

# Robust filtering and fixed-lag smoothing for uncertain discrete-time systems

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**Abstract**—Robust filtering and fixed-lag smoothing problems for uncertain discrete-time systems are treated in this paper. The problem is set in a  $\lambda$ -contaminated framework that does not require a model for the uncertainties. The results are presented in term of transfer function and in polynomials. The latter representation exhibits two spectral factors and a Diophantine equation in order to obtain the estimator. An example shows the effectiveness of the approach.

## I. INTRODUCTION

Modern process control strategies require a large amount of information concerning the state of the controlled process. In practice, it is not systematically possible to have access to these information. Hence, the physically inaccessible information is obtained from the available measurements together with model of the process. For linear systems, in the state-space form, this problem can be treated by the well-known Kalman filter.

The robustness of the Kalman filter is directly related to the correctness of the process model. Indeed, it has been shown that errors in the model lead to biased estimates [8]. Consequently, the Kalman filter should be modified in order to provide reliable estimates even in presence of model uncertainties. This objective has been reached through various approaches.

The  $H_\infty$  framework has been widely used in order to design robust estimators that guaranty a constant level of performance over a range of possible models. This objective has been fulfilled under the condition that the designer is able to provide a representation of the model uncertainties. Uncertainties can be modelled by means of so-called norm bounded uncertainties or polytopes. In the norm bounded representation, the problem is usually solved through Riccati equation [5] and game theoretic approach [17]. Concerning the polytopic representation, the problem is solved by LMIs [14][15]. The aforementioned representations are generally handled in a linear manner. Nevertheless, they can also be used to consider the uncertainties as multiplicative noises. For the first class of model, the problem is solved through a Riccati equation approach [19] while in the latter, the LMI framework has been successfully used [3][18].

On the other, the designer could consider that the uncertainty and biased estimation are consecutive to a lack

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of information on the model. Modelling the uncertainties is a way to introduce into the estimator design such an information but, in order to model the uncertainties, one should handle information on them. If the designer does not have any consistent idea that permits to model the uncertainties, then an alternative solution should be found. To the authors' knowledge, only few solutions have been provided in the literature.

This problem has been solved by:

- introducing the sensitivity of the estimation error with respect to the supposed uncertain parameters by means of a  $\lambda$ -contaminated approach in the frequency domain (see [10] for discretised systems and [11] for continuous systems) and the time domain (see [20] for discretised systems). In this context, no uncertainty modelling is required.
- Considering that state estimation is equivalent to a state feedback control problem. An integral action has been added to the Kalman filter structure [7] in order to provide a robust state estimate. The obtained filter is known as the Proportional-Integral Kalman filter (PI Kalman). Though interesting, this approach fails for systems with poles located close to the imaginary axis. An alternative has been proposed by means of a lead-lag controller in place of the PI controller [12].
- Uncertainty can be considered as unknown input or unknown bias. Both considerations have led to a series of development, namely, the Unknown Input Observer [1][2][9] and Bias Aware Filters [4][6][13].

In this article, we address the problem of causal robust filtering and fixed-lag smoothing for uncertain discrete-time systems with a  $\lambda$ -contaminated approach. This approach does not require any modelling of the uncertainties. This work is an extension of the result in [10] limited to a noncausal expression of the robust filter in the transfer function representation. Indeed, our objective is to design causal robust estimators in the transfer function framework and a polynomial representation as well.

In the sequel, the following notations will be used:

- $E\{\cdot\}$  is the mathematical expectation,
- $q^{-1}$  is the shift operator such that  $s(t-1) = q^{-1}s(t)$ ,
- $n_P$  is the degree of the polynomial  $P(q^{-1})$ ,
- $P^*(q^{-1})$  is the complex conjugate of the polynomial  $P(q^{-1})$ ,
- $\partial_X f = \frac{\partial f}{\partial X}$  is the partial derivative of the function  $f$  w.r.t.  $X$ ,

- $X(q^{-1}) = \{X(q^{-1})\}_+ + \{X(q^{-1})\}_-$  is the fraction expansion of the transfer function  $X(q^{-1})$  where  $\{X(q^{-1})\}_+$  has causal poles.

