{\lambda}_2 \\ (\star) & \tilde{\lambda}_1 & 0 \\ (\star) & (\star) & \tilde{\lambda}_2 \end{bmatrix} > 0 \tag{11}$$

$$0 < W < \tilde{\lambda}_2 I \tag{12}$$

$$\begin{bmatrix} \lambda_3 \mathbf{I} & \mathbf{I} \\ (\star) & Z \end{bmatrix} > 0, \quad \begin{bmatrix} \lambda_4 \mathbf{I} - K & K \\ (\star) & Z \end{bmatrix} > 0 \tag{13}$$

$$\begin{bmatrix} \tilde{Q} & S \\ (\star) & \tilde{\lambda}_1 R \end{bmatrix} > 0 \tag{14}$$

$$\lambda_3 + \lambda_4 < \lambda_{min}(R^{-1}) \tag{15}$$

$$\begin{bmatrix} K & \mathbf{0} & F' \\ (\star) & Z & C_z'(\alpha) - F' \\ (\star) & (\star) & \mathbf{I} \end{bmatrix} > 0 \tag{16}$$

with

$$\tilde{Q} = \begin{bmatrix} K & \mathbf{0} \\ \mathbf{0} & Z \end{bmatrix}, \quad S = \begin{bmatrix} K & \mathbf{I} \\ Z & \mathbf{0} \end{bmatrix},$$

then there exists a continuous-time robust filter in the form (3), such that the augmented system (4) is finite time bounded with respect to $(c_1, c_2, \mathscr{W}, T, R)$, with the estimation error

[2]The term $(\star)$ indicates symmetric blocks in the matrix inequality.

satisfying (7) for $\Omega = \Gamma\tilde{Q}_1\Gamma'$, $\Gamma = diag(\mathbf{I},\Gamma_{22})$, with Γ_{22} nonsingular. A realization of the filter is given by the matrices

$$A_f = -MK^{-1}, \quad B_f = L, \quad C_f = -F(\Gamma_{22}K)^{-1}.$$

Proof: As presented in [17] in the context of pole placement, consider the partitioned matrices

$$\tilde{Q}_1 = \begin{bmatrix} X & U' \\ U & \hat{X} \end{bmatrix}, \; \tilde{Q}_1^{-1} = \begin{bmatrix} Y & V' \\ V & \hat{Y} \end{bmatrix}, \; H = \begin{bmatrix} K+Z & \mathbf{I} \\ I & \mathbf{0} \end{bmatrix} \tag{17}$$

together with the following change of variables

$$\begin{bmatrix} M & L \\ F & W \end{bmatrix} = \begin{bmatrix} V' & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} A_f & B_f \\ C_f\Gamma_{22} & Q_2^{-1} \end{bmatrix} \begin{bmatrix} UZ & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{18}$$

where $Z = X^{-1}$ and $K = Y - Z$. By multiplying the LMI (10) to the left by $\bar{H}'$ and to the right by $\bar{H}$, and multiplying again the result to the left by $\tilde{H}'$ and to the right by $\tilde{H}$, with

$$\bar{H} = \begin{bmatrix} N & \mathbf{0} \\ \star & \mathbf{I} \end{bmatrix}, \; \tilde{H} = \begin{bmatrix} H^{-1} & \mathbf{0} \\ \star & \mathbf{I} \end{bmatrix}, \; N = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & X \end{bmatrix}$$

the LMI (8) is obtained. Moreover, it is easy to see that LMI (9) is satisfied if the conditions

$$\lambda_1\mathbf{I} < Q_1 < \mathbf{I} \tag{19}$$

$$\lambda_2\mathbf{I} < Q_2 \tag{20}$$

$$c_1/\lambda_1 + d/\lambda_2 < c_2 e^{-\beta T} \tag{21}$$

are guaranteed. By multiplying LMI (11) to the left and to the right by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{\lambda}_1^{-1} & 0 \\ 0 & 0 & \tilde{\lambda}_2^{-2} \end{bmatrix}$$

