

# Contraction of Riccati flows applied to the convergence analysis of the max-plus curse of dimensionality free method

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**Abstract**—Max-plus based methods have been recently explored for solution of first-order Hamilton-Jacobi-Bellman equations by several authors. In particular, McEneaney’s curse-of-dimensionality free method applies to the equations where the Hamiltonian takes the form of a (pointwise) maximum of linear/quadratic forms. In previous works of McEneaney and Kluberg, the approximation error of the method was shown to be  $O(1/(N\tau)) + O(\sqrt{\tau})$  where  $\tau$  is the time discretization step and  $N$  is the number of iterations. Here we use a recently established contraction result of the indefinite Riccati flow in Thompson’s metric to show that under different technical assumptions, still covering an important class of problems, the total error incorporating a pruning procedure of error order  $\tau^2$  is  $O(e^{-\alpha N\tau}) + O(\tau)$  for some  $\alpha > 0$  related to the contraction rate of the indefinite Riccati flow.

## I. INTRODUCTION

### A. Max-plus methods in optimal control

Dynamic Programming (DP) is a general approach to the solution of optimal control problems. In the case of deterministic optimal control, this approach leads to solving a first-order, nonlinear partial differential equation, the Hamilton-Jacobi-Bellman equation (HJB PDE). Various methods have been proposed for solving the HJB PDE, including finite difference schemes, the method of the vanishing viscosity [CL84], the antidiffusive schemes for advection [BZ07], and the so-called discrete dynamic programming method or semi-Lagrangian method [CD83], [Fal87], [CFF04]. These methods are all grid-based methods, i.e., they require a generation of a grid over some bounded region of the state space. These methods are known to suffer from the so called curse-of-dimensionality since the computational growth in the state-space dimension is exponential.

Recently a new class of methods has been developed after the work of Fleming and McEneaney [FM00], see in particular [McE07], [AGL08], [MDG08]. These methods are referred to as *max-plus basis methods* since they all rely on max-plus algebra. Their common idea is to approximate the value function by a supremum of finitely many “basis functions” and to propagate forward in time by exploiting the max-plus linearity of the Lax-Oleinik semi-group. Recall that the Lax-Oleinik semi-group  $(S_t)_{t \geq 0}$  associated to a Hamiltonian  $H(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is the evolution semi-

group of the following HJB PDE

$$-\frac{\partial v}{\partial t} + H(x, \nabla v) = 0, \quad (x, t) \in \mathbb{R}^n \times (0, T], \quad (1)$$

with initial condition

$$v(x, 0) = \phi(x), \quad x \in \mathbb{R}^n. \quad (2)$$

Thus,  $S_t$  maps the initial function  $\phi(\cdot)$  to the function  $v(\cdot, t)$ . Among several max-plus basis methods which have been proposed, the curse-of-dimensionality-free method introduced by McEneaney [McE07] is of special interest. This method applies to the special class of HJB PDE where the Hamiltonian  $H$  is given or approximated as a pointwise maximum of computationally simpler Hamiltonians:

$$H(x, \nabla V) = \max_{m \in \mathcal{M}} \{H^m(x, \nabla V)\} \quad (3)$$

with  $\mathcal{M} = \{1, 2, \dots, M\}$ . In particular, the author studied  $H^m$  of linear/quadratic forms, corresponding to linear quadratic optimal control problems:

$$H^m(x, p) = (A^m x)' p + \frac{1}{2} x' D^m x + \frac{1}{2} p' \Sigma^m p,$$

where  $(A^m, D^m, \Sigma^m)$  are all matrices meeting certain conditions. We denote by  $(S_t)_{t \geq 0}$  and  $(S_t^m)_{t \geq 0}$  for all  $m \in \mathcal{M}$  respectively the semi-group corresponding to  $H$  and  $H^m$  for all  $m \in \mathcal{M}$ . The essential idea (Section II) is to approximate the solution  $V$  of (3) by  $S_T[V_0]$  for some initial function  $V_0$  and a large  $T > 0$ . The error at point  $x$  of this finite horizon approximation is denoted by

$$\epsilon_0(x, T, V_0) := V(x) - S_T[V_0](x).$$

Next we approximate  $S_T[V_0]$  by  $\{\sup_{m \in \mathcal{M}} S_\tau^m\}^N[V_0]$  for a time discretization step  $\tau > 0$  and an iteration number  $N \in \mathbb{N}$  such that  $T = N\tau$ . The error at point  $x$  of this time discretization approximation is denoted by:

$$\epsilon(x, \tau, N, V_0) := S_T[V_0](x) - \{\sup_{m \in \mathcal{M}} S_\tau^m\}^N[V_0](x).$$