## II. PROBLEM STATEMENT

Consider the discrete-time Single Input - Single Output linear system described by the constitutive expression given below:

$$\begin{cases} s(t) = H(q^{-1})(u(t) + w(t)) \\ u(t) = L(q^{-1})r(t) \\ y(t) = s(t) + v(t) \end{cases} \quad (1)$$

where  $u(t)$  is a measured input,  $s(t)$  is the signal to be estimated,  $y(t)$  is the measured output,  $r(t)$ ,  $w(t)$  and  $v(t)$  are independent random signals.

The following assumptions are made:

- the transfer function of the system is:

$$H(q^{-1}) = q^{-\delta} \frac{B(q^{-1})}{A(q^{-1})} \quad (2)$$

- the measured input  $u(t)$  with known spectrum can be represented as follows :

$$u(t) = L(q^{-1})r(t) = \frac{G(q^{-1})}{F(q^{-1})}r(t) \quad (3)$$

with the generating noise  $r(t)$  is an independent zero mean processes with unit variance,

- the system noises  $w(t)$  and  $v(t)$  are described by two ARMA processes:

$$w(t) = \frac{D(q^{-1})}{C(q^{-1})}\eta(t), \quad v(t) = \frac{M(q^{-1})}{N(q^{-1})}n(t) \quad (4)$$

where  $\eta(t)$  and  $n(t)$  are two independent zero mean processes with unit variance,

- Polynomials  $A(q^{-1})$ ,  $C(q^{-1})$ ,  $F(q^{-1})$  and  $M(q^{-1})$  are stable,
- $D(q^{-1})/C(q^{-1})$ ,  $N(q^{-1})/M(q^{-1})$  are stable and causal with no common zeros on the unit circle,
- The system is detectable. Consequently, common factors of  $H(q^{-1})C(q^{-1})/D(q^{-1})$  must be stable, and those of  $L(q^{-1})H(q^{-1})$  as well,
- All polynomials are monic and coprime in  $q^{-1}$ .

We are interested in the design of a robust estimator (filter or fixed-lag smoother) for the signal  $s(t)$  while the transfer function  $H(q^{-1})$  is uncertain. This robust estimator will be optimal if it minimises the criterion:

$$J = (1 - \lambda)E\{\varepsilon^2(t)\} + \lambda E\{(\partial_H \varepsilon(t))^2\} \quad (5)$$

where:

- $\varepsilon(t) = s(t) - \hat{s}(t|t-m)$  is the estimation error,
- the optimal estimator is of the form:

$$\hat{s}(t|t-m) = q^{-m} [\mathcal{R}(q^{-1})y(t) + \mathcal{S}(q^{-1})u(t)] \quad (6)$$

where  $\mathcal{R}$ ,  $\mathcal{S}$  are stable and causal transfer functions to be determined.

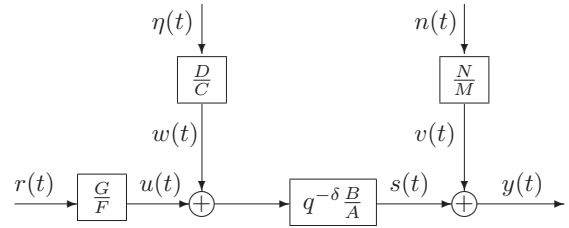


Fig. 1. The schematic view of the model of the system.

- $\lambda \in [0; 1[$  is a tuning parameter.

Criterion (5) belongs to the family of  $\lambda$ -contaminated models. The key idea consists in finding the equilibrium point between the action of two antagonist players, namely, the estimator minimising the estimation error and model uncertainties increasing the estimation error. The effect of uncertainties onto the estimation error is expressed by means of the sensitivity of the latter w.r.t. the uncertain transfer function. Parameter  $\lambda$  is the tuning parameter that should be adjusted in order to find the trade-off between the antagonist players.