with $\tilde{\lambda}_1^{-1} = \lambda_1$ and $\tilde{\lambda}_2^{-1} = \lambda_2$, and applying Schur complement in the resulting matrix, inequality (21) is obtained. Inequality (20), on the other hand, can be obtained by simply inverting LMI (12) and remembering that $\tilde{\lambda}_2^{-1} = \lambda_2$ and W is defined in (17). Finally, inequality (19) is obtained using the LMIs (13), (14), (15), and Lemma 2. In fact, by multiplying inequality (19) to the left and to the right by $R^{-1/2}$, and considering the partition of $\tilde{Q}_1$ given in (17) one has

$$\begin{bmatrix} X & U' \\ U & \hat{X} \end{bmatrix} < R^{-1} \tag{22}$$

$$\lambda_1 R^{-1} < \begin{bmatrix} X & U' \\ U & \hat{X} \end{bmatrix} \tag{23}$$

Inequality (22) is satisfied if the condition

$$\lambda_{max}(\tilde{Q}_1) < \lambda_{min}(R^{-1}) \tag{24}$$

is guaranteed. In accordance with Lemma 2 it follows that

$$\lambda_{max}(\tilde{Q}_1) \leq \lambda_{max}(X) + \lambda_{max}(\hat{X})$$

that is satisfied if the conditions

$$\lambda_{max}(\tilde{Q}_1) \leq \lambda_3 + \lambda_4, \quad X < \lambda_3\mathbf{I}, \quad \hat{X} < \lambda_4\mathbf{I}$$

are guaranteed. Consequently, inequality (24) is satisfied if

$$\lambda_3 + \lambda_4 < \lambda_{min}(R^{-1}) \tag{25}$$

$$X < \lambda_3\mathbf{I} \tag{26}$$

$$\hat{X} < \lambda_4\mathbf{I} \tag{27}$$

are verified. Inequality (25) is exactly the LMI (15), and (26) is equal to the first LMI in (13) after applying the Schur complement and using the fact that $Z = X^{-1}$.

Knowing that the identity $\tilde{Q}_1\tilde{Q}_1^{-1} = \mathbf{I}$ gives the equations $XY + U'V = \mathbf{I}$ and $UY + \hat{X}V = \mathbf{0}$, one has that, for $V = \mathbf{I}$, that

$$UZ = -K \text{ and } \hat{X} = K + KZ^{-1}K. \tag{28}$$

And, consequently, inequality (27) becomes

$$K + KZ^{-1}K < \lambda_5\mathbf{I}$$

that equals the second LMI in (13) after applying the Schur complement. By multiplying LMI (14) to the left by $\bar{H}'$ and to the right by $\bar{H}$, and multiplying again the result to the left by $\tilde{H}'$ and to the right by $\tilde{H}$, with matrices $\bar{H}$ and $\tilde{H}$ as defined before, inequality (23) is obtained after applying the Schur complement.

Finally, by multiplying LMI (16) to the left by $\bar{H}'$ and to the right by $\bar{H}$, multiplying again the result to the left by $\tilde{J}'$ and to the right by $\tilde{J}$, with

$$\tilde{J} = \begin{bmatrix} J^{-1}\Gamma^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \; J = \begin{bmatrix} \mathbf{I} & X \\ \mathbf{0} & U \end{bmatrix}$$

and then applying Schur complement in the resulting matrix, one has

$$e(t)'e(t) < \varsigma(t)'\Omega^{-1}\varsigma(t)$$

that, in accordance with Lemma 1, guarantees the constraint in the estimation error specified in Problem 1, and the finite time stability of the augmented system. The filter matrices are obtained from (18) and (28), $M = A_f UZ$, that is, $A_f = M(UZ)^{-1} = -MK^{-1}$, what also applies to B_f and C_f. ■

Remark 1: It should be noted that the filter is obtained by a solution of a LMI feasibility problem. Since the conditions of the LMI problem are only sufficient, the non-existence of a solution does not imply necessarily that a dynamic filter does not exist.