The total error at a point  $x$  is then simply  $\epsilon_0(x, T, V_0) + \epsilon(x, \tau, N, V_0)$ . Each  $S_\tau^m$  corresponds to solving a Riccati equation, requiring  $O(n^3)$  arithmetic operations. The total number of computational cost is  $O(|\mathcal{M}|^N n^3)$ , with a cubic growth in the state dimension  $n$ . In this sense it is considered

as a curse of dimensionality free method. However, we see that the computational cost is bounded by a number exponential to the number of iterations, which is referred to as the curse of complexity. In practice, a pruning procedure denoted by  $\mathcal{P}_\tau$  removing at each iteration a number of functions less useful than others is needed in order to reduce the curse of complexity. We denote the error at point  $x$  of the time discretization approximation incorporating the pruning procedure by:

$$\epsilon^{\mathcal{P}_\tau}(x, \tau, N, V_0) = S_T[V_0](x) - \{\mathcal{P}_\tau \circ \sup_{m \in \mathcal{M}} S_\tau^m\}^N[V_0](x).$$

### B. Main contributions

In this paper, we analyze the growth rate of  $\epsilon_0(x, T, V_0)$  as  $T$  tends to infinity and that of  $\epsilon^{\mathcal{P}_\tau}(x, \tau, N, V_0)$  as  $\tau$  tends to 0, incorporating a pruning procedure  $\mathcal{P}_\tau$  of error  $O(\tau^r)$  with  $r > 1$ . The growth rate of error  $\epsilon(x, \tau, N, V_0)$  as  $\tau$  tends to 0 is obtained as a corollary by letting  $r = +\infty$ .

We show that under technical assumptions (Assumption 2.1 and 3.2),

$$\epsilon_0(x, T, V_0) = O(e^{-\alpha T}), \quad \text{as } T \rightarrow +\infty$$

uniformly for all  $x \in \mathbb{R}^n$  and all initial function  $V_0$  in a certain compact (Theorem 4.1) and that given a pruning procedure generating error  $O(\tau^r)$  with  $r > 1$ ,

$$\epsilon^{\mathcal{P}_\tau}(x, \tau, N, V_0) = O(\tau^{\min\{1, r-1\}}), \quad \text{as } \tau \rightarrow 0$$

uniformly for all  $x \in \mathbb{R}^n$ ,  $N \in \mathbb{N}$  and  $V_0$  in a compact (Theorem 4.2). As a direct corollary, we have

$$\epsilon(x, \tau, N, V_0) = O(\tau), \quad \text{as } \tau \rightarrow 0$$

uniformly for all  $x \in \mathbb{R}^n$ ,  $N \in \mathbb{N}$  and  $V_0$  in a compact.

### C. Comparison with earlier estimates

It has been shown in [MK10, Thm 7.1] that under Assumption 2.1, for a given  $V_0$ ,

$$\epsilon_0(x, T, V_0) = O\left(\frac{1}{T}\right), \quad \text{as } T \rightarrow +\infty$$

uniformly for all  $x \in \mathbb{R}^n$ . They also showed [MK10, Thm 6.1] that if in addition to Assumption 2.1, the matrices  $\Sigma^m$  are all identical for  $m \in \mathcal{M}$ , then for a given  $V_0$ ,

$$\epsilon(x, \tau, N, V_0) = O(\sqrt{\tau}), \quad \text{as } \tau \rightarrow 0$$

uniformly for all  $x \in \mathbb{R}^n$  and  $N \in \mathbb{N}$ . Their estimates imply that to get a sufficiently small approximation error  $\epsilon$  we can use a horizon  $T = O(1/\epsilon)$  and a discretization step  $\tau = O(\epsilon^2)$ . Thus asymptotically the computational cost is:

$$O(|\mathcal{M}|^{O(1/\epsilon^3)} n^3), \quad \text{as } \epsilon \rightarrow 0. \quad (4)$$

The same reasoning applied to our estimates shows a considerably smaller asymptotic growth rate of the computational cost:

$$O(|\mathcal{M}|^{O(-\log(\epsilon)/\epsilon)} n^3), \quad \text{as } \epsilon \rightarrow 0 \quad (5)$$

McEneaney and Kluber [MK10] gave a technically difficult proof of the estimates (4) and (5), assuming that all the

$\Sigma^m$  are the same. They conjecture that the latter assumption can at least be released for a subclass of problems. This is supported by our results, showing that for the subclass of problems satisfying Assumption 3.2, this assumption can be omitted. To this end, we use a totally different approach. Our main ingredient is the strict local contraction property of the indefinite Riccati flow [GQ12], under Assumptions 2.1 and 3.2.

Our approach derives a tighter estimation of  $\epsilon_0(x, T, V_0)$  and  $\epsilon(x, \tau, N, V_0)$  compared to previous results as well as an estimation of  $\epsilon^{\mathcal{P}_\tau}(x, \tau, N, V_0)$  incorporating the pruning procedure. This new result justifies the use of pruning procedure of error  $O(\tau^2)$  without increasing the asymptotic total approximation error order.