The optimal estimator will be a filter for  $m = 0$  or a fixed-lag smoother for  $m < 0$ .

In order to simplify the presentation, the dependence with  $q^{-1}$  of the transfer functions and polynomials will be omitted.

## III. THE ROBUST FILTER AND FIXED-LAG SMOOTHER

The objective is to find the optimal estimator minimising criterion (5) under the structural constraint (6).

In order to present the estimation algorithm that minimises criterion (5), we express the estimation error  $\varepsilon(t)$  from (1) and (6). We obtain:

$$\varepsilon(t) = q^{-m} \{[(q^m - \mathcal{R})H - \mathcal{S}]u(t) + (q^m - \mathcal{R})Hw(t) - \mathcal{R}v(t)\} \quad (7)$$

From this relation, the sensitivity of the estimation error w.r.t. the uncertain transfer function  $H$  can be written as:

$$\partial_H \varepsilon(t) = q^{-m}(q^m - \mathcal{R})(u(t) + w(t)) \quad (8)$$

These two expressions, together with Parseval theorem [16], permit to write the criterion (5) in the  $z$ -domain. It turns out that:

$$J = \frac{1}{j2\pi} \oint_{|z|=1} \{(1 - \lambda)\mathcal{I}_1 + \lambda\mathcal{I}_2\} \frac{dz}{z} \quad (9)$$

with

$$\begin{cases} \mathcal{I}_1 &= [(z^m - \mathcal{R})H - \mathcal{S}]^* [(z^m - \mathcal{R})H - \mathcal{S}] \sigma_u^2 \\ &+ (z^m - \mathcal{R})^* H^* (z^m - \mathcal{R}) H \sigma_w^2 + \mathcal{R}^* \mathcal{R} \sigma_v^2 \\ \mathcal{I}_2 &= (z^m - \mathcal{R})^* (z^m - \mathcal{R}) (\sigma_u^2 + \sigma_w^2) \end{cases}$$

The optimal estimator  $(\mathcal{R}_o, \mathcal{S}_o)$  that minimises the criterion will be obtained by means of a variational technique. In that purpose, we introduce the following perturbations:

$$\mathcal{R} = \mathcal{R}_o + \alpha R \quad \mathcal{S} = \mathcal{S}_o + \beta S$$

where  $\alpha, \beta$  are real numbers, and  $R, S$  are stable transfer functions. The criterion is minimised if the first derivative of  $J$  in each direction is null for the optimal filters, and, if the second derivatives of  $J$  are positive. In other words, we should verify:

$$\begin{cases} \delta_{\mathcal{R}} = \partial_{\alpha} J|_{(\alpha, \beta)=(0,0)} \equiv 0 \\ \delta_{\mathcal{S}} = \partial_{\beta} J|_{(\alpha, \beta)=(0,0)} \equiv 0 \end{cases} \quad (10)$$

$$\begin{cases} \delta_{\mathcal{R}}^2 = \partial_{\alpha}^2 J|_{(\alpha, \beta)=(0,0)} \geq 0 \\ \delta_{\mathcal{S}}^2 = \partial_{\beta}^2 J|_{(\alpha, \beta)=(0,0)} \geq 0 \end{cases} \quad (11)$$

where  $\partial_{\alpha} J$  and  $\partial_{\alpha}^2 J$  are the first and second order partial derivative of  $J$  with respect to  $\alpha$  respectively.