Theorem 1 provides an infinite-dimensional condition in terms of the parameter α since the LMIs have to be tested at all points of the simplex $\mathscr{U}$. Convex finite-dimensional LMI conditions can be derived in terms of the vertices of the polytope (2) by developing the algebraic manipulations. However, instead of manipulating products of parameter-dependent matrices, the parser ROLMIP [18] has been used to construct the LMIs. This parser is available for download at http://www.dt.fee.unicamp.br/ agulhari/rolmip/rolmip.htm.

A technical difficulty in the solution of Problem 1 is concerned with the treatment of condition (9) since the minimum and the maximum eigenvalues of the Lyapunov matrix appear as variables of the problem. As can be seen in the proof of Theorem 1, a general partition is considered in the Lyapunov matrix introducing new matrix variables to the problem formulation that are used to provide convex design conditions. Consequently, the eigenvalues of the Lyapunov matrix should be characterized, or at least bounded, in terms of these partition blocks. One way of doing this is by fixing some of these matrix variables, as for instance by considering a diagonal Lyapunov matrix. The price to be paid is the increase of conservatism in the design conditions. To the best of the authors understanding, the works [8], [9] and [10], all based in Lemma 1, make use of particular structure for the Lyapunov matrix. However, by using Lemma 2, upper and lower bounds for the eigenvalues of positive definite matrices can be determined in such a way that a full Lyapunov matrix was considered in Theorem 1, reducing the conservatism of the proposed design conditions.

Finally, it is worth mentioning that although the FTB analysis conditions presented in [14] and [16] are necessary and sufficient, they involve the solution of differential linear matrix inequalities (DLMI) which are computationally prohibitive. In the context of FTS, the approach proposed in [3] considers dividing the time interval into several sub-intervals and then use piecewise affine Lyapunov functions to solve the DLMI. The quality of the solution depends on how big is the number of sub-intervals considered. It is worth mentioning that controllers obtained with these conditions will have time-varying structures even if the system is time-invariant [19], a characteristic that may not be desired from a practical point of view. For the particular class of constant Lyapunov matrices, the FTB sufficient conditions given by Lemma 1 represent an attractive alternative from the computational point of view, providing time-invariant robust filters with satisfactory results in many cases.

IV. Numerical Experiment

All the experiments have been performed in a PC equipped with: Linux Ubuntu 9.04, Athlon 64 X2 6000+ (3.0 GHz), 2GB RAM (800 MHz), using the SDP solver SeDuMi [20] interfaced by the parser ROLMIP 2.0 [18], MATLAB 7.0.1.

Consider system (1) with matrices modified from [9] as follows

$$A = \begin{bmatrix} -1 & 2 \\ -3 & \vartheta \end{bmatrix}, \quad B = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}$$
$$C_y = \begin{bmatrix} 1 & 0.5 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 \end{bmatrix}, \quad C_z = \begin{bmatrix} 0.6 & 1 \end{bmatrix}$$

The uncertainty ϑ is considered time-invariant belonging to the interval $-2 \leq \vartheta \leq -1$. This system can be described by a polytope as in (2) with two vertices.

By using Theorem 1 with $T = 2$, $\Gamma_{22} = 0.01$, $\mu = \beta = 0.5$, $c_1 = 0$, $c_2 = 2$, $d = 2$, and

$$R = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

a robust FTB filter as in (3) is obtained with matrices given by

$$A_f = \begin{bmatrix} -32.7976 & -0.7503 \\ -6.5276 & -39.9412 \end{bmatrix}, \quad B_f = \begin{bmatrix} 2.1506 \\ -1.7790 \end{bmatrix}$$
$$C_f = \begin{bmatrix} -6.1400 & -19.5852 \end{bmatrix}$$

For illustration purposes, a time-domain simulation was performed over the the interval t $\in [0,\ 2]$ considering zero initial conditions, three different values for ϑ and a perturbation input

$$w(t) = 1.2\mathrm{e}^{-0.2t} sen(t).$$

The results can be seen in Figure 1. For each value of ϑ, the estimated and desired signals are ploted in pairs using the same color.