The paper is organized as follows. In section II we recall the max-plus problem and the max-plus approximation method. In section III we state the contraction results on the indefinite Riccati flow, which is an essential ingredient to our main results. In section IV we present the main results and the sketch of proofs. And lastly in section V we give some remarks and some numerical illustrations on the theoretical estimates.

## II. PROBLEM STATEMENT

For the sake of completeness, we restate briefly the problem class and present some basic concepts and necessary assumptions in this section. The reader can find in [McE07] the same description with more details.

### A. Problem class

Let  $\mathcal{M} = \{1, \dots, M\}$  be a finite index set. We are interested in finding the value function of the following switching optimal control problem:

$$V(x) = \sup_{w \in \mathcal{W}} \sup_{\mu \in \mathcal{D}_\infty} \sup_T \int_0^T \frac{1}{2} (\xi_t^{\mu_t})' D^{\mu_t} \xi_t^{\mu_t} - \frac{\gamma^2}{2} |w_t|^2 dt$$

where

$$\mathcal{D}_\infty := \{\mu : [0, \infty) \rightarrow \mathcal{M} : \mu \text{ measurable}\},$$

$$\mathcal{W} := \{w : [0, \infty) \rightarrow \mathbb{R}^k : \int_0^T |w_t|^2 dt < \infty, \forall T < \infty\},$$

and  $\xi$  is subject to:

$$\dot{\xi} = A^{\mu_t} \xi + \sigma^{\mu_t} w_t, \quad \xi_0 = x. \quad (6)$$

As in [McE07], we make the following assumptions throughout the paper to guarantee the existence of  $V$ .

*Assumption 2.1:*

- There exists  $c_A > 0$  such that:

$$x' A^m x \leq -c_A |x|^2, \quad \forall x \in \mathbb{R}^n, m \in \mathcal{M}$$

- There exists  $c_\sigma > 0$  such that:

$$|\sigma^m| \leq c_\sigma, \quad \forall m \in \mathcal{M}$$

- All  $D^m$  are positive definite, symmetric, and there is  $c_D$  such that:

$$x' D^m x \leq c_D |x|^2, \quad \forall x \in \mathbb{R}^n, m \in \mathcal{M},$$

and

$$c_A^2 > \frac{c_D c_\sigma^2}{\gamma^2}$$

### B. Steady HJB equation

For any  $\delta \in (0, \gamma)$ , define

$$G_\delta := \{V_0 \text{ semiconvex}, 0 \leq V_0(x) \leq \frac{c_A(\gamma - \delta)^2}{c_\sigma^2} |x|^2, \forall x\}. \quad (7)$$

Then the value function  $V$  is the unique viscosity solution of the following corresponding HJB PDE in the class  $G_\delta$  for sufficiently small  $\delta$ :

$$0 = -H(x, \nabla V) = -\max_{m \in \mathcal{M}} H^m(x, \nabla V), \quad V(0) = 0 \quad (8)$$

where

$$H^m(x, p) = (A^m x)' p + \frac{1}{2} x' D^m x + \frac{1}{2} p' \Sigma^m p.$$

Denote by  $\{S_t\}_{t \geq 0}$  the evolution semi-group corresponding to  $H$ . It has been proved in [McE07] that for all  $V_0 \in G_\delta$ ,

$$\lim_{T \rightarrow \infty} S_T[V_0] = V \quad (9)$$

uniformly on compact sets.

### C. Max-plus based approximation

We review the basic steps of the algorithm proposed in [McE07] to approximate the value function  $V$ . Firstly using (9) we are allowed to approximate  $V$  by  $S_T[V_0]$  for some sufficiently large  $T$ . We then choose a time-discretization step  $\tau > 0$  and a number of iterations  $N$  such that  $T = N\tau$  to approximate  $S_T$  by

$$S_T = \{S_\tau\}^N \simeq \{\tilde{S}_\tau\}^N$$

where  $\tilde{S}_\tau = \sup_{m \in \mathcal{M}} S_\tau^m$  and  $\{S_t^m\}_{t \geq 0}$  is the evolution semi-group corresponding to  $H^m$  for all  $m \in \mathcal{M}$ . At the end of  $N$  iterations, we get our approximated value function represented by:

$$V \simeq \sup_{i_1, \dots, i_N} S_\tau^{i_N} \dots S_\tau^{i_1}[V_0].$$

If we choose  $V_0(x) = \frac{1}{2} x' P x$  as a quadratic function, then the approximated value function will be the maximum of  $|\mathcal{M}|^N$  quadratic functions. This so-called curse-of-complexity can be reduced by performing a pruning process at each iteration of the algorithm to remove some quadratic functions, see [MDG08].