The optimality condition (10) entails the relations:

- Condition for  $\mathcal{R}_o$

$$\delta_{\mathcal{R}} = \frac{-2}{2\pi} \oint_{|z|=1} R^* P_{\mathcal{R}} \frac{dz}{z} \equiv 0 \quad (12)$$

with

$$P_{\mathcal{R}} = H^* P_{\mathcal{S}} + (z^m - \mathcal{R}_o) \Phi^* \Phi - (1 - \lambda) \mathcal{R}_o \sigma_v^2 \quad (13)$$

and  $\Phi$  a transfer function with poles and zeros in the unit circle defined by:

$$\Phi^* \Phi = (1 - \lambda) H^* H \sigma_w^2 + \lambda (\sigma_u^2 + \sigma_w^2) \quad (14)$$

- Condition for  $\mathcal{S}_o$

$$\delta_{\mathcal{S}} = \frac{-2}{2\pi} \oint_{|z|=1} S^* P_{\mathcal{S}} \frac{dz}{z} \equiv 0$$

with

$$P_{\mathcal{S}} = (1 - \lambda) [(z^m - \mathcal{R}_o) H - \mathcal{S}_o] \sigma_u^2 \quad (15)$$

Our objective is to find a causal estimator. Hence, it appears that (15) should be equal to zero as all terms in these expressions are causal functions.

We obtain the expression of  $\mathcal{S}_o$  as follows:

$$P_{\mathcal{S}} = 0 \Rightarrow \mathcal{S}_o = (z^m - \mathcal{R}_o) H \quad (16)$$

Consequently, condition (13) becomes:

$$P_{\mathcal{R}} = (z^m - \mathcal{R}_o) \Phi^* \Phi - (1 - \lambda) \mathcal{R}_o \sigma_v^2 \quad (17)$$

Gathering the terms in  $\mathcal{R}_o$  leads to:

$$P_{\mathcal{R}} = z^m \Phi^* \Phi - \mathcal{R}_o [\Phi^* \Phi + (1 - \lambda) \sigma_v^2] \quad (18)$$

At this point of the development, for further investigation, the design framework should be specified. In a first step, the transfer function framework will offer a general formulation of the robust estimator. Then, in a second step, a polynomial solution to this optimisation problem will be presented.

#### A. Transfer function formulation

Let introduce the transfer function  $\Gamma$  with poles and zeros inside the unit circle, such that:

$$\Gamma \Gamma^* = \Phi^* \Phi + (1 - \lambda) \sigma_v^2 \quad (19)$$

Then, we can write that:

$$P_{\mathcal{R}} = z^m \Phi^* \Phi - \mathcal{R}_o \Gamma \Gamma^* \quad (20)$$

Setting this expression to zero leads to the expression of the noncausal estimator, i.e.

$$\mathcal{R}_o^{nc} = z^m \frac{\Phi^* \Phi}{\Phi^* \Phi + (1 - \lambda) \sigma_v^2} \quad (21)$$

In order to obtain the expression of the causal estimator, let consider the fraction expansion:

$$z^m \Phi^* \Phi \Gamma^{-*} = \{z^m \Phi^* \Phi \Gamma^{-*}\}_- + \{z^m \Phi^* \Phi \Gamma^{-*}\}_+ \quad (22)$$

Combining this expansion with (20), we get:

$$P_{\mathcal{R}} = \Gamma^* [\{z^m \Phi^* \Phi \Gamma^{-*}\}_- + \{z^m \Phi^* \Phi \Gamma^{-*}\}_+ - \mathcal{R}_o^c \Gamma] \quad (23)$$

Setting the causal part of this expression to zero leads to expression of the causal robust estimator:

$$\mathcal{R}_o^c = \{z^m \Phi^* \Phi \Gamma^{-*}\}_+ \Gamma^{-1} \quad (24)$$

Now, replacing this expression into (12), we obtain that:

$$\delta_{\mathcal{R}} = \frac{-2}{2\pi} \oint_{|z|=1} R^* \Gamma^* \{z^m \Phi^* \Phi \Gamma^{-*}\}_- \frac{dz}{z} \quad (25)$$

The integrand has poles and zeros outside the unit circle, then  $\delta_{\mathcal{R}}$  is null. The condition is verified.