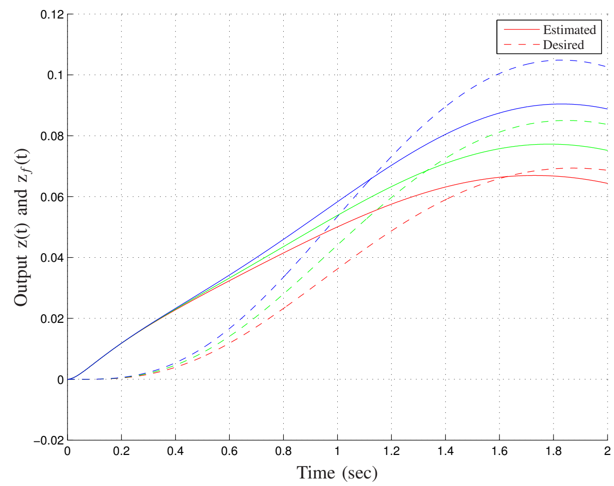

Fig. 1. Time simulation considering $\theta = -2, -1.5, -1$.

The state space variables behavior of the augmented system can be seen in Figure 2 considering zero initial conditions and ϑ varying from -2 to -1 with an increment step of 0.025. By analyzing the curve in Figure 2, it is evident that the robust filter can guarantee the finite time boundedness condition $\varsigma(t)'R\varsigma(t) < c_2 = 2$ for all set of uncertainties.

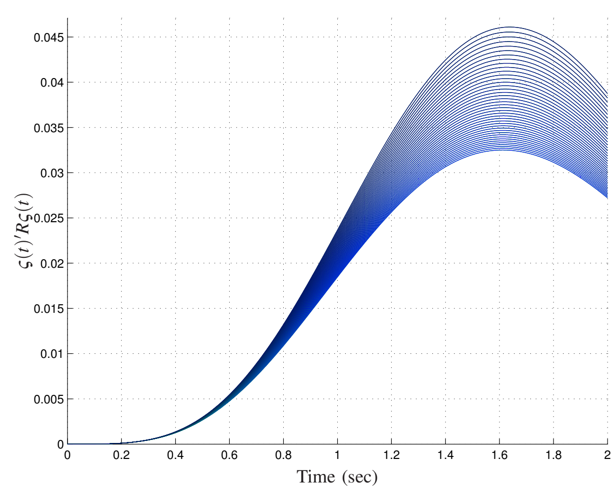

Fig. 2. Weighted squared norm of the augmented system states.

Finally, it is important to mention that the parameters $(c_1, c_2, d, T, R, \mu, \beta)$ represent design specifications and performance criteria that can be considered during the filter design. Furthermore, Figure 2 reveals a degree of conservatism of the presented results, probably related to the use of Lemma 2. Strategies for improving the presented design conditions and extensions to the problem of parameter-dependent filter design are under investigation by the authors.

V. Conclusion

In this paper, a design procedure of continuous-time robust filters for time-varying systems is presented. The time-varying parameters are modeled using a polytope and may be considered uncertain. The filter is obtained in such a way that the augmented system is stable in finite time and present bounded estimation error. The main feature of the proposed conditions is to use the result of N. Aronszajn that relates the eigenvalues of Hermitian positive semidefinite matrices with the eigenvalues of theirs constituent blocks. This result allowed to consider the Lyapunov matrix without any further restrictions, a fact apparently not explored in the literature within the framework considered here. The filter matrices are obtained from the solution of a feasibility problem with LMIs constraints. The numerical experiment performed reinforces the discussions presented throughout the text.

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