### D. Error estimate

As pointed out in [MK10], the approximation error comes from two parts. The first error source is

$$\epsilon_0(x, T, V_0) := V(x) - S_T[V_0](x).$$

This is due to approximate the infinite-horizon problem by a finite-horizon problem. The second one is

$$\epsilon(x, \tau, N, V_0) := S_{N\tau}[V_0](x) - \{\tilde{S}_\tau\}^N[V_0](x),$$

due to the approximation of the semi-group by a time-discretization. If we take into account the pruning procedure, then the second error source should be written as:

$$\epsilon^{\mathcal{P}_\tau}(x, \tau, N, V_0) = S_{N\tau}[V_0](x) - \{\mathcal{P}_\tau \circ \tilde{S}_\tau\}^N[V_0](x), \quad (10)$$

where  $\mathcal{P}_\tau$  represents a pruning action. We mark the subscript  $\tau$  since it is expected that the pruning procedure be adapted with the time step  $\tau$ . In particular, we say that  $\mathcal{P}_\tau$  is a pruning procedure generating an error  $O(\tau^r)$  if there is  $L > 0$  such that for all function  $f$ ,

$$\mathcal{P}_\tau[f] \leq f \leq (1 + L\tau^r)\mathcal{P}_\tau[f]. \quad (11)$$

The special case without pruning procedure can be recovered by considering  $r = +\infty$ .

## III. CONTRACTION PROPERTIES OF THE INDEFINITE RICCATI FLOW

Before showing the main results, we present here the essential ingredient to our proof: the contraction properties of the indefinite Riccati flow.

### A. Loewner order and the Thompson's part metric

We recall some basic notions and terminologies. We refer the readers to [Nus88] for more background.

We consider the space of  $n$ -dimensional symmetric matrices  $S_n$  equipped with the operator norm  $\|\cdot\|$ . The space of positive semi-definite (resp. positive definite) matrices is denoted by  $S_n^+$  (resp.  $\hat{S}_n^+$ ). The *Loewner order* " $\leq$ " and the *strict Loewner order* " $<$ " on  $S_n$  are defined by:

$$A \leq B \Leftrightarrow B - A \in S_n^+, \quad A < B \Leftrightarrow B - A \in \hat{S}_n^+.$$

For  $A \leq B$  we define the order intervals:

$$[A, B] := \{P \in S_n \mid A \leq P < B\}.$$

For  $P_1, P_2 \in \hat{S}_n^+$ , following [Nus88], we define

$$M(P_1/P_2) := \inf\{t > 0 : P_1 \leq tP_2\}$$

*Definition 3.1:* The *Thompson part metric* between two elements  $P_1$  and  $P_2$  of  $\hat{S}_n^+$  is

$$d_T(P_1, P_2) := \log(\max\{M(P_1/P_2), M(P_2/P_1)\}).$$

It is known [Nus88] that the metric space  $(\hat{S}_n^+, d_T(\cdot, \cdot))$  is complete.

### B. Contraction rate of the indefinite Riccati flow

For each  $m \in \mathcal{M}$ , define the function  $\Phi^m: S_n \rightarrow S_n$ :

$$\Phi^m(P) = (A^m)' P + P A^m + P \Sigma^m P + D^m \quad (12)$$

where  $\Sigma^m := \frac{\gamma^2}{2} \sigma^m (\sigma^m)'$ . Associated to each  $\Phi^m$  we define the flow map by:

$$M_t^m(P_0) = P(t), \quad t \in [0, T]$$

where  $P(t) : [0, T] \rightarrow S_n$  is a maximal solution of the following initial value problem on  $S_n$ :

$$\dot{P} = \Phi^m(P), \quad P(0) = P_0. \quad (13)$$

When  $V_0(x) = \frac{1}{2}x'P_0x$  with  $P_0 \in S_n$ , a classical result [YZ99] states that

$$S_t^m[V_0](x) = \frac{1}{2}x'M_t^m(P_0)x, \quad t \in [0, T]. \quad (14)$$

The standard Riccati equation refers to a vector field of the form (12) with  $-\Sigma^m$  and  $D^m$  positive semi-definite. Here we are concerned with the *indefinite* Riccati equation since the matrix coefficient  $\Sigma^m$  is positive semi-definite.