In order to verify that  $\mathcal{R}_o^c$  minimises criterion (5), conditions (11) should be evaluated. It turns out that:

$$\begin{aligned} \delta_{\mathcal{R}}^2 &= \frac{2}{2\pi} \oint_{|z|=1} R^* R \{ \Phi^* \Phi + (1 - \lambda) (H^* H \sigma_u^2 + \sigma_v^2) \} \frac{dz}{z} \\ \delta_{\mathcal{S}}^2 &= \frac{2(1-\lambda)}{2\pi} \oint_{|z|=1} S^* S \sigma_u^2 \frac{dz}{z} \end{aligned}$$

Both integrands are symmetric w.r.t. the unit circle. Hence, the quantities  $\delta_{\mathcal{R}}$  and  $\delta_{\mathcal{S}}$  are positive. The estimator  $(\mathcal{R}_o, \mathcal{S}_o)$  minimises criterion (5).

The previous results are synthesised in the following theorem:

*Theorem 1:* The robust estimator of the form (6) minimising criterion (5) is given by :

$$\hat{s}(t|t-m) = Hu(t) + q^{-m} \mathcal{R}_o^c (y(t) - Hu(t)) \quad (26)$$

where  $\mathcal{R}_o^c$  is obtained from equation (24) together with:

- the spectral factors  $\Phi$  in (14) and  $\Gamma$  in (19),
- the fraction expansion of  $z^m \Phi^* \Phi \Gamma^{-*}$  in (22),
- the tuning parameter  $\lambda \in [0; 1[$ .

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*Remark 1:* Setting  $\lambda$  to zero, the estimator  $\mathcal{R}_o^c$  is equivalent to the Wiener filter [16].

*Remark 2:* If the designer sets  $\lambda$  to 1, then  $\mathcal{R}_o^n = z^m$  and  $\mathcal{S}_o = 0$ . In this case, the estimate is exactly the measured output. This means that if  $\lambda = 1$ , the designer considers that transfer function  $H$  does not represent the system under consideration and no data processing should be operated.

### B. Polynomial formulation

In this subsection, the result in Theorem 1 will be presented in the polynomial framework. Consequently, let rewrite the expression (14) in a polynomial form:

$$\Phi^* \Phi = \frac{\tau \varphi^* \varphi}{(ACF)^* ACF} \quad (27)$$

together with the spectral factor:

$$\begin{aligned} \tau \varphi^* \varphi &= (1 - \lambda)(BDF)^* BDF \\ &+ \lambda A^* A [(CG)^* CG + (DF)^* DF] \end{aligned} \quad (28)$$

Hence, the expression (18) becomes:

$$P_{\mathcal{R}} = \frac{z^m \tau \varphi^* \varphi M^* M - \mathcal{R}_o^c \kappa \gamma^* \gamma}{(ACFM)^* ACFM} \quad (29)$$

with  $\gamma$  being the spectral factor defined as:

$$\kappa \gamma^* \gamma = \tau \varphi^* \varphi M^* M + (1 - \lambda)(ACFN)^* ACFN \quad (30)$$

Let consider:

- the optimal robust estimator is given by:

$$\mathcal{R}_o^c = \frac{M\theta}{\gamma} \quad (31)$$

- the Diophantine equation:

$$z^m \tau \varphi^* \varphi M^* - \kappa \gamma^* \theta = z ACF \chi^* \quad (32)$$

where  $\theta$  and  $\chi$  are stable polynomials in  $z^{-1}$  satisfying to the following degree conditions:

$$\begin{aligned} n_{\theta} &= \max(n_{\varphi} - m, n_A + n_C + n_F - 1) \\ n_{\chi} &= \max(n_{\varphi} + n_M + m, n_{\gamma}) - 1 \end{aligned} \quad (33)$$

Consequently, the expression of  $P_{\mathcal{R}}$  becomes:

$$P_{\mathcal{R}} = \frac{z \chi^*}{(ACFM)^*} \quad (34)$$

Replacing this expression into the integrand of condition (10) leads to :

$$\delta_{\mathcal{R}} = \frac{-2}{2\pi} \oint_{|z|=1} \left( \frac{R\chi}{ACFM} \right)^* dz$$

The integrand has all its poles outside the unit circle then this expression is null. The optimality condition (10) is verified.

From (26), the optimal solution represents a minimum. As a direct consequence, the robust estimator (24) minimises the criterion (5).

The previous results are synthesised in the following theorem:

*Theorem 2:* The robust estimator of the form (6) minimising criterion (5) is given by :

$$\hat{s}(t|t-m) = Hu(t) + q^{-m} \mathcal{R}_o^c (y(t) - Hu(t)) \quad (35)$$

where  $\mathcal{R}_o^c$  is obtained from equation (31) together with:

- the spectral factors  $\varphi$  in (28) and  $\gamma$  in (30),
- the resolution of the Diophantine equation (32) w.r.t. to polynomials  $\theta$  and  $\chi$  under condition (33),
- the tuning parameter  $\lambda \in [0; 1[$ .

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### IV. EXTENSION TO UNCERTAIN $L(q^{-1})$

From the results in Theorem 1 and Theorem 2, the expression of the optimal robust filter depends on the transfer function  $L$  which represents the contribution of the measured input into the sensitivity of the estimation error. Unfortunately, the a-priori knowledge of the spectrum of the input signal may not be perfect entailing a certain level of uncertainty in the transfer function  $L$ . Consequently, the designer should take into account this additional information in the design of the robust estimator. Finally, the original criterion (5) should be modified as follows:

$$J = (1 - \lambda - \mu) E \{ \varepsilon^2(t) \} + \lambda E \{ (\partial_H \varepsilon(t))^2 \} + \mu E \{ (\partial_L \varepsilon(t))^2 \} \quad (36)$$

where  $\lambda \in [0; 1[$  and  $\mu \in [0; 1[$  are a tuning parameters such that:  $0 \leq 1 - \lambda - \mu < 1$ .

From (7), we can write:

$$\partial_L \varepsilon(t) = q^{-m} [(q^m - \mathcal{R})H - S]r(t) \quad (37)$$

Hence, criterion (5) can be written in the  $z$ -domain as follows:

$$J = \frac{1}{j2\pi} \oint_{|z|=1} \{ (1 - \lambda - \mu) \mathcal{I}_1 + \lambda \mathcal{I}_2 + \mu \mathcal{I}_3 \} \frac{dz}{z}$$

with  $\mathcal{I}_1$  and  $\mathcal{I}_2$  defined in the previous section, and,  $\mathcal{I}_3$  defined as:

$$\mathcal{I}_3 = [(z^m - \mathcal{R})H - S]^* [(z^m - \mathcal{R})H - S] \quad (38)$$

The optimality condition (10) entails the relations:

- Condition for  $\mathcal{R}_o$

$$\delta_{\mathcal{R}} = \frac{-2}{2\pi} \oint_{|z|=1} R^* P'_{\mathcal{R}} \frac{dz}{z} \equiv 0$$

with

$$\begin{aligned} P'_{\mathcal{R}} &= H^* P'_S + (z^m - \mathcal{R}_o) \Phi'^* \Phi' \\ &- (1 - \lambda - \mu) \mathcal{R}_o \sigma_v^2 \end{aligned} \quad (39)$$

and

$$\Phi'^* \Phi' = (1 - \lambda - \mu) H^* H \sigma_w^2 + \lambda (\sigma_u^2 + \sigma_w^2) \quad (40)$$

- Condition for  $\mathcal{S}_o$

$$\delta_{\mathcal{S}} = \frac{-2}{2\pi} \oint_{|z|=1} S^* P'_S \frac{dz}{z} \equiv 0$$

with

$$P'_S = [(1 - \lambda - \mu)\sigma_u^2 + \mu][(z^m - \mathcal{R}_o)H - \mathcal{S}_o] \quad (41)$$

From  $P'_S$ , the expression of the filter  $\mathcal{S}_o$  is derived. It appears that it is exactly the same as in Theorems 1 and 2.

Finally, compared to the results presented in Theorems 1 and 2, poor modifications have to be made,  $(1 - \lambda)$  should be replaced by  $(1 - \lambda - \mu)$ .