The contraction property and the contraction rate calculus of the standard Riccati flow in Thompson's metric have been given in [LW94] and [LL07]. However, their approach depends on the algebraic property of the associated symplectic operator, which fails in the indefinite case. In [GQ12], the authors use a tangent characterization of invariant sets to show that under additional constraints on the matrix coefficients the local strict contraction still holds in the indefinite case. Below is the additional assumption needed to apply this new contraction result:

*Assumption 3.2:* There is  $m_D > 0$  such that

$$x'D^m x \geq m_D |x|^2, \quad \forall x \in \mathbb{R}^n, m \in \mathcal{M}$$

and

$$\frac{c_\sigma^2}{\gamma^2} m_D > (c_A - \sqrt{c_A^2 - c_D c_\sigma^2 / \gamma^2})^2.$$

In the sequel we denote

$$\lambda_1 = \frac{\gamma^2 (c_A - \sqrt{c_A^2 - c_D c_\sigma^2 / \gamma^2})}{c_\sigma^2}, \quad \lambda_2 := \sqrt{\frac{m_D \gamma^2}{c_\sigma^2}}.$$

*Remark 3.3:* Under Assumption 3.2, there is  $c > 0$  such that  $\Phi^m(\epsilon I) \geq 0$  for all  $\epsilon \in [0, c]$  and  $m \in \mathcal{M}$ . We can chose an  $\epsilon \leq c$  sufficiently small such that for some  $t_0 > 0$  and  $m \in \mathcal{M}$  we have  $S_{t_0}^m[0] \geq \epsilon I$ . Besides, for any  $\lambda \in [\lambda_1, \lambda_2]$ , we have  $\Phi^m(\lambda I) \leq 0$  for all  $m \in \mathcal{M}$ . Then it follows from a standard result on the Riccati equation that:

$$M_t^m(P_0) \in [\epsilon I, \lambda I], \quad \forall m \in \mathcal{M}, t \geq 0, P_0 \in [\epsilon I, \lambda I]. \quad (15)$$

The main ingredient to make our proofs is the following theorem:

*Theorem 3.4 (Corollary in [GQ12]):* Under Assumptions 2.1 and 3.2, for any  $\lambda \in [\lambda_1, \lambda_2]$ , there is  $\alpha > 0$  such that for all  $P_1, P_2 \in (0, \lambda I]$ ,

$$d_T(M_t^m(P_1), M_t^m(P_2)) \leq e^{-\alpha t} d_T(P_1, P_2), \quad \forall t \geq 0, m \in \mathcal{M}.$$

*C. Extension of the contraction result to the space of functions*

Now we extend the definition of Thompson's metric to the space of functions. For two functions  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ , we define the *Loewner order* " $\leq$ " by:

$$f \leq g \Leftrightarrow f(x) \leq g(x), \quad \forall x \in \mathbb{R}^n.$$

Similarly, for  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}_+$  we define

$$M(f/g) := \inf\{t > 0 : f \leq tg\}$$

We say that  $f$  and  $g$  are comparable if  $M(f/g)$  and  $M(g/f)$  are finite. In that case, we can define the "*Thompson metric*" between  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}_+$  by:

$$d_T(f, g) = \log(\max\{M(f/g), M(g/f)\}). \quad (16)$$

The following result is a consequence of the order-preserving character of the Riccati flow and of the contraction property in Theorem 3.4.

*Corollary 3.5:* Under Assumptions 2.1 and 3.2, let  $\lambda \in [\lambda_1, \lambda_2]$  and  $\epsilon > 0$  such that (15) holds. Then there is  $\alpha > 0$  such that for any two functions  $V_1$  and  $V_2$  of the form:

$$V_1(x) = \sup_{j \in J} \frac{1}{2} x' P_j x, \quad V_2(x) = \frac{1}{2} x' Q x,$$

where  $J$  is an index set and  $Q, P_j \in [\epsilon I, \lambda I]$  for all  $j \in J$ , we have

$$d_T(S_{t/N}^{i_N} \cdots S_{t/N}^{i_1}[V_1], S_{t/N}^{i_N} \cdots S_{t/N}^{i_1}[V_2]) \leq e^{-\alpha t} \log\left(\frac{\lambda}{\epsilon}\right)$$

for all  $t \geq 0$ ,  $N \in \mathbb{N}$  and  $\{i_1, \dots, i_N\} \in \mathcal{M}$ .

#### IV. MAIN RESULTS

Here are the two main results of this paper:

*Theorem 4.1:* Under Assumptions 2.1 and 3.2, let  $\lambda \in [\lambda_1, \lambda_2]$  and  $\epsilon > 0$  such that (15) holds. There exist  $\alpha > 0$  and  $K > 0$  such that,

$$\epsilon_0(x, T, V_0) \leq K e^{-\alpha T} |x|^2, \quad \forall x,$$

for all  $T > 0$  and  $V_0(x) = \frac{1}{2}x'P_0x$  with  $P_0 \in [\epsilon I, \lambda I]$ .