## V. EXAMPLE

The nominal system considered for the design of the estimators is given by the transfer functions:

$$H(q^{-1}) = 0.0257q^{-1} \frac{q^{-1} + 1.034}{q^{-2} - 2.053q^{-1} + 1.105}$$

$$L(q^{-1}) = \frac{0.1578}{q^{-2} - 2.051q^{-1} + 1.052}$$

We consider that  $w(t) = \eta(t)$  and  $v(t) = n(t)$ , i.e.,  $C(q^{-1}) = D(q^{-1}) = M(q^{-1}) = N(q^{-1}) = 1$ .

In the following, we consider that the real transfer function  $H$  is:

$$H_{real}(q^{-1}) = 0.0257q^{-1} \frac{q^{-1} + 1.034}{q^{-2} + a_2q^{-1} + 1.105}$$

with  $a_2 \in [-2.063; -2.042]$ .

The performance of the proposed robust estimator will be evaluated according to the relative root mean square error ( $rRMSE$ ) criterion defined as follows:

$$rRMSE = \sqrt{\frac{E\{\varepsilon^2(t)\}}{E\{s^2(t)\}}}$$

Furthermore, we define the boolean variable:

$$X = rRMSE(\lambda \neq 0) \leq rRMSE(\lambda = 0) \quad (42)$$

Clearly, it turns out that if the robust  $rRMSE$  (i.e.  $\lambda \neq 0$ ) is inferior to the  $rRMSE$  of the nominal filter (i.e.  $\lambda = 0$ ), then  $X = 1$ , else  $X = 0$ . Hence, it permits to evaluate quickly whether the robust filter outperforms or not the nominal filter.

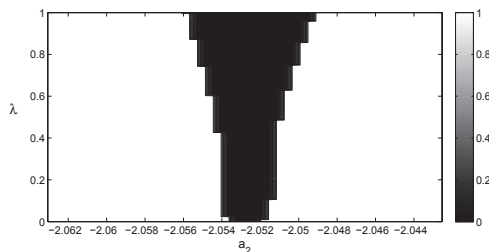


Fig. 2. Map of  $X$ .

From figure 2, it appears that in most cases, the robust filter outperforms the nominal filter for any chosen value of

parameter  $\lambda$ . Unsurprisingly, when parameter  $a_2$  gets close to its nominal value, the nominal filter performs better than the robust filter. This feature is standard in robust estimation in presence of parametric uncertainty.

In order to complete the analysis, we have exhibited a particular case. As a matter of fact, we have evaluated the performance of the robust filter with  $\lambda = 0.234$  and when parameter  $a_2$  is  $-2.044$ . These values have been chosen randomly.

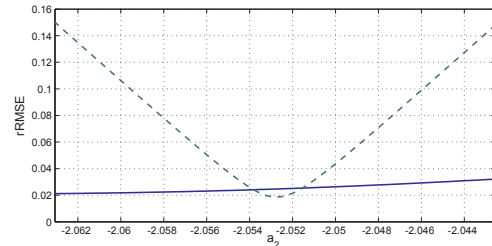


Fig. 3. Evolution with  $a_2$  of the  $rRMSE$  for the robust filter with  $\lambda = 0.234$  (solid) and the nominal filter (dashed).

From figure 3, it appears that the  $rRMSE$  of the robust filter is nearly constant over the range of possible values of  $a_2$ . From this plot, we can see that the  $rRMSE$  of the robust filter presents its largest value for  $a_2$  close to  $-2.044$ . On the other hand, the  $rRMSE$  of the nominal filter has its largest values in the same area as well. Consequently, the choice of this particular value of  $a_2$  is a least favourable case for both filters. Figure 4 confirms that this choice entails a large deviation from the nominal behaviour of the system.

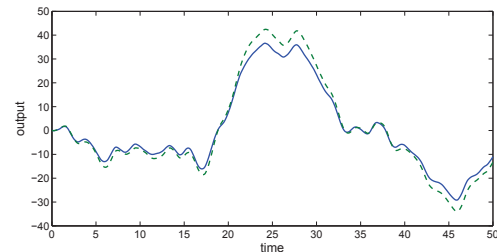


Fig. 4. Nominal (dashed) - Real (solid) for  $a_2 = -2.044$ .