*Theorem 4.2:* Let  $r > 1$ . Suppose that for each  $\tau > 0$  the pruning operation  $\mathcal{P}_\tau$  generates an error  $O(\tau^r)$  (see (11)). Under Assumptions 2.1 and 3.2, let  $\lambda \in [\lambda_1, \lambda_2]$  and  $\epsilon > 0$  such that (15) holds. Then there exist  $\tau_0 > 0$  and  $L > 0$  such that

$$\epsilon^{\mathcal{P}_\tau}(x, \tau, N, V_0) \leq L \tau^{\min\{1, r-1\}} |x|^2, \quad \forall x,$$

for all  $N \in \mathbb{N}$ ,  $\tau \leq \tau_0$  and  $V_0(x) = \frac{1}{2}x'P_0x$  with  $P_0 \in [\epsilon I, \lambda I]$ .

*A. Key lemma*

In addition to Theorem 3.4, another key lemma in our proofs is:

*Lemma 4.3:* For all  $T > 0$  and  $V_0 \in G_\delta$  locally Lipschitz,

$$S_T[V_0] = \sup_N \sup_{i_1, \dots, i_N} S_{T/N}^{i_N} \cdots S_{T/N}^{i_1}[V_0].$$

Lack of space, we briefly present the main idea of the proof for this lemma. First we show by elementary tools that the functional

$$J(x, T; V^0; \mu, w) := \int_0^T \frac{1}{2} \xi_t' D^{\mu_t} \xi_t - \frac{\gamma^2}{2} |w_t|^2 dt + V^0(\xi_T)$$

is continuous with respect to

$$\mu \in \mathcal{D}_T := \{\mu : [0, T] \rightarrow \mathcal{M} \mid \mu \text{ measurable}\}.$$

Then we apply the Lusin's theorem [Fol99] which states that every measurable function is a continuous function on nearly all its domain. This theorem allows to construct a piecewise constant function in  $\mathcal{D}_T$  which is arbitrarily close to a given

measurable function in  $\mathcal{D}_T$ . Thus an optimal control  $\mu \in \mathcal{D}_T$  can be approximated arbitrarily well by a piecewise constant functions in  $\mathcal{D}_T$ , which is an interpretation of the above lemma.

### B. Sketch of proof of Theorem 4.1

We show that:

*Lemma 4.4:* For any  $\lambda$  and  $\epsilon$  be as in Theorem 4.1. There is  $\alpha > 0$  such that

$$d_T(V, S_T[V_0]) \leq e^{-\alpha T} \log\left(\frac{\lambda}{\epsilon}\right)$$

for all  $V_0(x) = \frac{1}{2}x'P_0x$  with  $P_0 \in [\epsilon I, \lambda I]$ .

*Proof:* By Remark 3.3 there is  $t_0 > 0$  and  $m \in \mathcal{M}$  such that  $S_{t_0}^m[0] \in [\epsilon I, \lambda I]$ . Therefore,

$$V = \sup_{T>0} S_T[0] \leq \sup_{T>0} S_T[S_{t_0}^m[0]].$$

We also have,

$$V \geq \sup_{T>0} S_{T+t_0}[0] = \sup_{T>0} S_T[S_{t_0}[0]] \geq \sup_{T>0} S_T[S_{t_0}^m[0]].$$

Therefore using Lemma 4.3, we can write:

$$V = \sup_T S_T[S_{t_0}^m[0]] = \sup_T \sup_N \sup_{i_1, \dots, i_N} S_{T/N}^{i_N} \cdots S_{T/N}^{i_1}[S_{t_0}^m[0]].$$

By (15),  $V$  is a supremum of quadratic functions given by matrices in  $[\epsilon I, \lambda I]$ . Now we apply Corollary 3.5 and get  $\alpha > 0$  such that

$$\begin{aligned} d_T(V, S_T[V_0]) &= d_T(S_T[V], S_T[V_0]) \\ &\leq \sup_N \sup_{i_1, \dots, i_N} d_T(S_{T/N}^{i_N} \cdots S_{T/N}^{i_1}[V], S_{T/N}^{i_N} \cdots S_{T/N}^{i_1}[V_0]) \\ &\leq e^{-\alpha T} \log\left(\frac{\lambda}{\epsilon}\right). \end{aligned}$$

We omit the few steps left to deduce Theorem 4.1 from Lemma 4.4, essentially using the definition (16) and the upper boundedness of the solution.  $\blacksquare$

### C. Sketch of proof of Theorem 4.2

Before all we give the one step error estimation:

*Proposition 4.5:* Let  $\mathcal{K} \subset S_n$  be a compact convex subset. There exists  $\tau_0 > 0$  and  $L > 0$  such that

$$S_\tau[V_0](x) \leq \tilde{S}_\tau[V_0](x) + L\tau^2|x|^2, \quad \forall x,$$

for all  $\tau \in [0, \tau_0]$  and  $V_0(x) = \frac{1}{2}x'P_0x$  with  $P_0 \in \mathcal{K}$ .