The estimation error of both filters are plotted in figure 5. Clearly, the estimation error of the robust filter is approximately centred on zero while the nominal filter appears to be strongly biased. This shows the efficiency of the proposed approach.

In the previous case, only constant parameter error has been considered. In order to generalise the previous results, we consider that parameter  $a_2$  is time-varying as shown in figure 6. From the same figure, it can be seen that the effect on the output signal is not trivial.

The performance of the proposed estimator is plotted in figure 7. The  $rRMSE$  along all possible values of  $\lambda$  is inferior to the nominal  $rRMSE$  for any value of the tuning

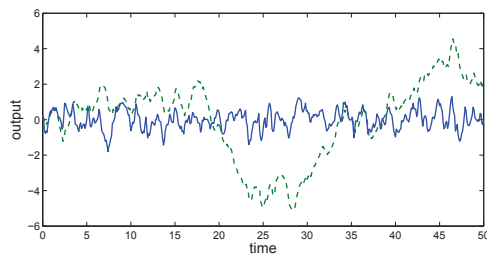


Fig. 5. Estimation error for the nominal filter (dashed) and for the robust filter with  $\lambda = 0.234$  (solid) for  $a_2 = -2.044$ .

parameter. Again, the estimation error for the optimal choice of  $\lambda$  behaves as a centred random process while the nominal estimator is completely biased (see figure 7). These results confirm the efficiency of the proposed robust estimator.

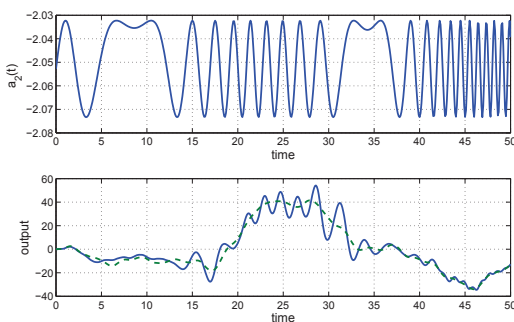


Fig. 6. Upper plot - Evolution of  $a_2(t)$ . Lower plot - Comparison of the nominal output (dashed) with the actual output (solid).

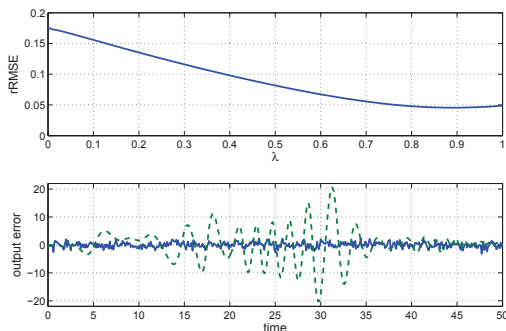


Fig. 7. Upper plot -  $rRMSE$  of the proposed estimator for  $a_2(t)$ . Lower plot - Estimation error of the nominal estimator (dashed) vs the optimal robust estimator (solid).

## VI. CONCLUDING REMARKS

The problem of robust filtering and fixed-lag smoothing has been tackled in the present work. The approach consists in introducing the sensitivity of the estimation error with respect to the uncertain transfer function within the design procedure by means of a  $\lambda$ -contaminated technique. The

advantage of this technique is that no modelisation of the uncertainty is required compared to standard approaches. The solution is expressed in the transfer function representation and a polynomial framework as well. In the latter, the design requires spectral factors and a Diophantine equation. Unlike standard approaches, the filter equations depend upon the spectrum of the known input. In order to overcome optimality problems due to uncertainties in this spectrum, these uncertainties have been introduced design of the robust estimator. It results in a slight modification of the constitutive equations of the robust filter. An example has shown the efficiency of the proposed approach to handle uncertainties in the transfer function of a system and to provide a reliable estimate of the desired signal.

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