*Proof:* [sketch] The key step is to prove the existence of some  $L > 0$  and  $\tau_0 > 0$  such that:

$$S_{\tau/N}^{i_N} \cdots S_{\tau/N}^{i_1}[V^0](x) \leq \sup_m S_\tau^m[V^0](x) + L\tau^2|x|^2, \quad \forall x,$$

for all  $N \in \mathbb{N}$ ,  $\{i_1, \dots, i_N\} \in \mathcal{M}^N$  and  $\tau \leq \tau_0$ . This is done by proving with induction on  $k \in \{1, \dots, N\}$  the following inequalities:

$$\begin{aligned} &M_{\tau/N}^{i_k} \cdots M_{\tau/N}^{i_1}[P_0] \\ &\leq P_0 + \frac{\tau}{N} \Phi^{i_1}(P_0) + \cdots + \frac{\tau}{N} \Phi^{i_k}(P_0) + \frac{L}{e} \left(1 + \frac{1}{N}\right)^k \frac{\tau^2 k^2}{N^2} I \end{aligned}$$

where  $I$  is the identity matrix. The case  $k = 1$  can be proved by showing the existence of  $L > 0$  and  $\tau_0 > 0$  such that:

$$\|M_\tau^m[P] - P - \tau \Phi^m(P_0)\| \leq L\tau^2 + L\tau\|P - P_0\|,$$

for all  $\tau \leq \tau_0$ ,  $P, P_0 \in \mathcal{K}$  and  $m \in \mathcal{M}$ . This is obtained by using the Mean value Theorem.  $\blacksquare$

Now we take into account the pruning procedure and analyze the error of

$$S_\tau \simeq \mathcal{P}_\tau \circ \tilde{S}_\tau.$$

Below is a direct consequence of Proposition 4.5.

*Corollary 4.6:* Let  $\epsilon, \lambda, r$  and  $\mathcal{P}_\tau$  be as in Theorem 4.2. Then there exists  $\tau_0 > 0$  and  $L > 0$  such that:

$$S_\tau[V_0](x) \leq (1 + L\tau^{\min\{2, r\}}) \mathcal{P}_\tau \circ \tilde{S}_\tau[V_0](x), \quad \forall x,$$

for all  $\tau \in [0, \tau_0]$  and  $V_0(x) = \frac{1}{2}x'P_0x$  with  $P_0 \in [\epsilon I, \lambda I]$ . Let  $s = \min\{2, r\}$ . Finally, Theorem 4.2 can be proved via the following inequalities:

$$\begin{aligned} &d_T(S_{k\tau}[V_0], \{\mathcal{P}_\tau \circ \tilde{S}_\tau\}^k[V_0]) \\ &\leq (1 + e^{-\alpha\tau} + \cdots + e^{-(k-1)\alpha\tau}) L\tau^s, \quad \forall k \in \mathbb{N}, \end{aligned} \quad (17)$$

where the constant  $\alpha$  follows from Theorem 3.4. This is proved by induction on  $k \in \mathbb{N}$ . The case  $k = 1$  is given by Corollary 4.6. For  $k > 1$  we use the following induction relations:

$$\begin{aligned} &d_T(S_{(k+1)\tau}[V_0], \{\mathcal{P}_\tau \circ \tilde{S}_\tau\}^{k+1}[V_0]) \\ &\leq d_T(S_{(k+1)\tau}[V_0], \mathcal{P}_\tau \circ \tilde{S}_\tau[S_{k\tau}[V_0]]) \\ &\quad + d_T(\mathcal{P}_\tau \circ \tilde{S}_\tau[S_{k\tau}[V_0]], \mathcal{P}_\tau \circ \tilde{S}_\tau[\{\mathcal{P}_\tau \circ \tilde{S}_\tau\}^k V_0]) \\ &\leq L\tau^s + d_T(\mathcal{P}_\tau \circ \tilde{S}_\tau[S_{k\tau}[V_0]], \mathcal{P}_\tau \circ \tilde{S}_\tau[\{\mathcal{P}_\tau \circ \tilde{S}_\tau\}^k V_0]) \\ &\leq L\tau^s + e^{-\alpha\tau} d_T(S_{k\tau}[V_0], \{\mathcal{P}_\tau \circ \tilde{S}_\tau\}^k[V_0]) \end{aligned} \quad (18)$$

where the second inequality follows from Corollary 4.6 and the invariance of the interval  $[\epsilon I, \lambda I]$  under each flow  $M_t^m$ . Lack of space, we omit the details for the last inequality which is a nontrivial consequence of the contraction result in Theorem 3.4. The difficulty lies in the fact that the uniform contraction only occurs on a compact set. From above we have shown that for all  $N \in \mathbb{N}$

$$d_T(S_{N\tau}[V_0], \{\mathcal{P}_\tau \circ \tilde{S}_\tau\}^N[V_0]) \leq L\tau^s(1 - e^{-\alpha\tau})^{-1}.$$

Finally note that

$$L\tau^s(1 - e^{-\alpha\tau})^{-1} \sim O(\tau^{s-1}) \quad \text{as } \tau \rightarrow 0.$$

Therefore we obtain the existence of  $L > 0$  such that:

$$d_T(S_{N\tau}[V_0], \{\mathcal{P}_\tau \circ \tilde{S}_\tau\}^N[V_0]) \leq L\tau^{s-1}, \quad \forall N \in \mathbb{N},$$

for all sufficiently small  $\tau$  and  $V_0(x) = \frac{1}{2}x'P_0x$  with  $P_0 \in [\epsilon I, \lambda I]$ . As in the sketch of proof of Theorem 4.1, the remains steps can be easily obtained by the definition of  $d_T$  (16) and the upper boundedness of the value function.

*Remark 4.7:* It should be pointed out that the crucial point is having  $\alpha > 0$ . If this is not the case ( $\alpha = 0$ ), then the iteration (18) only leads to:

$$d_T(S_{N\tau}[V_0], \{\mathcal{P}_\tau \circ \tilde{S}_\tau\}^N[V_0]) \leq LN\tau^s, \quad \forall N \in \mathbb{N}.$$

## V. FURTHER DISCUSSIONS

### A. Linear quadratic Hamiltonians

The contraction result being crucial to our analysis (see Remark 4.7), it is impossible to extend the result to the general case with linear terms as in [McE09]. However, the one step error analysis (Proposition 4.5) is not restricted to the pure quadratic Hamiltonian. Interested reader can verify that the one step error  $O(\tau^2)$  still holds in the case of [McE09] following the same lines of proof. Then by simply adding up the errors to time  $T$ , we get that:

$$\epsilon(x, \tau, N, V_0) \leq L(1 + |x|^2)N\tau^2 = L(1 + |x|^2)T\tau.$$

This estimation is of the same order as in [McE09] with much weaker assumption, especially the assumption on  $\Sigma^m$ .

### B. Convergence time

By Theorem 4.1, the finite horizon approximation error  $\epsilon_0(x, T, V_0)$  decreases exponentially with the time horizon  $T$ . Theorem 4.2 shows that for a sufficiently small  $\tau$  and a pruning procedure of error  $\tau^2$ , the discrete-time approximation error  $\epsilon^{\mathcal{P}\tau}(x, \tau, N, V_0)$  is  $O(\tau)$  uniformly for all  $N > 0$ . Therefore, for a fixed sufficiently small  $\tau$ , the total error decreases at each propagation step and becomes stationary after a time horizon  $T$  such that  $\epsilon_0(x, T, V_0) \leq \epsilon^{\mathcal{P}\tau}(x, \tau, N, V_0)$ .

To give an illustration, we implemented this max-plus approximation method, incorporating a pruning algorithm in [GMQ11] to a problem instance satisfying Assumption 2.1 and 3.2 in dimension  $n = 2$  and with  $|\mathcal{M}| = 3$  switches. The pruning algorithm generates at most  $\tau^2$  error at each step. We use the value of  $|H|_\infty$  on the region  $[-2, 2] \times [-2, 2]$  to measure the approximation. We observe that for each  $\tau$ , the backsubstitution error  $|H|_\infty$  becomes stationary after a number of iterations, see Figure 1 for  $\tau = 0.0006$ . We run

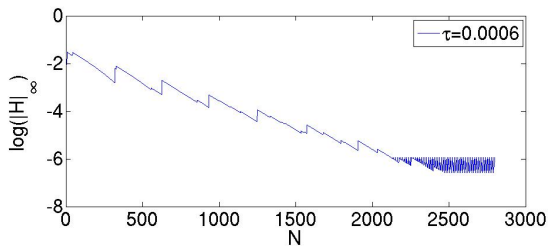


Fig. 1: Plot of  $\log |H|_\infty$  w.r.t. the iteration number  $N$

the instance for different  $\tau$  and for each  $\tau$  we collect the time horizon  $T$  when the backsubstitution error becomes stationary. The plot shows a linear growth of  $T$  with respect to  $-\log(\tau)$ , which is an illustration of the exponential decreasing rate in Theorem 4.1.

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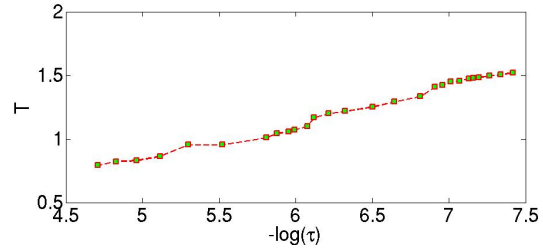


Fig. 2: Plot of the convergence time  $T$  w.r.t.  $-\log(\tau)$

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